Solution to Drazenr's January 2007 Comp Question
(Sketch of Solutions)

Part a

Agent's savings in the first period is given by $s_t = x_t - T_t^y - c_t^y$. Replacing it into consumption when old we get

$$c_{t+1}^o = R\left[x_t - T_t^y - c_t^y\right] - T_{t+1}^o$$

Rearranging

$$c_t^y + T_t^y + \frac{c_{t+1}^o}{R} + \frac{T_{t+1}^o}{R} = x_t$$

Part b

Assuming balanced budget, the Lagrangian for the worker's problem is

$$L = u(x_t - \bar{g} - s_t) + \beta E_t\left[u(Rs_t - \breve{g})\right]$$

FOC

$$s_t\rfloor \qquad -u'(x_t - \bar{g} - s_t) + \beta R E_t\left[u'(Rs_t - \breve{g})\right] = 0$$

Assuming $\beta R = 1$

$$u'(x_t - \bar{g} - s_t) = E_t\left[u'(Rs_t - \breve{g})\right]$$

Using functional form

$$a - b(x_t - \bar{g} - s_t) = E_t\left[a - b(Rs_t - \breve{g})\right]$$

$$x_t - \bar{g} - s_t = E_t\left[Rs_t - \breve{g}\right]$$

$$x_t - \bar{g} - s_t = Rs_t - E_t\left[\breve{g}\right]$$

$$(1+R)s_t = x_t - \bar{g} + E_t\left[\breve{g}\right]$$

$$s_t = \frac{1}{1+R}\left[x_t - \bar{g} + E_t\left(\breve{g}\right)\right]$$

Consumption when young is

$$c_t^y = x_t - \bar{g} - s_t$$

$c_t^y = x_t - \bar{g} - \frac{1}{1+R}\left[x_t - \bar{g} + E_t\left(\breve{g}\right)\right]$

$c_t^y = \frac{R}{1+R}\left(x_t - \bar{g}\right) - \frac{1}{1+R}E_t\left(\breve{g}\right)$

Consumption when old is a random variable

$c_{t+1}^o = \frac{R}{1+R}\left[x_t - \bar{g} + E_t\left(\breve{g}\right)\right] - \breve{g}$

<u>Part c</u>

Given that the utility function is quadratic, a mean preserving spread won't affect consumption and saving (certainty-equivalence). With a mean preserving spread the first order condition above comes

$x_t - \bar{g} - s_t = Rs_t - E_t\left[\breve{g} - \varepsilon\right] = Rs_t - E_t\left[\breve{g}\right]$ given that $E_t\left[\varepsilon\right] = 0$

Therefore, the optimal saving level is the same as above.

Even if consumption and saving do not change, utility after the mean preserving spread is smaller. Let's call s_t^{**} the savings level that maximize utility after the mean-preserving spread is introduced. Agent's indirect lifetime utility as of period t is

$$\begin{aligned} V &\equiv u(x_t - \bar{g} - s_t^{**}) + \beta \int\int u(Rs_t^{**} - \breve{g} - \varepsilon)dF(\breve{g})d\varepsilon \\ &< u(x_t - \bar{g} - s_t^{**}) + \beta \int u(Rs_t^{**} - \breve{g} - \int \varepsilon d\varepsilon)dF(\breve{g}) \\ &= u(x_t - \bar{g} - s_t^{**}) + \beta \int u(Rs_t^{**} - \breve{g})dF(\breve{g}) \equiv W \end{aligned}$$

where the inequality sign is due to the concavity of the utility function. Therefore, the agent is worse-off as consequence of the mean-preserving spread.

<u>Part d</u>

In the first period the government needs to issue debt for an amount of m. Under this new setting, agent's savings in the first period is given by

$s_t = x_t - T_t^y - c_t^y = x_t - \bar{g} + m - c_t^y.$

Consumption when old then becomes

$c_{t+1}^o = R\left[x_t - \bar{g} + m - c_t^y\right] - \breve{g} - Rm = R\left[x_t - \bar{g} - c_t^y\right] - \breve{g}$

Rearranging

$c_t^y + \bar{g} + \frac{c_{t+1}^o}{R} + \frac{\breve{g}}{R} = x_t$

which is the same intertemporal budget constraint as the one obtained in part a. Therefore, consumption does not change. Household saving goes up by m, government savings goes down by m and national saving does not change. Agent's utility as of period t remains the same.

Part e

Assuming balanced budget, the Lagrangian for the worker's problem is

$$L = u(x_t - \bar{g} - s_t) + \beta E_t \left[u(Rs_t - \breve{g}_{t+1} + \bar{g})\right]$$

FOC

$$s_t\rfloor \qquad -u'(x_t - \bar{g} - s_t) + \beta R E_t \left[u'(Rs_t - \breve{g}_{t+1} + \bar{g})\right] = 0$$

Assuming $\beta R = 1$

$$u'(x_t - \bar{g} - s_t) = E_t \left[u'(Rs_t - \breve{g}_{t+1} + \bar{g})\right]$$

Using the functional form of the utility function we get

$$a - b(x_t - \bar{g} - s_t) = E_t \left[a - b(Rs_t - \breve{g} + \bar{g})\right]$$

$$x_t - \bar{g} - s_t = Rs_t - E_t \left[\breve{g}\right] + \bar{g}$$

$$s_t = \tfrac{1}{1+R} \left[x_t - 2\bar{g} + E_t \left(\breve{g}\right)\right].$$

Consumption of the young is then

$$c_t^y = x_t - \bar{g}_t - \tfrac{1}{1+R} \left[x_t - 2\bar{g} + E_t \left(\breve{g}\right)\right]$$

$$c_t^y = \tfrac{1}{1+R} \left[Rx_t - (R-1)\bar{g}_t - E_t \left(\breve{g}\right)\right]$$

Consumption of the old

$$c_t^o = \tfrac{R}{1+R} \left[x_t - 2\bar{g} + E_t \left(\breve{g}\right)\right] - \breve{g} + \bar{g}$$

Consumption of the initial old

$$c_0^o = \tfrac{R}{1+R} x_t$$

Part f

In the case when the return on saving is stochastic, our first order condition becomes

$$-u'(x_t - \bar{g} - s_t) + \beta E_t \left[R_t u'(R_t s_t - \breve{g}_{t+1} + \bar{g}) \right] = 0$$

$$u'(c_t^y) = \beta E_t \left[R_t u'(c_t^o) \right]$$

$$u'(c_t^y) = \beta cov_t \left[R_t, u'(c_t^o) \right] + \beta E_t \left[R_t \right] E_t \left[u'(c_t^o) \right]$$

If $\breve{g}_t$ and R_t are uncorrelated, when R_t increases, c_t^o increases as well. Given the concavity of the utility function $u'(c_t^o)$ decreases. Then $cov_t \left[R_t, u'(c_t^o) \right] < 0$

Let $E_t \left[R_t \right] = R$ and c_t^u the optimal consumption level of the old in the case with return uncertainty. Given that

$$\beta cov_t \left[R_t, u'(c_t^u) \right] + \beta R E_t \left[u'(c_t^u) \right] < \beta R E_t \left[u'(c_t^u) \right]$$

the first order condition implies that the optimal consumption level when old in the case with return uncertainty is bigger than in the case with return uncertainty.

Part g

Using the same type of reasoning as above, if $\breve{g}_t$ and R_t are positively correlated, when R_t increases $\breve{g}_t$ increases as well. If c_t^o increases (decreases), the optimal consumption level when old in the case with return uncertainty is bigger (smaller) than in the case with return uncertainty. if $\breve{g}_t$ and R_t are negatively correlated, when R_t increases $\breve{g}_t$ decreases as well. c_t^o increases which implies the optimal consumption level when old in the case with return uncertainty is bigger than in the case with return uncertainty.