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COMP QUESTION
(Sketch of the solution)

This question was based on Blanchard and Fisher textbook (page 305 – 308)

$$Min \quad E_0\left[\sum_{t=0}^{\infty}\beta^t\left((a/2)(Y_t-u_t)^2+(b/2)(Y_t+I_{t-1}-I_t^*)^2\right)\right] \qquad (1)$$

Subject to:

$$I_t^* = E_t(S_t)+k \qquad (2)$$

$$I_t = I_{t-1}+Y_t-S_t \qquad (3)$$

(i)

Let's assume that the productivity shock u_t and the sales S_t both display a markovian structure, so that at time t the following is true:

Control variable: Y_t
Endogenous state variable: I_{t-1}
(Time-varying) Exogenous state variables: u_t, S_{t-1}

Bellman Equation:

$$V(u_t,S_{t-1},I_{t-1}) = \underset{Y_t}{Min}\left\{(a/2)(Y_t-u_t)^2+(b/2)(Y_t+I_{t-1}-I_t^*)^2+\beta E_t V(u_{t+1},S_t,I_t)\right\} \quad (4)$$

Subject to (2) and (3)

Plugging (2) and (3) into (4), the Bellman Equation can be rewritten as:

$$V(u_t,S_{t-1},I_{t-1}) = \underset{Y_t}{Min}\begin{Bmatrix}(a/2)(Y_t-u_t)^2+(b/2)(Y_t+I_{t-1}-E_t(S_t)-k)^2 \\ +\beta E_t V(u_{t+1},S_t,I_{t-1}+Y_t-S_t)\end{Bmatrix} \qquad (5)$$

Where u_t and S_t are assumed to follow markovian processes.

First-order condition:

$$Y_t: \quad a(Y_t - u_t) + b(Y_t + I_{t-1} - E_t(S_t) - k) + \beta E_t V_3(u_{t+1}, S_t, I_t) = 0 \qquad (6)$$

Envelope Condition:

$$V_3(u_t, S_{t-1}, I_{t-1}) = b(Y_t + I_{t-1} - E_t(S_t) - k) + \beta E_t V_3(u_{t+1}, S_t, I_t) \qquad (7)$$

From (6):

$$\beta E_t V_3(u_{t+1}, S_t, I_t) = -\left[a(Y_t - u_t) + b(Y_t + I_{t-1} - E_t(S_t) - k)\right] \qquad (6')$$

Plug (6') into (7):

$$V_3(u_t, S_{t-1}, I_{t-1}) = b(Y_t + I_{t-1} - E_t(S_t) - k) - \left[a(Y_t - u_t) + b(Y_t + I_{t-1} - E_t(S_t) - k)\right] \qquad (8)$$

Simplifying:

$$V_3(u_t, S_{t-1}, I_{t-1}) = -a(Y_t - u_t) \qquad (8')$$

Update (8'):

$$V_3(u_{t+1}, S_t, I_t) = -a(Y_{t+1} - u_{t+1}) \qquad (8'')$$

Finally, plug (8'') into (6) in order to get the Euler Equation:

$$a(Y_t - u_t) + b(Y_t + I_{t-1} - E_t(S_t) - k) = \beta a E_t(Y_{t+1} - u_{t+1}) \qquad (9)$$

This equation can be further rewritten by substituting in for output using (3):

$$a(I_t - I_{t-1} + S_t - u_t) + b(I_t - I_{t-1} + S_t + I_{t-1} - E_t(S_t) - k) = \beta a E_t(I_{t+1} - I_t + S_{t+1} - u_{t+1})$$

(ii)

This model simultaneously features production-smoothing incentives, time-varying productivity shocks (or cost shocks), and stockout avoidance behavior. Changing the underlying parameters affects the balance of power among these three motives and leads to different dynamics.

As *a* increases, both the production-smoothing motive and the effects of cost shocks become stronger. While smoothing production implies a reduced responsiveness of output, cost shocks work in the opposite direction. So, the final impact of an increase in *a* to the variance of output to sales (RVOS) is ambiguous. If the production-smoothing motive dominates cost-shock effects, then RVOS falls.

When b increases, stockout avoidance gains importance. In this case, production responds more to sales so that the firm does not deviate much from the target inventory level. As a result the RVOS tends to increase. Finally, if the variance of the productivity shock increases, then production volatility also increases and the RVOS tends to go up as well.

(iii)

No Productivity shocks: u_t is constant

In the absence of cost shocks, sales become the only stochastic force impacting the firm's output decision. Even though stockout avoidance behavior is still present, there is less incentive for both production and inventory volatility. Sales are not affected at all since they are exogenously given.

There are no productivity shocks and b=0:

The Euler Equation collapses to

$$Y_t = \beta E_t Y_{t+1} \qquad (11)$$

Which can be written as:

$$Y_{t+1} = \beta^{-1} Y_t + \varepsilon_{t+1}, \quad E_t \varepsilon_{t+1} = 0 \qquad (12)$$

Since β^{-1} is strictly greater than one, equation (12) implies that production is postponed as much as possible. Indeed, because stockout avoidance is not present and also because future profits are discounted, the firm will prefer to incur in production costs in the future rather than today. As a result, inventories tend be negative during the period when production is being postponed. Once again, sales are not affected since they are exogenously given.

Sales are constant:

If sales are constant, then the firm can control the inventory level perfectly since there is no uncertainty coming from demand. In this case, cost shocks become the only stochastic force impacting the firm's output decision. A positive shock leads to higher output and inventory accumulation, generating pro-cyclical inventory behavior.

(iv)

Sales persistency and the markovian assumption made in part (i) imply that observing S_{t-1} conveys useful information for the conditional forecast of sales. In particular, highly persistent sales shocks means that the value of sales observed at t-1 is a good indication of the level of sales to be realized at t (assuming iid innovations with sufficiently small variance). In this case, output reacts more to sales shocks and the RVOS tends to increase.

In contrast, it is low persistency in productivity shocks that leads to stronger output response. The logic behind this argument can be better understood with an example. Consider a firm that has been hit by a positive productivity shock. If it believes that the shock is not going to stick around for very long (alternatively, that the shock displays low persistency), then the firm will make arrangements to exploit the (likely) temporary period of favorable shock as much as possible. In other words, the firm will increase its current production and accumulate inventories taking advantage of the momentary good times. Additionally, quoting Blanchard and Fischer (pg. 308): "The presence of inventories thus leads to stronger but less persistent effects of productivity shocks on production".

(v)

The model does feature incentives for a larger-than-one relative variance of output to sales and for pro-cyclical inventory behavior as well. Thus, at least in qualitative terms, it is inline with two major empirical regularities. Nevertheless, a complete assessment of how successful the model is would probably require a numerical simulation. Then, it would be possible to check whether the empirical regularities can be quantitatively captured for a credible parameterization of the model.