ECON 602 - Macroeconomic Analysis II

Comprehensive Exam

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Note that the timing convention of these questions are different from the one adopted in Spring 2008 and later.

3. Here is the description of the economy. Use the common framework for all the models we discussed in the lectures with the following changes:

- There is no capital in the economy. Just like the models we worked with before there are two assets in the economy: a one-period nominal bond that issued by the government and held by the households with gross nominal return R_t and fiat money. The beginning-of-the-period stocks are denoted by (b_t, m_t).
- A neoclassical firm has a constant-returns-to-scale production function.
- When an agent consumes c_t units of consumption goods when he has m_{t+1}/p_t units of real money balances (end-of-period, as usual) he incurs a **time** loss of $\phi_t \equiv \phi(c_t, m_{t+1}/p_t)$. All we know about the function ϕ is that $\phi(c, z) > 0$, $\phi(0, z) = 0$, $\phi_1 > 0$ and $\phi_2 < 0$. This is the **key** difference in this question.
- The agent gets utility from consumption and leisure, given by $u(c, \ell)$.
- There is also a government in this economy with a standard budget constraint using income from a labor income tax (τ_t^h), printing money and issuing bonds and spending G_t every period, where $\{G_t\}_{t=1}^{\infty}$ is a deterministic and known sequence. To be clear, there are no lump-sum taxes.
- When using partial derivatives of functions, use subscript 1 and 2 to denote the partial derivative with respect to the first and second arguments.

1. (a) (25 points) **(Households)**

   i. *(5 points) Write down the problem of the representative household in this economy, being careful about what it chooses and what it takes as given. Do not solve this problem yet, just state the problem.*
   The problem is, given $\left\{\tau_t^h, R_t, p_t, w_t\right\}_{t=0}^{\infty}$ and b_0, m_0 to solve

$$V(b_t, m_t) = \max_{c_t, h_t, b_{t+1}, m_{t+1}} u(c_t, \ell_t) + \beta V(b_{t+1}, m_{t+1}) \tag{1}$$

$$\left(1 - \tau_t^h\right) p_t w_t h_t + R_{t-1} b_t + m_t - p_t c_t - b_{t+1} - m_{t+1} \geq 0 \tag{2}$$

$$\ell_t + h_t + \phi(c_t, m_{t+1}/p_t) = 1 \tag{3}$$

ii. *(15 points) Write down the conditions that characterize the solution to this problem. (No multipliers should appear here.)*

The FOC from this problem, using λ_t as the multiplier, and substituting the second constraint in to the utility function are

$$\begin{aligned}
c_t &: \quad u_1(t) - u_2(t)\phi_1(t) - p_t\lambda_t = 0 && (4)\\
h_t &: \quad -u_2(t) + \lambda_t\left(1-\tau_t^h\right)p_t w_t = 0 && (5)\\
b_{t+1} &: \quad -\lambda_t + \beta V_1(t+1) = 0 && (6)\\
m_{t+1} &: \quad -\frac{u_2(t)\phi_2(t)}{p_t} - \lambda_t + \beta V_2(t+1) = 0 && (7)
\end{aligned}$$

and the EC are

$$\begin{aligned}
V_1(t) &= \lambda_t R_{t-1} && (8)\\
V_2(t) &= \lambda_t && (9)
\end{aligned}$$

which yield

$$\begin{aligned}
\lambda_t &= \beta\lambda_{t+1}R_t && (10)\\
\frac{u_2(t)\phi_2(t)}{p_t} + \lambda_t &= \beta\lambda_{t+1} && (11)
\end{aligned}$$

From the first two FOC, we can solve for the multiplier as

$$\begin{aligned}
\lambda_t &= \frac{u_1(t) - u_2(t)\phi_1(t)}{p_t} && (12)\\
\lambda_t &= \frac{u_2(t)}{\left(1-\tau_t^h\right)p_t w_t} && (13)
\end{aligned}$$

Using these, we can write the optimality conditions as

$$u_1(t) - u_2(t)\phi_1(t) = \frac{u_2(t)}{\left(1-\tau_t^h\right)w_t} \quad (14)$$

$$\frac{u_1(t) - u_2(t)\phi_1(t)}{p_t} = \beta\frac{u_1(t+1) - u_2(t+1)\phi_1(t+1)}{p_{t+1}}R_t \quad (15)$$

$$\frac{u_1(t) - u_2(t)\phi_1(t) + u_2(t)\phi_2(t)}{p_t} = \beta\frac{u_1(t+1) - u_2(t+1)\phi_1(t+1)}{p_{t+1}} \quad (16)$$

$$\left(1-\tau_t^h\right)p_t w_t h_t + R_{t-1}b_t + m_t = p_t c_t + b_{t+1} + m_{t+1} \quad (17)$$

along with the transversality conditions

$$\begin{aligned}
\lim_{t\to\infty}\beta^t\lambda_t b_{t+1} &= 0 && (18)\\
\lim_{t\to\infty}\beta^t\lambda_t m_{t+1} &= 0 && (19)
\end{aligned}$$

where we cannot have simply $u_1(t)$ in the TVC and we need to include the multiplier on the BC, as it is originally stated.

iii. *(5 points) Write down a condition (derived from the conditions above) that summarize the money demand decision on the margin and briefly interpret it. (MB=MC)*

Combining (10) and (11), we get

$$-\frac{u_2(t)\,\phi_2(t)}{p_t} = \beta\left[\frac{u_1(t+1) - u_2(t+1)\,\phi_1(t+1)}{p_{t+1}}\right](R_t - 1) \tag{20}$$

where the LHS is the benefit of having one unit more nominal balances in period t, which is the increase in utility (ϕ_2 is negative) by having to spend less time shopping and the RHS is what you can do with that one unit of nominal balances if you hold it as bonds and carry over to the next period: you discount, you get $R_t - 1$ net returns and use it to spend on consumption which increases your consumption but also increases your shopping time.

(b) (3 points) (**Firm**) *Write down the problem of the firm and solve it.*

Since there is no capital and the firm is a CRS firm the production function is $Y_t = H_t$ and the problem of the firm is

$$\max_{H_t^D} p_t Y_t - p_t w_t H_t^D \tag{21}$$

$$= \max_{H_t^D} p_t H_t^D - p_t w_t H_t^D \tag{22}$$

whose first order condition yields

$$w_t = 1 \tag{23}$$

(c) (2 points) (**Government**) *Write down the budget constraint of the government.*

$$p_t G_t + R_{t-1} B_t + M_t = \tau_t^h p_t w_t H_t + B_{t+1} + M_{t+1} \tag{24}$$

(d) (15 points) **(Equilibrium)** *Using all the information so far, carefully define the equilibrium. Your definition must be self-contained. [At the expense of being redundant, write down all the equations that need to be in this definition. Don't worry about simplifying the equations further.]*

Definition 1 *A* ***competitive equilibrium*** *is a list of sequences* $\{c_t, h_t\}_{t=0}^{\infty}$ *[allocations],* $\{m_t, b_t\}_{t=0}^{\infty}$ *[portfolio choices],* $\{p_t, w_t\}_{t=0}^{\infty}$ *[prices],* $\{B_t\}_{t=0}^{\infty}$ *taking as given* $\left\{\tau_t^h, R_t\right\}_{t=0}^{\infty}$ *[policy variables] ,*$\{G_t\}_{t=0}^{\infty}$*,* m_0*,* B_0 *and* b_0 *such that*

1. *Households optimize : Given prices and policy variables, allocations and portfolio choices solve the household's problem by satisfying (14)-(17), the transversality conditions (18) and (19).*

2. *Firms optimize :* $w_t = 1$.

3. *Government's budget constraint (24) holds.*

4. Consistency / Market clearing : $b_t = B_t$.

In order to write down the equilibrium conditions, we should impose the conditions above to the optimality conditions of the household. So the equilibrium is a list of sequences $\{c_t, h_t, m_t, b_t, p_t\}_{t=0}^{\infty}$ *that satisfy the following equations (plus the TVC) given* $\left\{G_t, R_t, \tau_t^h\right\}_{t=0}^{\infty}$, m_0, b_0

$$u_1(t) - u_2(t)\phi_1(t) = \frac{u_2(t)}{\left(1-\tau_t^h\right) w_t} \tag{25}$$

$$\frac{u_1(t) - u_2\left[c_t, 1-h_t-\phi\left(c_t, \frac{m_{t+1}}{p_t}\right)\right]\phi_1(t)}{p_t} = \beta\frac{u_1(t+1) - u_2(t+1)\phi_1(t+1)}{p_{t+1}}R_t \tag{26}$$

$$\frac{u_1(t) - u_2(t)\phi_1(t) + u_2(t)\phi_2(t)}{p_t} = \beta\frac{u_1(t+1) - u_2(t+1)\phi_1(t+1)}{p_{t+1}} \tag{27}$$

$$\left(1-\tau_t^h\right) p_t w_t h_t + R_{t-1}b_t + m_t = p_t c_t + b_{t+1} + m_{t+1} \tag{28}$$

$$G_t + R_{t-1}b_t + m_t = \tau_t^h H_t + b_{t+1} + m_{t+1} \tag{29}$$

where $\pi_{t+1} \equiv p_{t+1}/p_t$, $u(t) = u\left[c_t, 1-h_t-\phi\left(c_t, m_{t+1}/p_t\right)\right]$ *and* $\phi(t) = \left(c_t, m_{t+1}/p_t\right)$.

1. e. (20 points) **(Ramsey Planner's Problem)** *Write down the Ramsey planner's problem with the objective function, things that he chooses, constraints and multipliers carefully specified. Do not proceed further, just write down the problem. You will have specific functional forms to work with in the next question which will make life much easier. [Hint: In order to get the implementability constraint, think about the analogy between this model and a MIU model. This is an important part of this question.]*

   Using $z_t = m_{t+1}/p_t$, Ramsey planner's problem is

$$\max_{\{c_t, h_t, z_t\}} \sum_{t=0}^{\infty} \beta^t u\left[c_t, 1-h_t-\phi\left(c_t, z_t\right)\right] \tag{30}$$

$$h_t - G_t - c_t \geq 0 \ \ [\mu_t] \tag{31}$$

$$\sum_{t=0}^{\infty} \beta^t \left\{\left[u_1(t) - u_2(t)\phi_1(t)\right]c_t - h_t u_2(t) - z_t u_2(t)\phi_2(t)\right\} - A_0 \geq 0 \ \ [\lambda] \tag{32}$$

   where he chooses $\{c_t, h_t, z_t\}$, by maximizing the lifetime utility of the households, respecting feasibility (with multiplier μ_t) and the implementability constraint (with multiplier λ). In order to get the implementability condition, think about re-writing the utility function as

$$v\left(c_t, 1-h_t, z_t\right) \equiv u\left[c_t, 1-h_t-\phi\left(c_t, z_t\right)\right] \tag{33}$$

   From our work in the lectures we know the PVIC for the MIU model is given by

$$\sum_{t=0}^{\infty} \beta^t \left[v_1(t) c_t - h_t v_2(t) + z_t v_3(t)\right] - A_0 \geq 0 \tag{34}$$

and substituting the $v(.)$ above we get the expression above. The problem in Lagrangian form is

$$\max_{\{c_t,h_t,z_t\}} \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} u\left[c_t, 1-h_t-\phi(c_t,z_t)\right] \\ +\lambda\left\{\left[u_1(t)-u_2(t)\phi_1(t)\right]c_t - h_t u_2(t) - z_t u_2(t)\phi_2(t) - A_0\right\} \\ +\mu_t\left[h_t - G_t - c_t\right] \end{array} \right\} \tag{35}$$

f. (35 points) **(Optimal Policy)** *Assume the following functional forms*

$$\begin{aligned} u(c,\ell) &= c\ell \\ \phi(c,z) &= \gamma\left(\frac{z}{c}\right)c \end{aligned}$$

where $\gamma(.)$ *is a function such that* $\gamma \geq 0$, $\gamma' < 0$ *and* $\gamma'' > 0$ *(note the strict inequalities).*

Note that this is not a very nice utility function. $\log(c\ell)$ *would be better than just* $c\ell$.

1. (a) i. (5 points) *What property does the* $\phi(c,z)$ *function has with the given restriction?*

It is homogenous of order 1, or CRS.

ii. (15 points) *Solve the Ramsey problem in part (e). You need to write down all the equations and inequalities that characterize the solution. Do not proceed any further in this part.*

With this utility function we have

$$u_1 = 1-h-\gamma\left(\frac{z}{c}\right)c \tag{36}$$

$$u_2 = c \tag{37}$$

$$\phi_1 = \gamma\left(\frac{z}{c}\right) - \gamma'\left(\frac{z}{c}\right)\frac{z}{c} \tag{38}$$

$$\phi_2 = \gamma'\left(\frac{z}{c}\right) \tag{39}$$

and the Ramsey problem in Lagrangian form is

$$\max_{\{c_t,h_t,z_t\}} \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} c_t\left[1-h_t-\gamma\left(\frac{z_t}{c_t}\right)c_t\right]+ \\ +\lambda\left\{ \begin{array}{c} \left[1-h-\gamma\left(\frac{z_t}{c_t}\right)c - c\left\{\gamma\left(\frac{z_t}{c_t}\right) - \gamma'\left(\frac{z_t}{c_t}\right)\frac{z_t}{c_t}\right\}\right]c_t \\ -h_t c_t - z_t c_t \gamma'\left(\frac{z}{c}\right) - A_0 \end{array} \right\} \\ +\mu_t\left[h_t - G_t - c_t\right] \end{array} \right\} \tag{40}$$

$$\max_{\{c_t,h_t,z_t\}} \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} c_t - c_t h_t - \gamma\left(\frac{z_t}{c_t}\right)c_t^2 \\ +\lambda\left\{c_t - 2hc_t - 2\gamma\left(\frac{z_t}{c_t}\right)c^2 - A_0\right\} + \mu_t\left[h_t - G_t - c_t\right] \end{array} \right\} \tag{41}$$

with first order conditions (dropping time subscripts)

$$c_t : 1 - h - \left[-z\gamma'\left(\frac{z}{c}\right) + 2c\gamma\left(\frac{z}{c}\right)\right] \tag{42}$$

$$+\lambda\left\{1 - 2h - 2\left[-z\gamma'\left(\frac{z}{c}\right) + 2c\gamma\left(\frac{z}{c}\right)\right]\right\} - \mu = 0 \tag{43}$$

$$h_t : -c - 2\lambda c + \mu = 0 \tag{44}$$

$$z_t : -\gamma'\left(\frac{z}{c}\right)c - 2\lambda\gamma'\left(\frac{z}{c}\right)c = 0 \tag{45}$$

and the complementary slackness conditions

$$\lambda\left\{c_t - 2hc_t - 2\gamma\left(\frac{z_t}{c_t}\right)c^2 - A_0\right\} = 0 \tag{46}$$

$$\mu_t\left[h_t - G_t - c_t\right] = 0 \tag{47}$$

and the multipliers must be nonnegative

$$\lambda \geq 0 \text{ and } \mu_t \geq 0 \tag{48}$$

iii. (15 points) *Is the Friedman Rule optimal? If so, prove. If not, find additional restrictions on the functions that would make it optimal.*
The important FOC is (45). It simplifies to

$$(1 + 2\lambda)\gamma'\left(\frac{z}{c}\right)c = 0 \tag{49}$$

Now, we know $c > 0$ must hold. We also know $\lambda \geq 0$. So, for this equation to hold $\gamma'\left(\frac{z}{c}\right) = 0$ must hold. Note that this is exactly the expression for ϕ_2. Let's now assume the Friedman rule is in effect. From the equilibrium condition we derived in a(iii) this implies

$$u_2(t)\,\phi_2(t) = 0 \tag{50}$$

which can only hold if $\phi_2(t) = 0$ since u_2 will not equal zero. So, the Ramsey problem implies $\phi_2(t) = 0$ and this implies Friedman rule is optimal. The only problem, however is that in defining the functions, we clearly states $\gamma' < 0$, which means the cost function has no satiation point for real money balances. Therefore $\phi_2 = 0$ can never hold with this specification and FR is not optimal.
If we change $\gamma(.)$ to allow $\gamma(.)' = 0$, then FR is optimal.

iv. *(BONUS) Link the result in (iii) to the results you know from the MIU model. [This is optional. Attempt to answer this only if you have time.]*