3. Consider a firm that holds inventories to avoid stockouts and to smooth production. To make the analysis tractable, assume the firm takes the sales process as given and minimizes costs given by:

$$E[\sum_{t=0}^{\infty}\beta^t\{(a/2)(Y_t-u_t)^2+(b/2)(Y_t+I_{t-1}-I_t^*)^2\}]$$

with

$$I_t^* = E(S_t)+k$$

$$I_t = I_{t-1}+Y_t-S_t$$

where E is the expectational operator, Y_t is output in period t, S_t is sales (taken as exogenous here but stochastic), I_t is inventories, I_t* is the target inventory level, u_t is a productivity shock that impacts costs, β is the discount factor, and a, b and k are positive constants. The incentive for production smoothing is captured by the first term and stockout avoidance incentives are proxied in the second term in that it is costly to deviate from target inventories. Firms must choose output in each period before knowing the realization of sales but after the realization of the productivity shock. Answer the following questions:

(i) Set this problem up as a dynamic programming problem and derive the Euler equation.

(ii) What will happen to the relative variance of output to sales as (i) a increases; (ii) b increases; (iii) the variance of the productivity shock increases? Discuss (your answer may be either descriptive or analytical).

(iii) Consider the following special cases and describe (either intuitively or analytically) the output, inventory and sales dynamics in these cases:
   a. There are no productivity shocks;
   b. b=0 and there are no productivity shocks;
   c. Sales are constant;

(iv) What role does the persistence of either sales or productivity shocks play in your answer to (iii)? Discuss.

(v) Can this model account for the observed empirical behavior for inventories? Discuss.