

ECON 602 - Macroeconomic Analysis II

Solutions to Comprehensive Exam

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Note that the timing convention of these questions are different from the one adopted in Spring 2008 and later.

a. (30 points) Define competitive equilibrium.

The household's problem is

$$\max_{\{c_{1t}, c_{2t}, h_t, m_t, b_t\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_{1t}, c_{1,t-1}, c_{2t}, c_{2,t-1}, 1 - h_t) \quad (1)$$

$$m_t - (1 + \tau_t^c) p_t c_{1t} \geq 0 \quad (\text{with multiplier } \beta^t \mu_t) \quad (2)$$

$$R_{t-1} b_t + m_{t-1} + p_{t-1} (1 - \tau_{t-1}^h) w_{t-1} h_{t-1} - b_{t+1} - m_t \quad (3)$$

$$- (1 + \tau_{t-1}^c) p_{t-1} c_{1,t-1} - (1 + \tau_{t-1}^c) p_{t-1} c_{2,t-1} \geq 0 \quad (\text{with multiplier } \beta^t \lambda_t) \quad (4)$$

with the FOC

$$c_{1t} : u_1(t) + \beta u_2(t+1) - \mu_t (1 + \tau_t^c) p_t - \beta p_t (1 + \tau_t^c) E_t(\lambda_{t+1}) = 0 \quad (5)$$

$$c_{2t} : u_3(t) + \beta u_4(t+1) - \beta p_t (1 + \tau_t^c) E_t(\lambda_{t+1}) = 0 \quad (6)$$

$$h_t : -u_5(t) + \beta p_t (1 - \tau_t^h) w_t E_t(\lambda_{t+1}) = 0 \quad (7)$$

$$m_t : \mu_t - \lambda_t + \beta E_t(\lambda_{t+1}) = 0 \quad (8)$$

$$b_{t+1} : -\lambda_t + \beta R_t E_t(\lambda_{t+1}) = 0 \quad (9)$$

Following usual steps, we get

$$[u_3(t) + \beta u_4(t+1)] = \beta p_t (1 + \tau_t^c) E_t(\lambda_{t+1}) \quad (10)$$

$$[u_1(t) + \beta u_2(t+1)] - [u_3(t) + \beta u_4(t+1)] = \mu_t (1 + \tau_t^c) p_t \quad (11)$$

$$\frac{(1 + \tau_t^c)}{(1 - \tau_t^h) w_t} = \frac{[u_3(t) + \beta u_4(t+1)]}{u_5(t)} \quad (12)$$

$$\lambda_t = \frac{[u_1(t) + \beta u_2(t+1)]}{(1 + \tau_t^c) p_t} \quad (13)$$

$$\frac{[u_1(t) + \beta u_2(t+1)]}{(1 + \tau_t^c) p_t} = \beta R_t E_t \left\{ \frac{[u_1(t+1) + \beta u_2(t+2)]}{(1 + \tau_{t+1}^c) p_{t+1}} \right\} \quad (14)$$

The firm's problem is obvious yielding

$$w_t = A_t \quad (15)$$

A CE in this economy is $\{C_{1t}, C_{2t}, H_t, M_t, B_t, P_t, w_t\}$, given exogenous government policies $\{\tau_t^c, \tau_t^h, R_t\}$ satisfying

$$\frac{[u_1(t) + \beta u_2(t+1)]}{(1 + \tau_t^c) p_t} = \beta R_t E_t \left\{ \frac{[u_1(t+1) + \beta u_2(t+2)]}{(1 + \tau_{t+1}^c) p_{t+1}} \right\} \quad (16)$$

$$\frac{(1 + \tau_t^c)}{(1 - \tau_t^h) w_t} = \frac{[u_3(t) + \beta u_4(t+1)]}{u_5(t)} \quad (17)$$

$$\frac{[u_1(t) + \beta u_2(t+1)]}{[u_3(t) + \beta u_4(t+1)]} = R_t \quad (18)$$

$$M_t = (1 + \tau_t^c) p_t C_{1t} \quad (19)$$

$$w_t = A_t \quad (20)$$

$$R_{t-1} B_t + M_{t-1} + p_{t-1} (1 - \tau_{t-1}^h) w_{t-1} H_{t-1} - B_{t+1} - M_t = (1 + \tau_{t-1}^c) p_{t-1} C_{1t-1} + (1 + \tau_{t-1}^c) p_{t-1} C_{2t-1} \quad (21)$$

$$A_t H_t = C_{1t} + C_{2t} + G \quad (22)$$

along with the TVCs and initial conditions where the LOM for A_t given by

$$A_{t+1} = A(A_t) \quad (23)$$

b. (23 points) Characterize the optimal policy.

Let's stick to the general notation as much as we can. The PVIC is given by the equation that was given in the exam. The Ramsey planner's problem is given by

$$\max_{\{C_{1t}, C_{2t}, H_t\}} E_0 \sum_{t=0}^{\infty} \beta^t u(C_{1t}, C_{1,t-1}, C_{2t}, C_{2,t-1}, 1 - H_t) \quad (24)$$

$$E_0 \sum_{t=0}^{\infty} \beta^t \{ [u_1(t) + \beta u_2(t+1)] C_{1t} + [u_3(t) + \beta u_4(t+1)] C_{2t} - u_5(t) H_t \} = \Xi \quad (\text{with multiplier } \lambda) \quad (25)$$

$$A_t H_t - C_{1t} - C_{2t} - G \geq 0 \quad (\text{with multiplier } \beta^t \mu_t) \quad (26)$$

or

$$\max_{\{C_{1t}, C_{2t}, H_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \{ u(C_{1t}, C_{1,t-1}, C_{2t}, C_{2,t-1}, 1 - H_t) \} \quad (27)$$

$$+ \lambda [u_1(t) + \beta u_2(t+1)] C_{1t} + \lambda [u_3(t) + \beta u_4(t+1)] C_{2t} - \lambda u_5(t) H_t - \lambda \Xi \quad (28)$$

$$+ \mu_t [A_t H_t - C_{1t} - C_{2t} - G] \quad (29)$$

For now, we drop the (extra) constraint that $R_t \geq 1$, which would have been enforced by adding the constraint

$$\frac{[u_1(t) + \beta u_2(t+1)]}{[u_3(t) + \beta u_4(t+1)]} \geq 1 \quad (30)$$

Note that at this stage we are not worried about the exact tools that the Ramsey planner has access to.

Note that this utility function is not very well-behaved. It would be nicer to use $\log(c_{1t}c_{2t} - \alpha c_{1,t-1}c_{2,t-1})$.

Let's write the problem one more time with the functional form and

$$u_1(t) = c_{2t} \quad (31)$$

$$u_2(t) = -\alpha c_{2,t-1} \quad (32)$$

$$u_3(t) = c_{1t} \quad (33)$$

$$u_4(t) = -\alpha c_{1,t-1} \quad (34)$$

$$u_5(t) = B \quad (35)$$

$$\max_{\{C_{1t}, C_{2t}, H_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \{C_{1t}C_{2t} - \alpha C_{1,t-1}C_{2,t-1} + B(1 - H_t)\} \quad (36)$$

$$+\lambda \underbrace{\left[\frac{C_{2t} - \alpha\beta C_{2t}}{(1-\alpha\beta)C_{2t}} C_{1t} + \lambda \frac{C_{1t} - \alpha\beta C_{1t}}{(1-\alpha\beta)C_{1t}} C_{2t} \right]}_{2\lambda(1-\alpha\beta)C_{1t}C_{2t}} - \lambda B H_t - \lambda \Xi \quad (37)$$

$$+\mu_t [A_t H_t - C_{1t} - C_{2t} - G] \quad (38)$$

The FOC for the Ramsey planner's problem is

$$C_{1t} : (1 - \alpha\beta) C_{2t} + 2\lambda(1 - \alpha\beta) C_{2t} - \mu_t = 0 \quad (39)$$

$$C_{2t} : (1 - \alpha\beta) C_{1t} + 2\lambda(1 - \alpha\beta) C_{1t} - \mu_t = 0 \quad (40)$$

$$H_t : -B - \lambda B + \mu_t A_t = 0 \quad (41)$$

or

$$C_{1t} : (1 - \alpha\beta) C_{2t} (1 + 2\lambda) = \mu_t \quad (42)$$

$$C_{2t} : (1 - \alpha\beta) C_{1t} (1 + 2\lambda) = \mu_t \quad (43)$$

$$H_t : (1 + \lambda) B = \mu_t A_t \quad (44)$$

These imply

$$C_{1t} = C_{2t} \quad (45)$$

$$(1 + \lambda) B = A_t (1 - \alpha\beta) C_{1t} (1 + 2\lambda) \quad (46)$$

It should be clear that these equations (along with TVCs, complementary slackness and initial conditions) completely characterize the optimal allocations for the Ramsey planner and we can also solve for the λ .

Now, let's see how we can decentralize these allocations.

First, (18) implies

$$\frac{(1 - \alpha\beta) C_{2t}}{(1 - \alpha\beta) C_{1t}} = R_t \quad (47)$$

which means under the optimal policy, $R_t = 1$ will hold. Note that since $R_t = 1$ for all t , dropping the constraint from the Ramsey planner's problem was OK here.

Now, (17) implies

$$\frac{(1 + \tau_t^c)}{(1 - \tau_t^h) A_t} = \frac{(1 - \alpha\beta) C_{1t}}{B} \quad (48)$$

It should already be clear that we have a problem here. Combining this equation with (46) we get

$$\frac{(1 + \tau_t^c)}{(1 - \tau_t^h)} = \frac{(1 + \lambda)}{(1 + 2\lambda)} \quad (49)$$

The only conclusion we can reach from here is that the ratio $(1 + \tau_t^c) / (1 - \tau_t^h)$ will be constant. There will be infinitely many combinations of policies $\{\tau_t^c, \tau_t^h\}$ that satisfy this equation. As should be obvious, there are too many tax instruments here.

- c. (23 points) Characterize the optimal policy when $\tau_t^h = 0$ is a constraint on the Ramsey planner.

This is going to be quick. This is the “standard” case. We still have $R_t = 1$ and the equation

$$\frac{(1 + \tau_t^c)}{(1 - \tau_t^h)} = \frac{(1 + \lambda)}{(1 + 2\lambda)} \quad (50)$$

still hold, except now $\tau_t^h = 0$ which means

$$(1 + \tau_t^c) = \frac{(1 + \lambda)}{(1 + 2\lambda)} \quad (51)$$

and the unique policy $\tau_t^c > 0$ is pinned down. (and yes, the tax rate is constant). Here, you could have been more methodical and introduced the equilibrium conditions implied by $\tau_t^h = 0$ as an additional constraint. You would have found that it binds at the optimal allocation.

- d. (24 points) Characterize the optimal policy when $\tau_t^h = \tau_t^c = 0$ is a constraint on the Ramsey planner.

This is the “nonstandard” case. What you should notice is that the Ramsey planner is now constrained. The extra constraint is given by the equilibrium condition (17) with $\tau_t^h = \tau_t^c = 0$ imposed.

$$\frac{1}{A_t} = \frac{(1 - \alpha\beta) C_{1t}}{B} \quad (52)$$

Using $\beta^t \xi_t$ to denote the multiplier on this constraint, the Ramsey planner's problem is

$$\begin{aligned} \max_{\{C_{1t}, C_{2t}, H_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \{ & C_{1t} C_{2t} - \alpha C_{1,t-1} C_{2,t-1} + B(1 - H_t) + 2\lambda(1 - \alpha\beta) C_{1t} C_{2t} - \lambda B H_t - \\ & + \xi_t [B - (1 - \alpha\beta) C_{1t} A_t] \\ & + \mu_t [A_t H_t - C_{1t} - C_{2t} - G]\} \end{aligned} \quad (54)$$

and the FOC are

$$C_{1t} : (1 - \alpha\beta) C_{2t} (1 + 2\lambda) - \xi_t (1 - \alpha\beta) A_t = \mu_t \quad (56)$$

$$C_{2t} : (1 - \alpha\beta) C_{1t} (1 + 2\lambda) = \mu_t \quad (57)$$

$$H_t : (1 + \lambda) B = \mu_t A_t \quad (58)$$

Combining the first two equations we get

$$C_{2t} - C_{1t} = \frac{\xi_t A_t}{(1 + 2\lambda)} > 0 \text{ as long as } \xi_t > 0 \quad (59)$$

where since $A_t > 0$ for all t and $\lambda > 0$, as long as the constraint doesn't bind the RHS will be positive.

As for the optimal policy, a deviation from the FR will be optimal as long as the constraint doesn't bind. From our derivations above, we know the constraint can never bind because we found the unique optimal policy would be $\tau_t^c > 0$.