4. Consider a two -period OLG model. The preferences of an agent born at time t are

$$u\left(c_t^1, c_{t+1}^2\right) = \ln c_t^1 + \beta \ln c_{t+1}^2$$

where c^1 denotes consumption when young and c^2 consumption when middle-aged. There is no physical capital and output is produced according to the following Cobb-Douglas production function (where $0 < \gamma < 1$)

$$Y_t = S_t^{\gamma} L_t^{1-\gamma}$$

where L_t is labor of young, unskilled workers and S_t is labor of middle-aged, skilled workers. The number of births per period is fixed at N. When young, agents spend a fraction m_t of their time working and a fraction $1 - m_t$ investing in human capital. The wage per unit time worked is w_t. The human capital they acquire is given by

$$h_{t+1} = h_t + (1 - m_t)\,\theta h_t \tag{1}$$

where h_t is human capital of the current middle-aged and where θ is a parameter. When middle-aged, agents just work and earn a wage v_t per unit of human capital. Their labor supply at t depends on the human capital h_t they accumulated when young: $S_t = N h_t$. Wages of young and middle-aged are equal to their marginal products.

(a) Solve for the consumer's optimal consumption, saving, and human capital investment profile.

(b) What is this model and (1) meant to represent? Does the model display multiple equilibrium growth paths? Why or why not?

(c) What is the equilibrium growth rate (or growth rates) of human capital ? What is the equilibrium growth rate (or growth rates) of output?

(d) Suppose the government imposes tax rates τ^1 and τ^2 on the wages of the young and the middle-aged respectively, and transfers the revenue back to them as lump-sums: $T_t^1 = \tau^1 w_t m_t$ and $T_{t+1}^2 = \tau^2 v_{t+1} h_{t+1}$. If the tax is "progressive", so that $\tau^2 > \tau^1$, how will this affect long run growth? Explain the intuition behind this.

Now suppose individuals live three periods, where preferences of an agent born at time t are

$$u\left(c_t^1, c_{t+1}^2, c_{t+2}^3\right) = \ln c_t^1 + \beta \ln c_{t+1}^2 + \beta^2 \ln c_{t+2}^3$$

where c^1 and c^2 are as before and c^3 is consumption when old. The production structure is as before. The old do not work. Assume for this part of the question that a consumer can also hold one-period government bonds, paying a fixed interest rate r.

(e) *Using backwards recursion*, solve for the optimal consumption, saving, and human capital investment profile, given the parameters γ, θ, β, and r. Does this version of the model display multiple equilibrium growth paths?

(f) What is the equilibrium growth rate (or growth rates) of human capital? What is the equilibrium growth rate (or growth rates) of output? Explain your results.