

Solution to Hatiwanger's January 2006 Comp Question  
(Sketch of Solutions)

Part i

The problem of the firm is to maximize

$$\int [pg(L(p)) - w(p)L(p)] f(p) dp$$

choosing  $w(p)$ ,  $L(p)$  and  $m$  subject to

$$\int [U(w(p)L(p)/m) - V(L(p)/m)] f(p) dp \geq U_0$$

and

$$m \geq L(p) \tag{1}$$

The Lagrangian is given by

$$L = \int [pg(L(p)) - w(p)L(p)] f(p) dp + \lambda \left[ \int [U(w(p)L(p)/m) - V(L(p)/m)] f(p) dp - U_0 \right] + \int [\mu(p)(m - L(p))] f(p) dp$$

The first order conditions for a given  $p$  are

$L(p)$

$$pg'(L(p)) - w(p) + \frac{\lambda}{m} [w(p)U'((w(p)L(p)/m)) - V'(L(p)/m)] - \mu(p) = 0 \tag{2}$$

$w(p)$

$$-L(p) + \frac{\lambda}{m} L(p)U'((w(p)L(p)/m)) = 0 \tag{3}$$

$m$

$$-\lambda \left[ \int \frac{\lambda L(p)}{m^2} [w(p)U'((w(p)L(p)/m)) - V'(L(p)/m)] f(p) dp \right] + \int \mu(p) f(p) dp = 0 \tag{4}$$

Note that the rationality constraint always binds because a profit maximizing firm never finds optimal to write a contract that gives the worker an utility levels

greater than  $U_0$ . Therefore,  $\lambda > 0$ . With respect to the capacity constraint, it might bind for some realizations of  $p$ , so we still need the condition

$$\mu(p)[m - L(p)] = 0 \quad (5)$$

From equation (3) we get that

$$U'((w(p)L(p)/m)) = \frac{m}{\lambda} \quad (6)$$

Given that both  $m$  and  $\lambda$  are constant, then  $w(p)L(p)$  must also be a constant, that is

$$w(p)L(p) = W \quad (7)$$

where  $W = mU'^{-1}(\frac{m}{\lambda})$ . The optimal contract presents **income rigidity**.

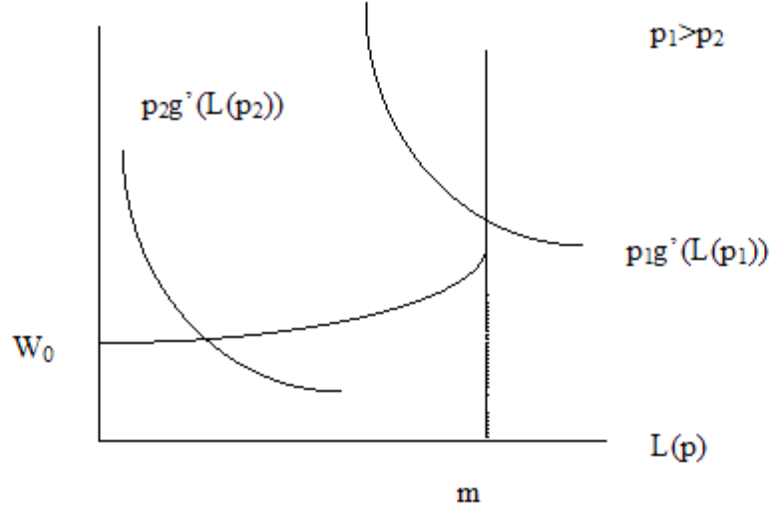
Replacing eq. (6) into equation (2)

$$pg'(L(p)) = w(p) - \frac{1}{U'((w(p)L(p)/m))} [w(p)U'((w(p)L(p)/m)) - V'(L(p)/m)] + \mu(p)$$

$$pg'(L(p)) = w(p) - \left[ w(p) - \frac{V'(L(p)/m)}{U'((w(p)L(p)/m))} \right] + \mu(p)$$

$$pg'(L(p)) = \frac{V'(L(p)/m)}{U'((w(p)L(p)/m))} + \mu(p) \quad (8)$$

Given the income rigidity result we obtained earlier and that  $m$  is a constant,  $U'((w(p)L(p)/m))$  is constant. Also note that given strict convexity of  $V$  the right hand side of the expression above is increasing in  $L(p)$ . (See graph below)



Finally, working with equation (4) we get the following:

$$E[\mu(p)] = \lambda \left[ \int \frac{\lambda L(p)}{m^2} [w(p)U'((w(p)L(p)/m)) - V'(L(p)/m)] f(p) dp \right]$$

$$E[\mu(p)] = \lambda \left[ \int \frac{\lambda}{m^2} [L(p)w(p)U'((w(p)L(p)/m)) - L(p)V'(L(p)/m)] f(p) dp \right]$$

$$m = \frac{\lambda}{E[\mu(p)]} \left[ \int \frac{\lambda}{m} [L(p)w(p)U'((w(p)L(p)/m)) - L(p)V'(L(p)/m)] f(p) dp \right]$$

$$m = \frac{\lambda}{E[\mu(p)]} \left[ \int \frac{\lambda}{U'((w(p)L(p)/m))} [L(p)w(p)U'((w(p)L(p)/m)) - L(p)V'(L(p)/m)] f(p) dp \right]$$

$$m = \frac{\lambda}{E[\mu(p)]} \left\{ \int \left[ L(p)w(p) - L(p) \frac{V'(L(p)/m)}{U'((w(p)L(p)/m))} \right] f(p) dp \right\} \quad (9)$$

The system of equation form by equations (7), (8) and (9) fully determine the optimal choice of  $L(p)$ ,  $w(p)$  and  $m$ .

#### Part ii

This is not the first best contract because of the firm's necessity to attach workers ex-ante. The allocations under the second-best contract in cases where

eq. (1) does not bind are the same as the ones under the first-best contract. When eq. (1) binds, allocations under the two contracts differ. Under the first best contract the firm hires more workers and pays lower wages.

This model does not explain wage rigidity. According to equation (7) income is rigid, not wage.

### Part iii

#### iii-a

The incentives compatibility constraints are given by

$$p_a g(L(p_a)) - w(p_a)L(p_a) \geq p_a g(L(p_b)) - w(p_b)L(p_b) \quad (10)$$

and

$$p_b g(L(p_b)) - w(p_b)L(p_b) \geq p_b g(L(p_a)) - w(p_a)L(p_a) \quad (11)$$

Summing these two inequalities

$$\begin{aligned} p_a g(L(p_a)) - w(p_a)L(p_a) + p_b g(L(p_b)) - w(p_b)L(p_b) \\ \geq p_a g(L(p_b)) - w(p_b)L(p_b) + p_b g(L(p_a)) - w(p_a)L(p_a) \end{aligned}$$

Rearranging

$$\begin{aligned} (p_a - p_b)g(L(p_a)) &\geq (p_a - p_b)g(L(p_b)) \\ g(L(p_a)) &\geq g(L(p_b)) \end{aligned}$$

or equivalently,

$$L(p_a) \geq L(p_b)$$

#### iii-b

For notational convenience let  $p_1 = p_A$  and  $p_2 = p_B$ . The Lagrangian in this case is given by

$$\begin{aligned} L = & \sum_{i=1}^2 \frac{1}{2} [p_i g(L(p_i)) - w(p_i)L(p_i)] + \lambda \left[ \left( \sum_{i=1}^2 \frac{1}{2} [U(w(p_i)L(p_i)/m) - V(L(p_i)/m)] \right) - U_0 \right] \\ & + \sum_{i=1}^2 \frac{1}{2} [\mu(p_i)(m - L(p_i)) + \phi [L(p_1) - L(p_2)]] \end{aligned}$$

The first order conditions for a given  $p$  are

$$L(p_1)]$$

$$\frac{1}{2} [p_1 g'(L(p_1)) - w(p_1)] + \frac{\lambda}{2m} [w(p_1) U'((w(p_1)L(p_1)/m)) - V'(L(p_1)/m)] - \frac{1}{2} \mu(p_1) + \phi = 0 \quad (12)$$

$$L(p_2)]$$

$$\frac{1}{2} [p_2 g'(L(p_2)) - w(p_2)] + \frac{\lambda}{2m} [w(p_2) U'((w(p_2)L(p_2)/m)) - V'(L(p_2)/m)] - \frac{1}{2} \mu(p_2) - \phi = 0 \quad (13)$$

$$w(p_i)]$$

$$-\frac{1}{2} L(p_i) + \frac{\lambda}{2m} L(p_i) U'((w(p_i)L(p_i)/m)) = 0 \quad (14)$$

$$m]$$

$$-\sum_{i=1}^2 \frac{\lambda L(p_i)}{2m^2} [w(p_i) U'(w(p_i)L(p_i)/m) - V'(L(p_i)/m)] + \sum_{i=1}^2 \frac{1}{2} \mu(p_i) = 0 \quad (15)$$

The rationality constraint always binds because a profit maximizing firm never finds optimal to write a contract that gives the worker expected utility levels greater than  $U_0$ . Therefore,  $\lambda > 0$ . With respect to the capacity constraint, it might bind for some realizations of  $p$ , so we still need the condition  $\mu(p_i)[m - L(p_i)] = 0$ , for  $i = 1, 2$ .

Note that the first order conditions for  $w(p)$  and  $m$  are essentially the same as obtained above. Therefore, we also obtain income rigidity in this case. The first order conditions for the different  $L(p)$  are not the same. Using eq (12) we get

$$p_1 g'(L(p_1)) = w(p_1) - \frac{\lambda}{m} [w(p_1) U'((w(p_1)L(p_1)/m)) - V'(L(p_1)/m)] + \mu(p_1) - 2\phi$$

$$p_1 g'(L(p_1)) = w(p_1) - \left[ w(p_1) - \frac{V'(L(p_1)/m)}{U'((w(p_1)L(p_1)/m))} \right] + \mu(p_1) - 2\phi$$

$$p_1 g'(L(p_1)) = \frac{V'(L(p_1)/m)}{U'((w(p_1)L(p_1)/m))} + \mu(p_1) - 2\phi$$

Similarly, using eq. (13)

$$p_2 g'(L(p_2)) = \frac{V'(L(p_2)/m)}{U'((w(p_2)L(p_2)/m))} + \mu(p_2) + 2\phi$$

### iii-c

To see whether there is over-employment in this economy or not, we have to see how the utility of consumption net of the disutility from effort when the agent is not working ( $Y_o \equiv U(w(p) * 0/m) - V(0/m)$ ) compares to  $Q(L(p)) \equiv \frac{V'(L(p)/m)}{U'((w(p)L(p)/m))}$ . That is, we have to compare the opportunity cost of labor ( $Y_o$ ) to value used by the firm while deciding  $L(p)$ . The answer is going to depend on the functional forms assumptions. But three different cases can emerge:

1)  $Y_o < Q(L(p))$  for all  $L(p) < m$ . In this case we have underemployment. Underemployment is going to be worse in bad states.

2)  $Y_o > Q(L(p))$  for all  $L(p) < m$ . For all  $L(p) < m$  we have over-employment. The bigger  $p$  is the more over-employment we find.

3)  $Y_o = Q(L(p))$  for some  $0 < L(p) < m$ . For those  $L(p)$  such that  $Y_o > Q(L(p))$  we are in case 2 above. The same type of reasoning applies. For that unique  $L(p)$  such that  $Y_o = Q(L(p))$  we get the efficient level of employment. For those  $L(p)$  such that  $Y_o < Q(L(p))$  we are again in case 1 above.

### iii-d

The contract with asymmetric information in this model is the same as the contract with symmetric information. Given that the optimal contract in both cases present income rigidity, the incentive compatibility constraint never binds. No matter what state of the world the firm ends up paying the same amount to workers. There is no need to lie.