Solution: Comps Question 2 August 2006
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(a) The individual objective may be written:

$$\max_{s_t \geq 0} u\left(e - s_t\right) + \beta z f\left(s_t\right)$$

implying an optimal individual holding of shmoos defined implicitly by:

$$u'\left(e - s_t^*\right) = \beta z f'\left(s_t^*\right)$$

Since $zf\left(\cdot\right)$ represents an individual storage technology, s_t^* is also the equilibrium. It is also the steady state. Differentiating FOC, one obtains:

$$\frac{\partial s^*}{\partial z} = \frac{\beta f'\left(s^*\right)}{-u''\left(\cdot\right) - \beta f''\left(\cdot\right)} > 0$$

and

$$\frac{\partial s^*}{\partial \beta} = \frac{z f'\left(s^*\right)}{-u''\left(\cdot\right) - \beta f''\left(\cdot\right)} > 0$$

(b) The budget constraints of a representative individual are (where either the x^j or the τ^j are zero, depending on the tax regime):

$$\begin{aligned} c_t^y &= \left(1 - \tau_t^y\right) e - x_t^y - s_t \\ c_{t+1}^o &= \left(1 - \tau_{t+1}^o\right) z f\left(s_t\right) - x_{t+1}^o \end{aligned}$$

For the "initial" old, one has

$$c_0^o = \left(1 - \tau_0^o\right) z f\left(s^*\right) - x_0^o$$

The government's budget constraint is given by

$$g = x_t^y + x_t^o + \tau_t^y e + \tau_{t+1}^o z f\left(s^*\right)$$

(c) With lump-sum taxes the FOC becomes

$$u'\left(e - \bar{s} - x^y\right) = \beta z f'\left(\bar{s}\right)$$

where $\bar{s}$ is the lump-sum tax equilibrium. Therefore i)

$$\frac{\partial \bar{s}}{\partial x^y} = \frac{-u''\left(\cdot\right)}{u''\left(\cdot\right) + \beta z f''\left(\cdot\right)} < 0$$

which is an income effect.

ii) Because utility is linear in second period consumption, x^o has no effect on saving and hence $\bar{s}$.

With distortionary taxes, the FOC becomes

$$u'\left(\left(1-\tau_t^y\right)e-\hat{s}\right)=\left(1-\tau_{t+1}^o\right)\beta z f'\left(\hat{s}\right)$$

where $\hat{s}$ is the distortionary tax equilibrium. Hence

iii) τ_t^y acts like a lump-sum tax, exactly like x^y.

iv) τ_{t+1}^o reduces $\hat{s}$ even with linear utility because it reduces the net return to accumulating shmoos:

$$\frac{\partial \hat{s}}{\partial \tau_{t+1}^o}=\frac{\beta z f'\left(\hat{s}\right)}{u''\left(\cdot\right)+\left(1-\tau_{t+1}^o\right)\beta z f''\left(\cdot\right)}<0$$

(d) Write down the indirect utility function

$$U\left(x_t^y,x_{t+1}^o\right)=u\left(e-s\left(x_t^y,x_{t+1}^o\right)-x_t^y\right)+\beta\left(zf\left(s\left(x_t^y,x_{t+1}^o\right)-x_{t+1}^o\right)\right)$$

Using the envelope theorem (or the first-order condition), one obtains the effect on steady state welfare:

$$\frac{\partial U}{\partial x^y}=-u'\left(\cdot\right) \qquad \frac{\partial U}{\partial x^o}=-\beta$$

Since g is not in the utility function (at least explicitly), the only effect is via the effect of taxes reducing income. Since the FOC implies that $u'\left(\cdot\right)<\beta$, we have $\left|\frac{\partial U}{\partial x^o}\right|>\left|\frac{\partial U}{\partial x^y}\right|$.

(e) Consider optimal choice of x^y to maximize welfare given g:

$$\max_{x^y} U=u\left(e-\bar{s}\left(x^y\right)-x^y\right)+\beta\left(zf\left(\bar{s}\left(x^y\right)\right)-\left(g-x^y\right)\right)$$

with FOC

$$u'\left(\bar{c}^y\right)=\beta$$

But the FOC for the no-government equilibrium is:

$$u'\left(c^{y*}\right)=\beta z f'\left(s^*\right)<\beta$$

when $zf'\left(s^*\right)<1$. Hence optimal policy is $x^y>0$ which reduces c^y.

The tax transfer policy allows income to be transferred across periods one-for-one. When $zf'\left(s^*\right)<1$, accumulating shmoos has a lower rate of return, so that everyone can be made better off by the government transferring income across periods via lump-sum taxation. Note that $zf'\left(s^*\right)<1$ is the equivalent here of the condition for dynamic inefficiency from class.

(f) The representative individual's optimization problem may be written:

$$\max_{s_t,b_t} u\left(e-x^y-s_t-b_t\right)+\beta\left[R_{t+1}b_t+zf\left(s_t\right)-x^o\right]$$

(where $R = 1 + r$) yielding first-order conditions for s_t and b_t That is *demand* for s and b:

$$\begin{aligned} u'(e - x^y - s_t - b_t) &= \beta R \\ zf'(s_t) &= \beta \end{aligned}$$

(using $R_t = R$ for all t in this stationary equilibrium). When the government keeps b constant over time, its budget constraints in the "initial" period is:

$$g = x_1^y + x_0^o + b$$

and in each subsequent period $t \geq 1$, when the government pays off maturing debt, it is:

$$g + Rb = x_t^y + x_{t-1}^o + b$$

Note that though x_t^y and x_t^o are constant x_0^o need not equal x_t^o (and in general $x_0^o \neq x_t^o$).

Consider first a policy of using only x^o to finance expenditure and debt repayment. Using the FOCs in this case, we have

$$u'(e - s - b) = \beta z f'(s)$$

so that equilibrium s is decreasing in b, but does not depend directly on x^o. $R = zf'(s)$ implies that increases in b that lower s thus raise R.

Holding g constant across different levels of b, the initial budget constraint is

$$x_0^o = g - b$$

For all future generations, the government budget constraint implies:

$$x^o = g + (R - 1) b$$

Therefore for given g an increase in b unambiguously lowers x_0^o and raises x^o, with welfare of initial old rising and of subsequent generations falling. What debt does is transfer resources to the initial old.

Now suppose only x^y is used. Note first that the government budget constraint for all individuals $t \geq 1$ is:

$$x^y = g + (R - 1) b$$

that is, identical to the case of taxes on the old. For the initial generation, one has:

$$\begin{aligned} x_0^o &= g - b - x^y \\ &= -Rb < 0 \end{aligned}$$

so that higher debt implies a positive lump-sum transfer to the old for any level of g. The FOCs imply

$$u'(e - s - x^y - b) = \beta z f'(s)$$

or

$$u'(e - s - g - Rb) = \beta z f'(s)$$

Since an increase in b raises R, the negative effect on s is larger. Hence using x^y makes the representative individual $t \geq 1$ worse off and the initial generation even better off than x^o.

(g) The tax policies in the previous part implied that debt resulted in a transfer of resources between generations. Ricardian equivalence would mean that issuance of debt does not yield a transfer of resources in equilibrium. Since both policies implied that x_0^o fell with increases in b, we would need first of all that

$$x_0^o = 0$$

which implies the initial old are unaffected by b. The condition $x_0^o = 0$ requires that the initial period government budget constraint yields

$$x^y = g - b$$

and this holds in every subsequent period. Therefore for $t \geq 1$,

$$x^o = Rb$$

The individual's utility becomes

$$u(e - g - s_t) + \beta z f(s_t)$$

so debt has no effect on utility or equilibrium s.