2. This question considers coups and other fun stuff. If a leader (or "government") is in office at the beginning of period t she enjoys fixed utility $\bar{u} > 0$ in the period and has a discount factor of $0 < \beta < 1$ on future utility. At the end of each period the leader may be removed from office, in which case she is imprisoned forever, which gives her present discounted utility of $\Psi < 0$. She survives as leader to the next period with probability $0 \leq \sigma_t \leq 1$.

She may increase her probability of surviving as leader to period $t+1$ by spending h_t in t. That is, let $\sigma_t \equiv \sigma(h_t, t)$ (where the second argument indicates that the function σ may change over time), where for given t, $\sigma(\cdot, t)$ is increasing and concave in h. Assume also that $\sigma(\cdot, t)$ satisfies the Inada conditions on h and that $\lim_{h_t \to \infty} \sigma(h_t, t) = 1$ and $\lim_{h_t \to 0} \sigma(h_t, t) = 0$.

The government begins period t with assets a_t (carried over from the previous period) that produce income $y(a_t)$, where $y(\cdot)$ is an increasing concave function. The leader may spend government income on h_t or may save, that is, accumulate a_{t+1}.

a) Write down the government's budget constraint in any period and its infinite horizon problem when $\sigma(h_t, t)$ changes over time.

b) Assume that the function $\sigma(h, t)$ does not change over time, that is, $\sigma(h, t) = \sigma(h)$ for all t. Discuss briefly why this assumption allows the leader's utility maximization problem to be written in terms of a time-invariant value function and write down the leader's dynamic programming problem.

c) Derive the first-order and envelope conditions for this problem and explain what each means in economic terms.

d) Derive the equation or equations determining the steady state. Is the steady state unique? Explain briefly what you think are the key economic characteristics of the steady state solution.

e) Show that the dynamic system converges to a steady state. (You may find it helpful to consider a diagram here with a_t on one axis and relevant functions of a_t on the other.)

f) In this part of the question,you are asked to consider an application of the above ideas. President S.-B. Sneetch of Seussland faces an election at the end of period 3. That is, she is in office for certain in periods 1, 2, and 3, and wins the election at the end of period 3 to remain in office with known probability $\sigma(h_3)$, which satisfies the above assumptions on $\sigma(h_t, t)$. If she is re-elected she will declare herself "President Forever After", that is, she will retain power with probability 1 until $t = \infty$.

She begins period 1 with given assets a_1. Suppose that at $t = 1, 2, 3$ her utility from holding office is $u(g_t)$, where $y(a_t)$ can be split between g_t and a_{t+1} at $t = 1, 2$ and among g_3, h_3 and a_4 at $t = 3$. $u(\cdot)$ is an increasing, concave function which satisfies the Inada conditions. For $t = \{4, \ldots, \infty\}$ Sneetch's single-period utility is $\bar{u} > 0$, as above (that is, independent of g) if she remains in office; the present discounted value of her utility from $t = 4$ onward if she is removed from office is $\Psi < 0$, also as above.

i) Solve for President Sneetch's choice of a_t in each period $t = \{2, 3, 4, \ldots\}$, that is, find equations that would determine a_t in each period when a_1 is given.

ii) Discuss the time path of a_t before the election depending on the value of a_1. How is the time path affected by an increase in the cost of losing office (an increase in the absolute value of Ψ)? (HINT: It may be useful to consider the first-order conditions rather than simply using "brute force" algebra.)