2. Read the whole question before starting to work. Consider an OLG model of a continuum of two-period lived individuals. Normalize the population size to 1 and assume there is no population growth 1. The utility function of a representative young person born at time t is given by

$$U\left(c_t^y, c_{t+1}^o\right) = \ln c_t^y + \beta \ln c_{t+1}^o$$

where subscripts refer to calendar time, superscripts to age, and $1 > \beta > 0$. Each young person is endowed with one unit of leisure when young, zero when old, and can save, where saving supports capital accumulation. Members of the initial old generation are each endowed with k_0 units of physical capital and the initial young generation with h_0 units of human capital. Human capital evolves according to the relation

$$h_{t+1} = B h_t^{\mu} E_t^{1-\mu}$$

where E_t is public expenditures on education in period t, $B > 0$, and $1 > \mu > 0$. There is a continuum of firms, with the number of firms normalized to 1. A representative firm produces output y_t, at time t according to the technology

$$y_t = A k_t^{\alpha} \left(n_t h_t\right)^{1-\alpha}$$

where $A > 0$, $0 < \alpha < 1$, k_t is capital rented by the firm, and n_t is "raw" labor input (so that $n_t h_t$ is "skilled labor input"). Assume that capital does not depreciate. Capital and labor income is taxed at a uniform rate τ_t in order to finance public expenditures on education. Assume initially that the tax rate is set at $t = 0$ and is constant over time.

(a) Write down the budget constraints for a representative young agent at t when young and old given the wage of skilled labor w_t and the interest rate r_{t+1}. Write down the individual's choice problem and the firm's choice problem.

(b) Write down the definition of competitive equilibrium.

(c) Derive the equations for evolution of human capital h_t and physical capital k_t. Derive the steady state ratio x^* of human to physical capital $x_t \equiv h_t / k_t$. (HINT: Derive $\ln x^*$.)

(d) Derive the long-run growth rate of the economy as a function of $\ln x^*$ and the parameters of the model. What constant tax rate maximizes the steady-state growth rate? Explain your result on the optimal tax rate.

(e) Could the economy have higher utility if the tax rate were chosen at the beginning of each period rather than once and for all at $t = 0$? Explain. If so, what would the evolution of the tax rate look like? What if the tax rate were chosen at the beginning of each generation, to apply to that generation?

(f) Why does this economy display intensive growth, whereas the Diamond OLG model does not? What assumptions about factor inputs would imply convergence to a steady state *level* of output, as in Diamond, rather than sustained *growth* of output?