ECON 602 - Macroeconomic Analysis II

Comprehensive Exam

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This question is from Correia (*J Pub E,* 1996).

a. Household's Problem. Taking as given $\left\{\tau_t^k, \tau_t^h, w_t, r_t, s_t, R_t^b\right\}$ and all stochastic processes the household solves

$$\max_{\{c_t,k_t,b_t,h_t,L_t\}} E_0 \sum_t \beta^t \left[\log(c_t) + \log(1-h_t) + \log(1-L_t)\right] \tag{1}$$

subject to

$$c_t + k_t + b_t \leq \left(1-\tau_t^h\right) w_t h_t + s_t L_t + R_t^k k_{t-1} + R_t^b b_{t-1} \tag{2}$$

where

$$R_t^k \equiv 1 + \left(1-\tau_t^k\right)(r_t - \delta) \tag{3}$$

FOC are

$$\frac{1}{c_t} = \xi_t \tag{4}$$

$$\frac{1}{1-h_t} = \xi_t \left(1-\tau_t^h\right) w_t \tag{5}$$

$$\frac{1}{1-L_t} = \xi_t s_t \tag{6}$$

$$\xi_t = \beta E_t \left(\xi_{t+1} R_{t+1}^k\right) \tag{7}$$

$$\xi_t = \beta E_t \left(\xi_{t+1} R_{t+1}^b\right) \tag{8}$$

where $\beta^t \xi_t$ is the multiplier on (2). Rearranging

$$\frac{c_t}{1-h_t} = \left(1-\tau_t^h\right) w_t \tag{9}$$

$$\frac{c_t}{1-L_t} = s_t \tag{10}$$

$$\frac{1}{c_t} = \beta E_t \left(\frac{R_{t+1}^k}{c_{t+1}}\right) \tag{11}$$

$$\frac{1}{c_t} = \beta E_t \left(\frac{R_{t+1}^b}{c_{t+1}}\right) \tag{12}$$

along with the appropriate TVCs characterize the solution to the household's problem.

b. The firm's problem yields

$$
\begin{aligned}
r_t &= F_K\left(k_{t-1}, h_t, L_t\right) && (13)\\
w_t &= F_H\left(k_{t-1}, h_t, L_t\right) && (14)\\
s_t &= F_L\left(k_{t-1}, h_t, L_t\right) && (15)
\end{aligned}
$$

c. Equilibrium is a list $\left\{c_t, k_t, h_t, L_t, b_t, w_t, r_t, s_t, R_t^b\right\}$, given $\left\{\tau_t^h, \tau_t^k, g_t\right\}$, k_0, b_0 and the processes for exogenous processes, that satisfy (2), (9)-(12) and (13)-(15). Collecting everything we get

$$
\begin{aligned}
\frac{c_t}{1-h_t} &= \left(1-\tau_t^h\right) F_H\left(k_{t-1}, h_t, L_t\right) && (16)\\
\frac{c_t}{1-L_t} &= F_L\left(k_{t-1}, h_t, L_t\right) && (17)\\
\frac{1}{c_t} &= \beta E_t\left\{\frac{1+\left(1-\tau_{t+1}^k\right)\left[F_K\left(k_t, h_{t+1}, L_{t+1}\right)-\delta\right]}{c_{t+1}}\right\} && (18)\\
\frac{1}{c_t} &= \beta E_t\left(\frac{R_{t+1}^b}{c_{t+1}}\right) && (19)\\
c_t+k_t+g_t &= F\left(k_{t-1}, h_t, L_t\right)+(1-\delta) k_{t-1} && (20)\\
c_t+k_t+b_t &= \left(1-\tau_t^h\right) F_H\left(k_{t-1}, h_t, L_t\right) h_t+F_L\left(k_{t-1}, h_t, L_t\right) L_t && (21)\\
&\quad +\left(1-\tau_t^k\right)\left[F_K\left(k_{t-1}, h_t, L_t\right)-\delta\right] k_{t-1}+R_t^b b_{t-1} && (22)
\end{aligned}
$$

to define $\left\{c_t, k_t, h_t, L_t, b_t, R_t^b\right\}$ which are 6 equations in 6 unknowns.

d. The naive Ramsey problem will be

$$
\max_{\left\{c_t, h_t, k_t, L_t\right\}} E_0 \sum_t \beta^t\left[\log \left(c_t\right)+\log \left(1-h_t\right)+\log \left(1-L_t\right)\right] \quad (23)
$$

subject to

$$
\begin{aligned}
E_0 \sum_{t=0} \beta^t\left[1-\frac{h_t}{1-h_t}-\frac{L_t}{1-L_t}\right] &= A_0 \text{ (w/ multiplier } \lambda) && (24)\\
F\left(k_{t-1}, h_t, L_t\right)+(1-\delta) k_{t-1}-c_t-g_t-k_t &\geq 0 \text{ (w/ multiplier } \beta^t \gamma_t) && (25)
\end{aligned}
$$

with FOC

$$
\frac{1}{c_t}-\gamma_t=0 \quad (26)
$$

$$
-\frac{1}{1-h_t}-\frac{\xi\left(1-h_t+h_t\right)}{\left(1-h_t\right)^2}+\gamma_t F_h\left(k_{t-1}, h_t, L_t\right)=0 \quad (27)
$$

$$
-\frac{1}{1-L_t}-\frac{\xi\left(1-L_t+L_t\right)}{\left(1-L_t\right)^2}+\gamma_t F_L\left(k_{t-1}, h_t, L_t\right)=0 \quad (28)
$$

$$
-\gamma_t+\beta E_t\left\{\gamma_{t+1}\left[F_k\left(k_t, h_{t+1}, L_{t+1}\right)+1-\delta\right]\right\}=0 \quad (29)
$$

and rearranging

$$\frac{c_t}{1-h_t}\left[1+\frac{\xi}{1-h_t}\right]=F_h\left(k_{t-1},h_t,L_t\right) \tag{30}$$

$$\frac{c_t}{1-L_t}\left[1+\frac{\xi}{1-L_t}\right]=F_L\left(k_{t-1},h_t,L_t\right) \tag{31}$$

$$\frac{1}{c_t}=\beta E_t\left\{\frac{\left[F_k\left(k_t,h_{t+1},L_{t+1}\right)+1-\delta\right]}{c_{t+1}}\right\} \tag{32}$$

Proposition 1 *The allocations from the naive Ramsey problem cannot be implemented under any competitive equilibrium.*

Proof. To see this, compare the equilibrium condition (17) with the Ramsey problem's solution (31)

$$\frac{c_t}{1-L_t} = F_L\left(k_{t-1},h_t,L_t\right) \tag{33}$$

$$\frac{c_t}{1-L_t}\left[1+\frac{\xi}{1-L_t}\right] = F_L\left(k_{t-1},h_t,L_t\right) \tag{34}$$

We see that both equations cannot hold unless $\xi=0$, which would be the case when Ramsey planner is not constrained by the PVIC, and this cannot happen. ■

e.

Proposition 2 *The long-run capital income tax rate is positive iff L and K are complements and negative iff they are substitutes.*

To prove this, we first need to solve the "proper" Ramsey problem with

$$\frac{c_t}{1-L_t}-F_L\left(k_{t-1},h_t,L_t\right)\geq \ \text{(w/ multiplier } \beta^t\mu_t) \tag{35}$$

to the problem. The FOC are

$$\frac{1}{c_t}-\gamma_t+\frac{\mu_t}{1-L_t}=0 \tag{36}$$

$$-\frac{1}{1-h_t}-\frac{\xi}{\left(1-h_t\right)^2}+\gamma_t F_h\left(k_{t-1},h_t,L_t\right)-\mu_t F_{LH}\left(k_{t-1},h_t,L_t\right)=0 \tag{37}$$

$$-\frac{1}{1-L_t}-\frac{\xi}{\left(1-L_t\right)^2}+\gamma_t F_L\left(k_{t-1},h_t,L_t\right)-\frac{\mu_t c_t}{\left(1-L_t\right)^2}=0 \tag{38}$$

$$-\gamma_t+\beta E_t\left\{\gamma_{t+1}\left[F_k\left(k_t,h_{t+1},L_{t+1}\right)+1-\delta\right]\right\}-\mu_t F_{LK}\left(k_{t-1},h_t,L_t\right)=0 \tag{39}$$

Simplifying and imposing steady state on (36) , we get

$$\gamma=\beta\gamma\left(F_k+1-\delta\right)-\mu F_{LK} \tag{40}$$

and combining with the equilibrium condition (18) at the steady state

$$1=\beta\left\{1+\left(1-\tau_{t+1}^k\right)\left[F_K\left(k_t,h_{t+1},L_{t+1}\right)-\delta\right]\right\} \tag{41}$$

we get

$$\tau^k=\frac{\mu F_{LK}}{\beta\gamma\left(F_K-\delta\right)} \tag{42}$$