Solution to Drazen's January 2006 Comp Question
(Sketch of Solutions)

Part a

Given that leisure is not an argument of the utility function, the assumption of $l < 1$ does not seem to be a good one. The agent can always do better by increasing her labor supply which allows her to achieve higher levels of consumption. For the assumption $l < 1$ to be reasonable one has to think of some kind of institution/regulation or inherent taste for non-market activities that keeps the young workers from working full-time. This institution or taste must not affect or apply to middle-aged workers.

Part b

$$\underset{c_t^y, c_{t+1}^m, c_{t+2}^o}{Max} \ln c_t^y + \beta \ln c_{t+1}^m + \beta^2 c_{t+2}^o \tag{1}$$

subject to

$$c_t^y = w_t l - s_t \tag{2}$$

$$c_{t+1}^m = w_{t+1} + (1 + r_{t+1})s_t - s_{t+1} \tag{3}$$

$$c_{t+2}^o = (1 + r_{t+2})s_{t+1} \tag{4}$$

Part c

Combining eq. (3) and eq. (4) we get

$c_{t+1}^m = w_{t+1} + (1 + r_{t+1})s_t - \frac{c_{t+2}^o}{(1+r_{t+2})}$

Equivalently,

$$s_t = \frac{c_{t+1}^m}{(1 + r_{t+1})} + \frac{c_{t+2}^o}{(1 + r_{t+1})(1 + r_{t+2})} - \frac{w_{t+1}}{(1 + r_{t+1})} \tag{5}$$

Replacing eq. (5) into eq. (2) we get

$c_t^y = w_t l + \frac{w_{t+1}}{(1+r_{t+1})} - \frac{c_{t+1}^m}{(1+r_{t+1})} - \frac{c_{t+2}^o}{(1+r_{t+1})(1+r_{t+2})}$

Rearranging,

$$c_t^y + \frac{c_{t+1}^m}{(1+r_{t+1})} + \frac{c_{t+2}^o}{(1+r_{t+1})(1+r_{t+2})} = w_t l + \frac{w_{t+1}}{(1+r_{t+1})} \tag{6}$$

Let's now solve for the optimal consumption levels. The Lagrangian is given by,

$L = \ln c_t^y + \beta \ln c_{t+1}^m + \beta^2 c_{t+2}^o + \lambda \left[w_t l + \frac{w_{t+1}}{(1+r_{t+1})} - c_t^y - \frac{c_{t+1}^m}{(1+r_{t+1})} - \frac{c_{t+2}^o}{(1+r_{t+1})(1+r_{t+2})} \right]$

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$c_t^y \rfloor$

$$\frac{1}{c_t^y} = \lambda \tag{7}$$

$c_{t+1}^m \rfloor$

$$\frac{\beta}{c_{t+1}^m} = \frac{\lambda}{(1+r_{t+1})} \tag{8}$$

$c_{t+2}^o \rfloor$

$$\frac{\beta^2}{c_{t+2}^o} = \frac{\lambda}{(1+r_{t+1})(1+r_{t+2})} \tag{9}$$

Combining eq (7) and eq. (8), and eq. (8) and eq. (9) we get

$c_{t+1}^m = \beta(1+r_{t+1})c_t^y$

$c_{t+2}^o = \beta(1+r_{t+2})c_{t+1}^m$

Re expressing in terms of c_t^y,

$$c_{t+1}^m = \beta(1+r_{t+1})c_t^y \tag{10}$$

$$c_{t+2}^o = \beta^2(1+r_{t+1})(1+r_{t+2})c_t^y \tag{11}$$

Replacing eq (10) and eq (11) into the intertemporal budget constraint

$c_t^y + \beta c_t^y + \beta^2 c_t^y = w_t l + \frac{w_{t+1}}{(1+r_{t+1})}$

$$c_t^y = \frac{1}{\left(1+\beta+\beta^2\right)}\left[w_t l + \frac{w_{t+1}}{(1+r_{t+1})}\right] \tag{12}$$

Replacing eq (12) into eq (10) and eq. (11)

$$c_{t+1}^m = \frac{\beta(1+r_{t+1})}{\left(1+\beta+\beta^2\right)}\left[w_t l + \frac{w_{t+1}}{(1+r_{t+1})}\right] \tag{13}$$

$$c_{t+2}^o = \frac{\beta^2(1+r_{t+1})(1+r_{t+2})}{\left(1+\beta+\beta^2\right)}\left[w_t l + \frac{w_{t+1}}{(1+r_{t+1})}\right] \tag{14}$$

Part d

To obtain $a^y \equiv s_t$, replace eq (12) into eq (2)

$$a^y = w_t l - \frac{1}{(1+\beta+\beta^2)}\left[w_t l + \frac{w_{t+1}}{(1+r_{t+1})}\right]$$

$$a^y = \frac{1}{\left(1+\beta+\beta^2\right)}\left[\left(\beta+\beta^2\right) w_t l - \frac{w_{t+1}}{(1+r_{t+1})}\right] \tag{15}$$

Similarly, to obtain $a^m \equiv s_{t+1}$, note that eq (3) implies

$$a^m = w_{t+1} + (1+r_{t+1})a^y - c_{t+1}^m$$

Replacing eq (13) and eq (15) into the equation above we get

$$a^m = w_{t+1} + \frac{(1+r_{t+1})}{(1+\beta+\beta^2)}\left[\left(\beta+\beta^2\right) w_t l - \frac{w_{t+1}}{(1+r_{t+1})}\right] - \frac{\beta(1+r_{t+1})}{(1+\beta+\beta^2)}\left[w_t l + \frac{w_{t+1}}{(1+r_{t+1})}\right]$$

$$a^m = \left[1 - \frac{1}{(1+\beta+\beta^2)} - \frac{\beta}{(1+\beta+\beta^2)}\right] w_{t+1} + \frac{(1+r_{t+1})}{(1+\beta+\beta^2)}\left[\left(\beta+\beta^2\right) - \beta\right] w_t l$$

$$a^m = \frac{\beta^2}{(1+\beta+\beta^2)} w_{t+1} + \frac{(1+r_{t+1})\beta^2}{(1+\beta+\beta^2)} w_t l$$

$$a^m = \frac{\beta^2\left[w_{t+1} + (1+r_{t+1})w_t l\right]}{\left(1+\beta+\beta^2\right)} \tag{16}$$

Note that r_{t+1} affect both a^y and a^m. There is a positive relation between r_{t+1} and a^y. When r_{t+1} goes up, the price of consumption when middle-aged becomes cheaper. Young people substitute present consumption for future consumption which makes savings to go up. Later, the agents smooth-out those extra savings. That is why r_{t+1} has also a positive effect on a^m.

Part e

We already know that $a^y = w_t l - \frac{1}{(1+\beta+\beta^2)}\left[w_t l + \frac{w_{t+1}}{(1+r_{t+1})}\right]$. For $a_t^y < 0$, we need that

$$w_t l < \frac{1}{(1+\beta+\beta^2)}\left[w_t l + \frac{w_{t+1}}{(1+r_{t+1})}\right]$$
$$w_t l < \frac{1}{(1+\beta+\beta^2)}[\text{PDV of Lifetime Income}]$$

Current period income has to be less than the present discounted value of lifetime income times a factor capturing the relative value of present and future consumption. Agents only borrow in the first period if their incomes are not big enough.

Part f

The equilibrium condition for capital is given by

$$K_{t+1} = S_t$$

In this model $S_t \equiv (1+n)N_t s_t^y + N_t s_t^m$. Therefore

$$K_{t+1} = S_t = (1+n)N_t s_t^y + N_t s_t^m$$

Total labor supply is given by

$$L_t = (1+n)lN_t + N_t.$$

Then

$$\frac{K_{t+1}}{L_{t+1}} = \frac{S_t}{(1+n)L_t}$$

$$k_{t+1} = \frac{(1+n)N_t s_t^y + N_t s_t^m}{(1+n)[(1+n)lN_t + N_t]}$$

$$k_{t+1} = \frac{(1+n)s_t^y + s_t^m}{(1+n)^2 l + (1+n)}$$

$$
\begin{aligned}
k_{t+1} &= \frac{1}{1+n+l+nl}\left\{\frac{1}{\left(1+\beta+\beta^2\right)}\left[\left(\beta+\beta^2\right)w_t l - \frac{w_{t+1}}{(1+r_{t+1})}\right]\right\} \\
&\quad + \frac{1}{(1+n)^2 l + (1+n)}\left\{\frac{\beta^2\left[w_t + (1+r_t)w_{t-1}l\right]}{\left(1+\beta+\beta^2\right)}\right\} \qquad (17)
\end{aligned}
$$

Note that because we are working with two different generations $s^m = \frac{\beta^2[w_t+(1+r_t)w_{t-1}l]}{(1+\beta+\beta^2)}$.

Now, firms optimality conditions are $r_t = f'(k_t)$ and $w_t = f(k_t) - f'(k_t)k_t \equiv \phi(k_t)$. Replacing these conditions into the expression above

$$
\begin{aligned}
k_{t+1} &= \frac{1}{1+n+l+nl}\left\{\frac{1}{\left(1+\beta+\beta^2\right)}\left[\left(\beta+\beta^2\right)\phi(k_t) l - \frac{\phi(k_{t+1})}{(1+f'(k_{t+1}))}\right]\right\} \\
&\quad + \frac{1}{(1+n)^2 l + (1+n)}\left\{\frac{\beta^2\left[\phi(k_t) + (1+f'(k_t))\phi(k_{t-1})l\right]}{\left(1+\beta+\beta^2\right)}\right\}
\end{aligned}
$$

which is a second order non-linear difference equation.

Part g

The dynamic shouldn't change much. What we are basically doing here is dividing the first period of two-period model studied in class into two subperiods.