Solutions to Shea's August 2006 Comp Question

Households maximize $E_0 \sum_{t=0}^{\infty} \beta^t U(C_t)$

where $U(C) = \frac{\sigma}{\sigma-1} C^{\frac{\sigma-1}{\sigma}}$

and $C = \left[n^{\frac{\varepsilon-1}{\varepsilon}} + \omega h^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}$

Part A

Replacing C into the utility function

$$U(C) = \frac{\sigma}{\sigma-1} \left[n^{\frac{\varepsilon-1}{\varepsilon}} + \omega h^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}\frac{\sigma-1}{\sigma}}$$

When $\varepsilon = \sigma$

$$U(C) = \frac{\sigma}{\sigma-1} \left[n^{\frac{\varepsilon-1}{\varepsilon}} + \omega h^{\frac{\varepsilon-1}{\varepsilon}} \right] = \frac{\sigma}{\sigma-1} n^{\frac{\varepsilon-1}{\varepsilon}} + \frac{\sigma}{\sigma-1} \omega h^{\frac{\varepsilon-1}{\varepsilon}}$$

Utility is additively separable in n and h.

To show that n and $h.$ are Edgeworth-Pareto substitutes in utility, first take the partial derivative of U *w.r.t.* n

$$\frac{\partial U(C)}{\partial n} = \frac{\partial U(C)}{\partial C} \frac{C}{\partial n}$$

$$\frac{\partial U(C)}{\partial n} = C^{-\frac{1}{\sigma}} \frac{\varepsilon}{\varepsilon-1} \left[n^{\frac{\varepsilon-1}{\varepsilon}} + \omega h^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{1}{\varepsilon-1}} \left(\frac{\varepsilon-1}{\varepsilon} n^{-\frac{1}{\varepsilon}} \right)$$

$$\frac{\partial U(C)}{\partial n} = C^{-\frac{1}{\sigma}} C^{\frac{1}{\varepsilon}} n^{-\frac{1}{\varepsilon}} = C^{\frac{\sigma-\varepsilon}{\sigma\varepsilon}} n^{-\frac{1}{\varepsilon}}$$

Now take derivative *w.r.t.* h

$$\frac{\partial U(C)}{\partial n \partial h} = \frac{\sigma-\varepsilon}{\sigma\varepsilon} n^{-\frac{1}{\varepsilon}} C^{\frac{\sigma-\varepsilon-\sigma\varepsilon}{\sigma\varepsilon}} \frac{\varepsilon}{\varepsilon-1} \left[n^{\frac{\varepsilon-1}{\varepsilon}} + \omega h^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{1}{\varepsilon-1}} \left(\frac{\varepsilon-1}{\varepsilon} \omega h^{-\frac{1}{\varepsilon}} \right)$$

$$\frac{\partial U(C)}{\partial n \partial h} = \frac{\sigma-\varepsilon}{\sigma\varepsilon} n^{-\frac{1}{\varepsilon}} C^{\frac{\sigma-\varepsilon-\sigma\varepsilon}{\sigma\varepsilon}} C^{\frac{1}{\varepsilon}} \omega h^{-\frac{1}{\varepsilon}}$$

$$\frac{\partial U(C)}{\partial n \partial h} = \frac{\sigma-\varepsilon}{\sigma\varepsilon} \omega \left(nh \right)^{-\frac{1}{\varepsilon}} C^{\frac{2\sigma-\varepsilon-\sigma\varepsilon}{\sigma\varepsilon}} < 0 \text{ if } \quad \sigma < \varepsilon$$

Part B

Households maximize $E_0 \sum_{t=0}^{\infty} \beta^t U(C_t)$ subject to a budget constraint. The budget constraint for this problem is
$n_t + p_t h_t + q_t z_{t+1} = (q_t + d_t) z_t$

$$V(z_t) = \max_{n_t, h_t, z_{t+1}} \left\{ \begin{array}{c} \frac{\sigma}{\sigma-1}\left[n_t^{\frac{\varepsilon-1}{\varepsilon}} + \omega h_t^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}\frac{\sigma-1}{\sigma}} + \beta E_t V(z_{t+1}) \\ +\lambda_t\left[(q_t + d_t) z_t - n_t - p_t h_t - q_t z_{t+1}\right] \end{array} \right\}$$

FOC

$$n_t\rfloor \qquad C_t^{\frac{\sigma-\varepsilon}{\sigma\varepsilon}} n_t^{-\frac{1}{\varepsilon}} - \lambda_t = 0$$

$$h_t\rfloor \qquad C_t^{\frac{\sigma-\varepsilon}{\sigma\varepsilon}} \omega h_t^{-\frac{1}{\varepsilon}} - \lambda_t p_t = 0$$

$$z_{t+1}\rfloor \qquad \beta E_t \frac{\partial V(h_{t+1}, z_{t+1})}{\partial z_{t+1}} - \lambda_t q_t = 0$$

From the first two FOC we get an intratemporal optimality condition $n_t = \left(\frac{\omega}{p_t}\right)^{-\varepsilon} h_t$

Take envelope condition

$$\frac{\partial V(z_t)}{\partial z_t} = \lambda_t (q_t + d_t)$$

Replacing in third FOC

$$\beta E_t \left[\lambda_{t+1}(q_{t+1} + d_{t+1})\right] = \lambda_t q_t$$

$$\beta E_t \left[\frac{\lambda_{t+1}}{\lambda_t} \frac{(q_{t+1} + d_{t+1})}{q_t}\right] = 1$$

Note that $\frac{(q_{t+1}+d_{t+1})}{q_t} = RR_t$, the return on the equity. Note that

$$\lambda_t = C_t^{\frac{\sigma-\varepsilon}{\sigma\varepsilon}} n_t^{-\frac{1}{\varepsilon}}$$
$$\lambda_t = \left[\left(n_t^{\frac{\varepsilon-1}{\varepsilon}} + \omega h_t^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]^{\frac{\sigma-\varepsilon}{\sigma\varepsilon}} n_t^{-\frac{1}{\varepsilon}}$$

Then,

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{\left[\left(n_{t+1}^{\frac{\varepsilon-1}{\varepsilon}} + \omega h_{t+1}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]^{\frac{\sigma-\varepsilon}{\sigma\varepsilon}} n_{t+1}^{-\frac{1}{\varepsilon}}}{\left[\left(n_t^{\frac{\varepsilon-1}{\varepsilon}} + \omega h_t^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right]^{\frac{\sigma-\varepsilon}{\sigma\varepsilon}} n_t^{-\frac{1}{\varepsilon}}}$$

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{\left(n_{t+1}^{\frac{\varepsilon-1}{\varepsilon}} + \omega h_{t+1}^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\sigma-\varepsilon}{\sigma(\varepsilon-1)}}}{\left(n_t^{\frac{\varepsilon-1}{\varepsilon}} + \omega h_t^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\sigma-\varepsilon}{\sigma(\varepsilon-1)}}} \left(\frac{n_{t+1}}{n_t}\right)^{-\frac{1}{\varepsilon}}$$

$$\frac{\lambda_{t+1}}{\lambda_t} = \left[\frac{n_{t+1}^{\frac{\varepsilon-1}{\varepsilon}}}{n_t^{\frac{\varepsilon-1}{\varepsilon}}} \frac{\left(1+\omega\left(\frac{h_{t+1}}{n_{t+1}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)}{\left(1+\omega\left(\frac{h_t}{n_t}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)}\right]^{\frac{\sigma-\varepsilon}{\sigma(\varepsilon-1)}} \left(\frac{n_{t+1}}{n_t}\right)^{-\frac{1}{\varepsilon}}$$

$$\frac{\lambda_{t+1}}{\lambda_t} = \left[\frac{\left(1+\omega\left(\frac{h_{t+1}}{n_{t+1}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)}{\left(1+\omega\left(\frac{h_t}{n_t}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)}\right]^{\frac{\sigma-\varepsilon}{\sigma(\varepsilon-1)}} \left(\frac{n_{t+1}^{\frac{\varepsilon-1}{\varepsilon}}}{n_t^{\frac{\varepsilon-1}{\varepsilon}}}\right)^{\frac{\sigma-\varepsilon}{\sigma(\varepsilon-1)}} \left(\frac{n_{t+1}}{n_t}\right)^{-\frac{1}{\varepsilon}}$$

$$\frac{\lambda_{t+1}}{\lambda_t} = \left[\frac{\left(1+\omega\left(\frac{h_{t+1}}{n_{t+1}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)}{\left(1+\omega\left(\frac{h_t}{n_t}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)}\right]^{\frac{\sigma-\varepsilon}{\sigma(\varepsilon-1)}} \left(\frac{n_{t+1}}{n_t}\right)^{\frac{\sigma-\varepsilon}{\sigma\varepsilon}} \left(\frac{n_{t+1}}{n_t}\right)^{-\frac{1}{\varepsilon}}$$

$$\frac{\lambda_{t+1}}{\lambda_t} = \left[\frac{\left(1+\omega\left(\frac{h_{t+1}}{n_{t+1}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)}{\left(1+\omega\left(\frac{h_t}{n_t}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)}\right]^{\frac{\sigma-\varepsilon}{\sigma(\varepsilon-1)}} \left(\frac{n_{t+1}}{n_t}\right)^{-\frac{1}{\sigma}}$$

$$\frac{\lambda_{t+1}}{\lambda_t} = \frac{M_{t+1}}{\beta}$$

Replacing into our Euler equation

$$\beta E_t\left[\frac{M_{t+1}}{\beta} RR_{t+1}\right] = 1$$

$$E_t\left[M_{t+1} RR_{t+1}\right] = 1$$

Part C

Let's $\alpha_t \equiv \frac{n_t}{n_t + p_t h_t}$.
$\alpha_t = \frac{1}{1+p_t\frac{h_t}{n_t}}$
$\alpha_t\left(1 + p_t\frac{h_t}{n_t}\right) = 1$
$p_t = \left(\frac{1-\alpha_t}{\alpha_t}\right)\left(\frac{h_t}{n_t}\right)^{-1}$

Now, remember that in part a we obtained that $n_t = \left(\frac{\omega}{p_t}\right)^{-\varepsilon} h_t$ or $\omega = p_t\left(\frac{h_t}{n_t}\right)^{\frac{1}{\varepsilon}}$. Substituting in p_t

$\omega = \frac{1-\alpha_t}{\alpha_t}\left(\frac{h_t}{n_t}\right)^{\frac{1}{\varepsilon}-1}$ for any t

We can now replace ω into

$$M_{t+1} \equiv \beta \left[\frac{\left(1+\omega\left(\frac{h_{t+1}}{n_{t+1}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)}{\left(1+\omega\left(\frac{h_t}{n_t}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)}\right]^{\frac{\sigma-\varepsilon}{\sigma(\varepsilon-1)}} \left(\frac{n_{t+1}}{n_t}\right)^{-\frac{1}{\sigma}}$$

$$M_{t+1} = \beta \left[\frac{\left(1+\frac{1-\alpha_{t+1}}{\alpha_{t+1}}\left(\frac{h_{t+1}}{n_{t+1}}\right)^{\frac{1}{\varepsilon}-1}\left(\frac{h_{t+1}}{n_{t+1}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)}{\left(1+\frac{1-\alpha_t}{\alpha_t}\left(\frac{h_t}{n_t}\right)^{\frac{1}{\varepsilon}-1}\left(\frac{h_t}{n_t}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)}\right]^{\frac{\sigma-\varepsilon}{\sigma(\varepsilon-1)}} \left(\frac{n_{t+1}}{n_t}\right)^{-\frac{1}{\sigma}}$$

$$M_{t+1} = \beta \left[\frac{1+\frac{1-\alpha_{t+1}}{\alpha_{t+1}}}{1+\frac{1-\alpha_t}{\alpha_t}}\right]^{\frac{\sigma-\varepsilon}{\sigma(\varepsilon-1)}} \left(\frac{n_{t+1}}{n_t}\right)^{-\frac{1}{\sigma}}$$

$$M_{t+1} = \beta \left[\frac{\alpha_t}{\alpha_{t+1}}\right]^{\frac{\sigma-\varepsilon}{\sigma(\varepsilon-1)}} \left(\frac{n_{t+1}}{n_t}\right)^{-\frac{1}{\sigma}}$$

$$M_{t+1} = \beta \left[\frac{\alpha_{t+1}}{\alpha_t}\right]^{\frac{\varepsilon-\sigma}{\sigma(\varepsilon-1)}} \left(\frac{n_{t+1}}{n_t}\right)^{-\frac{1}{\sigma}}$$

Finally, our Euler equation becomes

$$E_t\left[\beta \left[\frac{\alpha_{t+1}}{\alpha_t}\right]^{\frac{\varepsilon-\sigma}{\sigma(\varepsilon-1)}} \left(\frac{n_{t+1}}{n_t}\right)^{-\frac{1}{\sigma}} RR_{t+1}\right] = 1$$

Part D

The model with housing reduce to the standard C-CAPM when the utility is separable in n and h. Let's assume $U(n_t, h_t) = f(n_t) + g(h_t)$. In this case the FOC w.r.t. n becomes

$$f'(n_t) - \lambda_t = 0$$

Now, remember that the FOC with respect to the asset was

$$\beta E_t\left[\frac{\lambda_{t+1}}{\lambda_t}\frac{(q_{t+1}+d_{t+1})}{q_t}\right] = 1$$

Replacing λ_t and RR_t we get

$$\beta E_t\left[\frac{f'(n_{t+1})}{f'(n_t)}\frac{(q_{t+1}+d_{t+1})}{q_t}\right] = 1$$

This equation relates changes in marginal utility of consumption to returns as in the baseline C-CAPM model.

Equity-premium puzzle: for plausible degrees of risk aversion, the C-CAPM applied to aggregate data cannot explain the difference of returns between bonds and stocks.

Part E

We showed above that the household optimality conditions imply

$E_t\left[M_{t+1}R_t\right] = 1$

This expression can be rewritten as

$E_t\left[M_{t+1}\right]E_t\left[R_t\right] + Cov_t\left[M_{t+1}R_t\right] = 1$

Rearranging we get an expression for the equity premium

$E_t\left[R_t\right] = \frac{1-Cov_t\left[M_{t+1}R_t\right]}{E_t\left[M_{t+1}\right]}$

where $Cov_t\left[M_{t+1}R_t\right] = corr(M_{t+1}R_t) * st.dev(M_{t+1}) * st.dev(R_t)$

The pricing kernel with housing has a "two factor" structure. The standard CCAPM without housing is a one factor model: the pricing kernel depends only on consumption growth, and the expected returns therefore depend exclusively on their correlation with consumption growth. If the utility function is concave, the consumption smoothing motive makes it difficult to explain the observed equity premium. In a this model with housing and nonseparable utility, the change in the expenditure share emerges as a second factor. This composition risk factor drives the asset pricing performance of the model. If non-housing consumption is smooth as in the standard model, the higher correlation between M_{t+1} and R_t will help explain a bigger fraction of the equity premium. "Indeed, (in a baseline calibration of the model) non-housing consumption growth behaves much like NIPAR aggregate consumption growth: it is smooth, and its covariance with stock returns (denominated in units of numeraire) is small and positive. With separable utility, tiny values of the intertemporal elasticity would thus be need to generate high equity premia. Given that the covariance of stock returns with expenditure share growth $\Delta \ln \alpha_{t+1}$ is negative, stocks have low payoffs during recessions, when non-housing consumption growth is low, and especially low payoffs in severe recessions, when housing consumption is relative low (and α is high). This generates higher equity premia than under the standard model". (Piazzesi, Schneider and Tuzel)