Consider the standard Diamond OLG model (young supply labor; old don't work; etc.) extended to include government expenditure. Let G_t denote the government's purchases of goods in period t. Assume $G_t = (1+n)^t G_0$, where G_0 is a given positive number and $n \geq 0$ is the given constant rate of population growth. G_t is a free public good, such as free fireworks displays. To finance its purchases, the government levies taxes and its budget is balanced every period. To begin, assume that only labor income is taxed, where the labor income tax rate is μ_t.

Let the aggregate production function satisfy all the "standard" conditions (including the Inada conditions) and suppose there is no technical progress. Assume that capital depreciates at rate $\delta = 1$. For simplicity, normalize initial labor supply L_0 to 1.

Assume that an individual born at time t has the utility function

$$U\left(c_t^y, c_{t+1}^o, G_t, G_{t+1}\right) = \ln c_t^y + \gamma \ln G_t + \beta\left[\ln c_{t+1}^o + \gamma \ln G_{t+1}\right]$$

where γ is a positive parameter (an indicator of how strongly the public good is desired).

a) Write down the individual's budget constraint and derive the saving function of the young. How would an increase in γ affect the saving function?

b) Write down the economy-wide constraint on output and its uses in terms of per-capita magnitudes. Write down the government budget constraint.

Assume for the rest of the question that the aggregate production function is $F(K_t, L_t) = K_t^\alpha L_t^{1-\alpha}$, where $1 > \alpha > 0$.

c) Find the tax rate μ_t as a function of k_t. What is the simplest assumption on the capital stock that you can make to ensure that $\mu_t < 1$ for all t?

d) Derive explicitly the fundamental difference equation of the model and illustrate the dynamics in a diagram.

e) Show both algebraically and diagrammatically how the steady-state capital-labor ratio depends on: i)K_0; ii)G_0; iii)α.

f) Assume that the economy has been in steady state until t, with only a labor income tax. Then there is an unanticipated change in government tax policy so that capital income is also taxed. Specifically, from period t onwards, income of the old is taxed at the same proportional rate as labor income, the common tax rate being τ_t. Assume that the path of government expenditure is unchanged and the budget is still balanced each period. Find the new tax rate τ_t for each period, that is, τ_t, τ_{t+1}, $\tau_{t+2}, \ldots$.

g) Show the effect on the steady-state capital-labor ratio in this case relative to the case with only a tax on labor income. Explain carefully the intuition of your result, including the effect on saving and the general equilibrium effect.