Solution Q2– August 2007 Comps

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a) The government's budget constraint in any period is

$$y(a_t) + (1-\delta)\, a_t = h_t + a_{t+1} \tag{1}$$

I assume that the depreciation rate $\delta = 1$, so that we have $y(a_t) = h_t + a_{t+1}$. The government's infinite horizon problem is

$$V(a_t, t) = \bar{u} + \beta\sigma(h_t, t)\, V(a_{t+1}, t+1) + \beta(1 - \sigma(h_t, t))\,\Psi \tag{2}$$

b) Since $\sigma(h)$ does not change over time, it is clear from (2) that $\Omega(a_t, t)$ is time-invariant as well, so we may write

$$V(a_t) = \bar{u} + \beta(1 - \sigma(h_t))\,\Psi + \beta\sigma(h_t)\, V(a_{t+1})$$

subject to the bc (1).

A key thing I was looking for was the ability to take a description of a situation and write it down as a problem that could use a technique you had learned to solve it. (Remember the example of using a two-sector model to explain why barbers in the U.S. are paid more than barbers in Africa.) What is crucial is to get the sequencing of events right, where many people wrote down

$$V(a_t) = \sigma(h_t)\,\bar{u} + (1 - \sigma(h_t))\,\Psi + \beta V(a_{t+1})$$

which isn't consistent with the verbal description of what happens when.

c) The first-order condition is

$$\sigma'(h_t)\,(V(a_{t+1}) - \Psi) = \sigma(h_t)\, V'(a_{t+1}) \tag{3a}$$

equating the marginal value of the uses of a_t. On the LHS, the marginal value of h_t used to increase the survival probability is the marginal effect on $\sigma(h_t)$ multiplied by the the present discounted value of surviving versus not surviving to next period, namely $V(a_{t+1}) - \Psi$. The RHS is the marginal value of carrying another unit of a_{t+1} on $V(a_{t+1})$ itself, which must be multiplied by the probability $\sigma(h_t)$ of surviving to $t+1$. β "falls out" of both sides since both uses of a_t affect welfare in the following period.

The envelope condition is

$$V'(a_t) = \beta y'(a_t)\,\sigma'(h_t)\,(V(a_{t+1}) - \Psi) \tag{4}$$

which says simply that the value of another unit of a_t optimally used is the discounted (by β) value of the the higher expected PDV welfare it allows.

d) Combining (3a) and (4) one obtains

$$V'(a_t) = \beta y'(a_t)\,\sigma(h_t)\,V'(a_{t+1}) \tag{5}$$

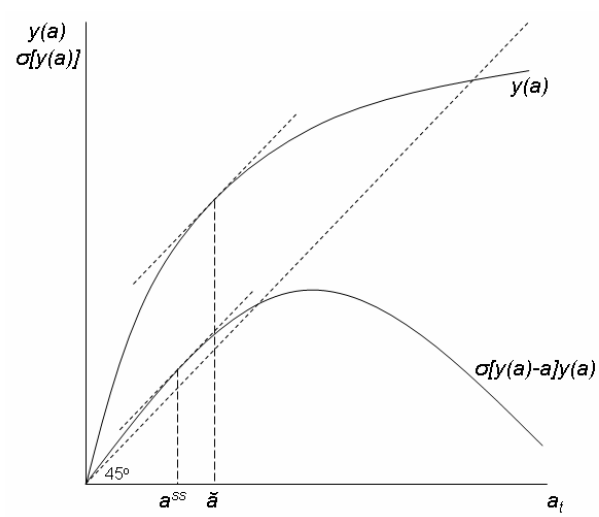
so that the steady state a^{SS} will be determined by

$$\sigma\left(y\left(a^{SS}\right) - a^{SS}\right) y'\left(a^{SS}\right) = \frac{1}{\beta} \tag{6}$$

where h^{SS} is given by

$$h^{SS} = y\left(a^{SS}\right) - a^{SS}$$

This may be represented in a diagram showing showing $\sigma(\cdot)\,y(a)$ (which always lies below $y(a)$ since $\sigma < 1$ for $a < \infty$) versus a_t with steady state being represented by the a^{SS} which gives $\sigma(\cdot)\,y'(a) = 1/\beta > 1$. Note that $\sigma(y(a) - a)$ peaks at $\breve{a}$ such that $y'(\breve{a}) = 1$ so that $\sigma(\cdot)\,y(a)$ peaks at some point to the right of $\breve{a}$ but before $a = y(a)$, where $\sigma = 0$. a^{SS} lies to the left of $\breve{a}$, and one may see that a^{SS} is unique.



Steady state may be characterized as having a survival probability less than one, but a sort of modified GR condition holds.

e) In the same diagram, we see that the slope of $\sigma(y(a) - a)\beta y(a_t)$ is greayer than $1/\beta$ to the left of the steady state and greater to the right. Using (5) and the concavity of $\Omega(a_t)$, one sees that $a_{t+1} > a_t$ for $a_t < a^{SS}$ and $a_{t+1} < a_t$ for $a_t > a^{SS}$.

f) The key point here was to realize that the optimal $a_t = 0$ for $t \geq 4$, so this can be seen as a finite-horizon problem with an endpoint of $a_4 = 0$. One can then use the backwards optimization techniques.

i) For $t \geq 4$, $\sigma = 1$ so that period utility is $\bar{u}$ implying $V_4 = V_5 = \ldots = \frac{\bar{u}}{1-\beta}$. No a is carried over, that is, optimal $a_4 = 0$. We may then write

$$V_3(a_3) = u(f(a_3) - h_3) + \beta(1 - \sigma(h_3))\Psi + \beta\sigma(h_3)\frac{\bar{u}}{1-\beta}$$

with FOC for h_3 of

$$u'(f(a_3) - h_3) = \beta\sigma'(h_3)\Omega$$

where $\Omega \equiv \left(\frac{\bar{u}}{1-\beta} - \Psi\right)$. This may be solved for a function $h_3 = H(a_3)$. We may then write $V_3(a_3)$ as

$$V_3(a_3) = u(f(a_3) - H(a_3)) + \beta + \beta\sigma(H(a_3))\Omega \tag{7}$$

This gives the value of a_3 optimally used. We may then write

$$V_2(a_2) = u(f(a_2) - a_3) + \beta V_3(a_3) \tag{8}$$

the FOC for a_3 is

$$u'(f(a_2) - a_3) = \beta V_3'(a_3) \tag{9}$$

where, from (7),

$$V_3'(a_3) = u'[f(a_3) - H(a_3)]\left(f'(a_3) - H'(a_3)\right) + \beta\sigma'\left(H'(a_3)\right)\Omega \tag{10}$$

Substituting (10) into (9) we can solve for optimal a_3 chosen at $t = 2$ as a function of a_2 (the "policy" function), namely $a_3 = A_2(a_2)$. Eliminate $V_3(\cdot)$ from (8) and use $a_3 = A_2(a_2)$ to obtain

$$\begin{aligned} V_2(a_2) &= u\left(f(a_2) - A_2(a_2)\right) + \qquad (11) \\ &\quad \beta\left(u\left(f\left(A_2(a_2)\right) - H\left(A_2(a_2)\right)\right) + \beta\left(1 - \sigma\left(H\left(A_2(a_2)\right)\right)\right)\Psi + \beta\sigma\left(H\left(A_2(a_2)\right)\right)\frac{\bar{u}}{1-\beta}\right) \end{aligned}$$

that is $V_2(\cdot)$ as a function only of a_2. For period 1 we may then write We may then write

$$V_1(a_1) = u\left(f(a_1) - a_2\right) + \beta V_2(a_2) \qquad (12)$$

with FOC for a_2 of

$$u'\left(f(a_1) - a_2\right) = \beta V_2'(a_2) \qquad (13)$$

which using (11) becomes an equation in a_1 and a_2 which can be denoted by $a_2 = A_1(a_1)$. Then we have a sequence of values for given a_1 of

$$a_t = \left\{a_1, A_1(a_1), A_2\left(A_1(a_1)\right), 0, 0, \ldots\right\}_{t=1}^{t=\infty}$$

ii) To consider the time path of a_t, consider the sequence of first-order conditions

$$\begin{aligned} u'\left(f(a_1) - a_2\right) &= \beta f'(a_2)\, u'\left(f(a_2) - a_3\right) \qquad (14) \\ u'\left(f(a_2) - a_3\right) &= \beta f'(a_3)\, u'\left(f(a_3) - h_3\right) \qquad (15) \\ u'\left(f(a_3) - h_3\right) &= \beta \sigma'(h_3)\, \Omega \qquad (16) \end{aligned}$$

Note first that $u'\left(f(a_t) - a_{t+1}\right) \gtreqless u'\left(f(a_{t+1}) - a_{t+2}\right)$ as $\beta f'(a_{t+1}) \gtreqless 1$, as in the standard C-K growth problem. Hence, for a_1 sufficiently low, $\beta f'(a_{t+1}) > 1$, so that a_t is rising over time.

The last FOC indicates the level of this path. As $|\Psi| \to \infty$, then $\Omega \to \infty$ so that $u'\left(f(a_3) - h_3\right) \to \infty$ and $h_3 \to f(a_3)$. Intiuitively, as losing office becomes infinitely costly, all resources are spent on trying to retain office. Similarly, in earlier periods almost all income $y(a_t)$ is saved.