MD Comps August 2004 – SOLUTION

(a)

$$c_t^1 = w_t m_t \qquad c_{t+1}^2 = v_{t+1} h_{t+1} \equiv v_{t+1} h_t \left(1 + (1 - m_t)\,\theta\right)$$

so on may write maximization problem as

$$\underset{m_t}{Max} \ln w_t m_t + \beta \ln v_{t+1} h_t \left(1 + (1 - m_t)\,\theta\right)$$

leading to FOC

$$\frac{w_t}{w_t m_t} = \beta \frac{v_{t+1}\theta h_t}{v_{t+1} h_t \left(1 + (1 - m_t)\,\theta\right)}$$

implying a constant m_t

$$m = \frac{1+\theta}{\theta\,(1+\beta)} \qquad c_t^1 = \frac{1+\theta}{\theta\,(1+\beta)} w_t \qquad c_{t+1}^2 = v_{t+1} h_t \frac{\beta}{1+\beta}\,(1+\theta)$$

where one must assume that $\theta \geq 1/\beta$.

(b) The model represents an externality from human capital accumulation as represented by (1). There is a unique equilibrium growth path. From the individual's point of view, since the cost of a unit of training (w_t) and the return to a unit of training ($v_{t+1} h_t \theta$) are both independent of m, so that there is a unique solution for $1 - m$, the fraction of his time spent getting educated. In general equilibrium ($w_t = \frac{\partial Y_t}{\partial L_t}$ and $v_{t+1} = \frac{\partial Y_{t+1}}{\partial S_{t+1}}$), the ratio $\frac{v_{t+1} h_t \theta}{w_t}$ is a function of (constant) m, but (due in part to the form of the utility function) independent of h, so there is no effect as in Azariadis and Drazen. The return to training in aggregate falls with $1 - m$, in the aggregate, that is, there is standrad decreasing returns in the aggregate.

(c) Growth rate of human capital is

$$\frac{h_{t+1}}{h_t} - 1 = (1 - m_t)\,\theta = \frac{\beta\theta - 1}{1+\beta}$$

which is positive as long as $\theta > 1/\beta$. Growth rate of output is

$$\frac{Y_{t+1}}{Y_t} - 1 = \frac{N h_{t+1}^{\gamma} m_{t+1}^{1-\gamma}}{N h_t^{\gamma} m_t^{1-\gamma}} - 1 = \left(\frac{h_{t+1}}{h_t}\right)^{\gamma} - 1 = \left(\frac{\beta\,(1+\theta)}{1+\beta}\right)^{\gamma} - 1$$

(d) The consumer's budget constraints may be written

$$c_t^1 = \left(1 - \tau^1\right) w_t m_t + T_t^1 \qquad c_{t+1}^2 = \left(1 - \tau^2\right) v_{t+1} h_t \left(1 + (1 - m_t)\,\theta\right) + T_{t+1}^2$$

Intuitively, if $\tau^2 = \tau^1$, the tax lowers the cost of human capital accumulation and the return equally, so m would be unaffected. If $\tau^2 > \tau^1$, then the tax lowers the return more than the cost, so that m will fall, as will the growth rate.

(This could be shown by maximizing the consumer's objective function with T_t^1 and T_{t+1}^2 treated parametrically to obtain

$$m = \frac{1+\theta}{\theta(1+\beta)} + \frac{\left(1-\tau^1\right) w_t T_{t+1}^2 - \beta\left(1-\tau^2\right) v_{t+1}\theta h_t T_t^1}{\theta(1+\beta)(1-\tau^1) w_t (1-\tau^2) v_{t+1} h_t}$$

and then substitute $T_t^1 = \tau^1 w_t m_t$ and $T_{t+1}^2 = \tau^2 v_{t+1} h_{t+1}$.)

(e) Since the return to bonds and to human capital investment are both parametric (see part (b)), when young the consumer only accumulates one of the two assets, depending on parameter values. When middle aged the consumers saves via bonds. Call saving via bonds when young s_t and when middle-aged j_{t+1}.

CASE 1: $1 + r \equiv R < \frac{v_{t+1} h_t \theta}{w_t}$ (return to HK exceeds return to bonds), so that $s_t = 0$ and $1 - m_t > 0$. The consumer's objective is

$$\ln w_t m_t + \beta \ln(v_{t+1} h_t (1 + (1 - m_t)\theta) - j_{t+1}) + \beta^2 \ln R j_{t+1}$$

By backwards recursion, solve first

$$\underset{j_{t+1}}{Max} \ln(v_{t+1} h_{t+1} - j_{t+1}) + \beta \ln R j_{t+1}$$

that is, for given m_t, yielding

$$j_{t+1} = \frac{\beta}{1+\beta} v_{t+1} h_{t+1}$$

Then solve, as above, for m_t conditional on this solution for j_{t+1}, yielding

$$m_t = \frac{1+\theta}{\theta(1+\beta(1+\beta))}$$

CASE 2: $R > \frac{v_{t+1} h_t \theta}{w_t}$ (return to bonds exceeds return to HK), so that $m_t = 1$, $h_{t+1} = h_t$, and $s_t > 0$. The consumer's objective is

$$\ln(w_t - s_t) + \beta \ln(v_{t+1} h_t + R s_t - j_{t+1}) + \beta^2 \ln R j_{t+1}$$

By backwards recursion, solve first

$$\underset{j_{t+1}}{Max} \ln(v_{t+1} h_t + R s_t - j_{t+1}) + \beta \ln R j_{t+1}$$

for given s_t, yielding

$$j_{t+1} = \frac{\beta}{1+\beta}(v_{t+1} h_t + R s_t)$$

Then solve for s_t conditional on this solution for j_{t+1}.

Note that though there are two different types of solutions, there is NOT multiple equilibrium, since the solution depends on exogenous parameters and not on initial conditions in terms of h_t. That is, an economy with given parameter values has a unique path independent of h_0.

(f) Case 1 as in (c) but with different values. Case 2 has no growth.