Macroeconomic Comprehensive Theory Examination (August 2002)

Sketch of the Solution - Shea's Question (#1)

Parent i's utility: $\log C_1(i,p) + \beta(i,p)\log C_2(i,p) + \gamma \log T(i)$ (1)

Parent i's budget constraint: $C_1(i,p) + C_2(i,p) + T(i) = Y(i,p)$ (2)

Child i's utility: $\log C_2(i,c) + \beta(i,c)\log C_3(i,c)$ (3)

Child i's budget constraint: $C_2(i,c) + C_3(i,c) = Y(i,c) + T(i)$ (4)

(a)

Part I

Optimal decisions:

Parent: Maximize (1) subject to (2)

$$L = \log C_1(i,p) + \beta(i,p)\log C_2(i,p) + \gamma \log T(i) + \lambda\left[Y(i,p) - C_1(i,p) - C_2(i,p) - T(i)\right]$$

First-order conditions:

$$C_1(i,p):\quad \frac{1}{C_1(i,p)} = \lambda \qquad (5)$$

$$C_2(i,p):\quad \frac{\beta(i,p)}{C_2(i,p)} = \lambda \qquad (6)$$

$$T(i):\quad \frac{\gamma}{T(i)} = \lambda \qquad (7)$$

Combine (5) and (6):

$$\frac{1}{C_1(i,p)} = \frac{\beta(i,p)}{C_2(i,p)} \quad \therefore \quad C_2(i,p) = \beta(i,p)C_1(i,p) \qquad (8)$$

Combine (5) and (7):

$$\frac{1}{C_1(i,p)} = \frac{\gamma}{T(i)} \quad \therefore \quad T(i) = \gamma C_1(i,p) \qquad (9)$$

Plug (8) and (9) into the budget constraint (2):

$$C_1(i,p) + C_2(i,p) + T(i) = Y(i,p) \quad \therefore$$
$$C_1(i,p) + \beta(i,p)C_1(i,p) + \gamma C_1(i,p) = Y(i,p) \quad \therefore$$
$$C_1(i,p)(1 + \beta(i,p) + \gamma) = Y(i,p) \quad \therefore$$

Finally, we obtain the optimal first-period consumption of the parent:

$$C_1^*(i,p) = \frac{Y(i,p)}{(1 + \beta(i,p) + \gamma)} \qquad (10)$$

Optimal second-period consumption and optimal transfers can also be obtained by plugging (10) into (8) and (9) respectively:

$$C_2^*(i,p) = \frac{\beta(i,p)Y(i,p)}{(1 + \beta(i,p) + \gamma)} \qquad (11)$$

$$T^*(i) = \frac{\gamma Y(i,p)}{(1 + \beta(i,p) + \gamma)} \qquad (12)$$

<u>Child: Maximize (3) subject to (4)</u>

$$L = \log C_2(i,c) + \beta(i,c)\log C_3(i,c) + \lambda\left[Y(i,c) + T(i) - C_2(i,c) - C_3(i,c)\right]$$

First-order conditions:

$$C_2(i,c): \quad \frac{1}{C_2(i,c)} = \lambda \qquad (13)$$

$$C_3(i,c): \quad \frac{\beta(i,c)}{C_3(i,c)} = \lambda \qquad (14)$$

Combine (13) and (14):

$$\frac{1}{C_2(i,c)} = \frac{\beta(i,c)}{C_3(i,c)} \quad \therefore \quad C_3(i,c) = \beta(i,c)C_2(i,c) \qquad (15)$$

Plug (15) into the budget constraint (4):

$$C_2(i,c)+C_3(i,c)=Y(i,c)+T(i) \quad \therefore$$
$$C_2(i,c)+\beta(i,c)C_2(i,c)=Y(i,c)+T(i) \quad \therefore$$
$$C_2(i,c)(1+\beta(i,c))=Y(i,c)+T(i) \quad \therefore$$

Finally, we obtain the optimal period-two consumption of the child:

$$C_2^*(i,c)=\frac{Y(i,c)+T^*(i)}{(1+\beta(i,c))} \qquad (16)$$

Where $T^*(i)$ is given by (12).

By combining (16) and (15), we get the optimal period-three consumption of the child:

$$C_3^*(i,c)=\frac{\beta(i,c)(Y(i,c)+T^*(i))}{(1+\beta(i,c))} \qquad (17)$$

Where $T^*(i)$ is given by (12) as before.

<u>Summary of optimal decisions:</u>

$$C_1^*(i,p)=\frac{Y(i,p)}{(1+\beta(i,p)+\gamma)} \qquad (10)$$

$$C_2^*(i,p)=\frac{\beta(i,p)Y(i,p)}{(1+\beta(i,p)+\gamma)} \qquad (11)$$

$$T^*(i)=\frac{\gamma Y(i,p)}{(1+\beta(i,p)+\gamma)} \qquad (12)$$

$$C_2^*(i,c)=\frac{Y(i,c)+T^*(i)}{(1+\beta(i,c))} \qquad (16)$$

$$C_3^*(i,c)=\frac{\beta(i,c)(Y(i,c)+T^*(i))}{(1+\beta(i,c))} \qquad (17)$$

Part II

Define the following:

<u>Parent's wealth holdings at the end of period 1</u>

$$W(i,p) \equiv Y(i,p) - C_1^*(i,p) = Y(i,p) - \frac{Y(i,p)}{(1+\beta(i,p)+\gamma)} = \frac{Y(i,p)(\beta(i,p)+\gamma)}{(1+\beta(i,p)+\gamma)} \qquad (18)$$

<u>Child's wealth holdings at the end of period 2</u>

$$W(i,c) \equiv Y(i,c) - C_2^*(i,c) = Y(i,p) - \frac{Y(i,c)+T^*(i)}{(1+\beta(i,c))} = \frac{\beta(i,c)Y(i,c)-T^*(i)}{(1+\beta(i,c))} \qquad (19)$$

Where $T^*(i)$ is given by (12).

Now suppose that $\gamma=0$ (such that $T^*(i)=0$), definitions (18) and (19) become:

$$W(i,p) \equiv \frac{Y(i,p)\beta(i,p)}{(1+\beta(i,p))} \qquad (20)$$

$$W(i,c) \equiv \frac{\beta(i,c)Y(i,c)}{(1+\beta(i,c))} \qquad (21)$$

Applying log in both sides of (20) and (21):

$$\log W(i,p) \equiv \log\left(\frac{\beta(i,p)}{(1+\beta(i,p))}\right) + \log Y(i,p) \qquad (22)$$

$$\log W(i,c) \equiv \log\left(\frac{\beta(i,c)}{(1+\beta(i,c))}\right) + \log Y(i,c) \qquad (23)$$

The definitions above can be further rewritten as:

$$w(i,p) \equiv b(i,p) + y(i,p) \qquad (24)$$

$$w(i,c) \equiv b(i,c) + y(i,c) \qquad (25)$$

Where

$$w(i,p) \equiv \log W(i,p); \quad b(i,p) \equiv \log\left(\frac{\beta(i,p)}{(1+\beta(i,p))}\right); \quad y(i,p) \equiv \log Y(i,p)$$

And,

$$w(i,c) \equiv \log W(i,c); \quad b(i,c) \equiv \log\left(\frac{\beta(i,c)}{(1+\beta(i,c))}\right); \quad y(i,c) \equiv \log Y(i,c)$$

(b)

$$Corr(w(i,p),w(i,c)) = \frac{\overbrace{Cov(w(i,p),w(i,c))}^{I}}{\underbrace{(Var(w(i,p)))^{1/2}}_{II}\underbrace{(Var(w(i,c)))^{1/2}}_{III}} \qquad (26)$$

Plug (24) and (25) into *I* in the formula above:

$$I = Cov(w(i,p),w(i,c)) = Cov(b(i,p)+y(i,p),b(i,c)+y(i,c)) \qquad (27)$$

Using the hint provided, expression (27) can be written as:

$$Cov(b(i,p),b(i,c)) + Cov(b(i,p),y(i,c)) + Cov(y(i,p),b(i,c)) + Cov(y(i,p),y(i,c)) \quad (28)$$

Furthermore, assumptions made in the question tell us that:

$$Corr(b(i,p),b(i,c)) = \rho(b) \qquad (29)$$

$$Corr(b(i,p),y(i,c)) = 0 \qquad (30)$$

$$Corr(y(i,p),b(i,c)) = 0 \qquad (31)$$

$$Corr(y(i,p),y(i,c)) = \rho(y) \qquad (32)$$

Once we also know the variances of the variables above, we can recover the covariances that appear in equation (28). Indeed,

$$Cov(b(i,p),b(i,c)) = Corr(b(i,p),b(i,c))(\sigma^2(b))^{1/2}(\sigma^2(b))^{1/2} = \rho(b)\sigma^2(b) \qquad (33)$$

$$Cov(b(i,p),y(i,c)) = Corr(b(i,p),y(i,c))(\sigma^2(b))^{1/2}(\sigma^2(b))^{1/2} = 0 \qquad (34)$$

$$Cov(y(i,p),b(i,c)) = Corr(y(i,p),b(i,c))(\sigma^2(b))^{1/2}(\sigma^2(b))^{1/2} = 0 \qquad (35)$$

$$Cov\left(y(i,p),y(i,c)\right)=Corr\left(y(i,p),y(i,c)\right)\left(\sigma^2(y)\right)^{1/2}\left(\sigma^2(y)\right)^{1/2}=\rho(y)\sigma^2(y) \qquad (36)$$

Plugging (33) – (36) into (28), we establish that:

$$I=Cov\left(w(i,p),w(i,c)\right)=\rho(b)\sigma^2(b)+\rho(y)\sigma^2(y) \qquad (37)$$

Now, let's deal with term II of formula (26):

$$II=\left(Var(w(i,p))\right)^{1/2}=\left(Var(b(i,p)+y(i,p))\right)^{1/2}=\left(\sigma^2(b)+\sigma^2(y)\right)^{1/2} \qquad (38)$$

Above we use the fact that $b(i,p)$ and $y(i,p)$ are uncorrelated.

Similarly, we can calculate III:

$$III=\left(Var(w(i,c))\right)^{1/2}=\left(Var(b(i,c)+y(i,c))\right)^{1/2}=\left(\sigma^2(b)+\sigma^2(y)\right)^{1/2} \qquad (39)$$

Above we use the fact that $b(i,c)$ and $y(i,c)$ are uncorrelated.

Plugging (37), (38), and (39) in to (26):

$$Corr\left(w(i,p),w(i,c)\right)=\frac{Cov\left(w(i,p),w(i,c)\right)}{\left(Var(w(i,p))\right)^{1/2}\left(Var(w(i,c))\right)^{1/2}} \quad \therefore$$

$$Corr\left(w(i,p),w(i,c)\right)=\frac{\rho(b)\sigma^2(b)+\rho(y)\sigma^2(y)}{\sigma^2(b)+\sigma^2(y)} \qquad (40)$$

We are now asked to prove that:

$$Corr\left(w(i,p),w(i,c)\right)>\rho(y) \quad \Leftrightarrow \quad \rho(b)>\rho(y)$$

<u>Proof:</u>

($\Rightarrow$)

If $Corr\left(w(i,p),w(i,c)\right)>\rho(y)$, then from (40) we know that:

$$\frac{\rho(b)\sigma^2(b)+\rho(y)\sigma^2(y)}{\sigma^2(b)+\sigma^2(y)}>\rho(y) \quad \therefore$$

$$\rho(b)\sigma^2(b) + \rho(y)\sigma^2(y) > \rho(y)\sigma^2(b) + \rho(y)\sigma^2(y) \quad \therefore$$

$$\rho(b)\sigma^2(b) > \rho(y)\sigma^2(b) \quad \therefore$$

Finally,

$$\rho(b) > \rho(y)$$

$(\Leftarrow)$

If $\rho(b) = \rho(y)$, then from (40)

$$\frac{\rho(b)\sigma^2(b) + \rho(y)\sigma^2(y)}{\sigma^2(b) + \sigma^2(y)} = \frac{\rho(y)\sigma^2(b) + \rho(y)\sigma^2(y)}{\sigma^2(b) + \sigma^2(y)} = \rho(y) \quad (41)$$

And thus $Corr\big(w(i,p), w(i,c)\big) = \rho(y)$

From (41), if $\rho(b) > \rho(y)$

$$\frac{\rho(b)\sigma^2(b) + \rho(y)\sigma^2(y)}{\sigma^2(b) + \sigma^2(y)} > \frac{\rho(y)\sigma^2(b) + \rho(y)\sigma^2(y)}{\sigma^2(b) + \sigma^2(y)} = \rho(y) \quad (42)$$

And thus $Corr\big(w(i,p), w(i,c)\big) > \rho(y)$

Hence, we proved that:

$$Corr\big(w(i,p), w(i,c)\big) > \rho(y) \quad \Leftrightarrow \quad \rho(b) > \rho(y)$$

(c)

Recall that

$$W(i,p) \equiv \frac{Y(i,p)(\beta(i,p) + \gamma)}{(1 + \beta(i,p) + \gamma)} \qquad (18)$$

$$W(i,c) \equiv \frac{\beta(i,c)Y(i,c) - T^*(i)}{(1 + \beta(i,c))} \qquad (19)$$

Where

$$T^*(i) = \frac{\gamma Y(i,p)}{(1+\beta(i,p)+\gamma)} \qquad (12)$$

Plug (12) into (19):

$$W(i,c) \equiv \frac{\beta(i,c)Y(i,c)}{(1+\beta(i,c))} - \frac{\gamma Y(i,p)}{(1+\beta(i,c))(1+\beta(i,p)+\gamma)} \qquad (19')$$

We now assume that:

$$\beta(i,p) = \beta(i,c) = \beta \qquad (43)$$

And

$$\frac{\beta}{1+\beta} \cong \frac{\beta+\gamma}{1+\beta+\gamma} \qquad (44)$$

Imposing (43) and (44) into (18) and (19'), respectively:

$$W(i,p) \equiv \frac{Y(i,p)(\beta+\gamma)}{(1+\beta+\gamma)} = \frac{Y(i,p)\beta}{(1+\beta)} \qquad (45)$$

$$W(i,c) \equiv \frac{\beta Y(i,c)}{(1+\beta)} - \frac{\gamma Y(i,p)}{(1+\beta)(1+\beta+\gamma)} \qquad (46)$$

We can now calculate the covariance between (45) and (46):

$$Cov\big(W(i,p),W(i,c)\big) = Cov\left(\frac{Y(i,p)\beta}{(1+\beta)}, \frac{\beta Y(i,c)}{(1+\beta)} - \frac{\gamma Y(i,p)}{(1+\beta)(1+\beta+\gamma)}\right) \qquad (47)$$

Again, resorting to the hint provided, we can rewrite (47) as:

$$Cov\left(\frac{Y(i,p)\beta}{(1+\beta)}, \frac{\beta Y(i,c)}{(1+\beta)}\right) + Cov\left(\frac{Y(i,p)\beta}{(1+\beta)}, -\frac{\gamma Y(i,p)}{(1+\beta)(1+\beta+\gamma)}\right) \quad \therefore$$

$$\left(\frac{\beta}{(1+\beta)}\right)^2 Cov\big(Y(i,p),Y(i,c)\big) - \left(\frac{\beta\gamma}{(1+\beta)^2(1+\beta+\gamma)}\right) Cov\big(Y(i,p),Y(i,p)\big) \qquad (48)$$

Using (48), the covariance formula (47) becomes:

$$Cov\left(W(i,p),W(i,c)\right)=\left[\left(\frac{\beta}{(1+\beta)}\right)^2-\left(\frac{\beta\gamma}{(1+\beta)^2(1+\beta+\gamma)}\right)\right]Cov\left(Y(i,p),Y(i,c)\right) \qquad (49)$$

Assuming that $Cov\left(Y(i,p),Y(i,c)\right)$ is positive, formula (49) tells us that the intergenerational covariance of the level of wealth is lower in the presence of bequests (γ slightly positive) than if there are no bequests ($\gamma = 0$).

Note that the concept of parent's wealth adopted includes resources to be transferred and so it overestimates parent's deferred consumption. Similarly, the concept of child's wealth does not include transfers to be received and so it underestimates child's deferred consumption. Indeed, a rational child internalizes any bequest yet to be received, and her measured wealth will be smaller when transfers are positive because there is less need to intertemporally reshuffle resources. At the same time, positive transfers imply a greater level of parent's wealth because provisions for both future consumption and bequest have to be made. Summing up, giving the definitions of wealth adopted, in the presence of bequests parent's wealth tends to increase, and child's wealth tends to decrease, leading to a reduction of the intergenerational wealth covariance.