Solutions to Shea's August 2008 Comp Question

Utility function is $U(c_t) = a + bc_{i1} - \frac{d}{2}c_t^2$

The household i problem is to maximize

$V \equiv a + bc_{i1} - \frac{d}{2}c_{i1}^2 + \beta E_t \left[a + bc_{i2} - \frac{d}{2}c_{i2}^2\right]$ subject to $c_{i2} = y_{i2} + R(y_{i1} - c_{i1})$

The Lagrangian can then be written as

$$L = a + bc_{i1} - \frac{d}{2}c_{i1}^2 + \beta E_t \left[a + bc_{i2} - \frac{d}{2}c_{i2}^2\right] + \lambda \left[y_{i2} + R(y_{i1} - c_{i1}) - c_{i2}\right]$$

FOC

$c_{i1}\rfloor \qquad b - dc_{i1} - \lambda R = 0$

$c_{i2}\rfloor \qquad \beta \left[b - dE_t c_{i2}\right] - \lambda = 0$

Combining these two FOC we get

$$b - dc_{i1} = \beta R \left[b - dE_t c_{i2}\right]$$

$$\left[1 - \beta R\right] b - dc_{i1} = -\beta R d E_t c_{i2}$$

$$E_t c_{i2} = \frac{c_{i1}}{\beta R} - \frac{1-\beta R}{\beta R d} b$$

Taking expectation of the dynamic budget constraint and replacing in the equation above we get

$$E_t y_{i2} + R(y_{i1} - c_{i1}) = \frac{c_{i1}}{\beta R} - \frac{1-\beta R}{\beta R d} b$$

$$E_t y_{i2} + R y_{i1} = \frac{1+\beta R^2}{\beta R} c_{i1} - \frac{1-\beta R}{\beta R d} b$$

$$c_{i1} = \frac{\beta R}{1+\beta R^2} \left[E_t y_{i2} + R y_{i1} + \frac{1-\beta R}{\beta R d} b\right]$$

Aggregate consumption is

$$\sum_{i=1}^{N} c_{i1} = \frac{\beta R}{1+\beta R^2} \sum_{i=1}^{N} \left[E_t y_{i2} + R y_{i1} + \frac{1-\beta R}{\beta R d} b\right]$$

$$\sum_{i=1}^{N} c_{i1} = \frac{\beta R}{1+\beta R^2}\left[E_t Y_2 + R Y_1 + \frac{1-\beta R}{\beta R d} bN\right]$$

For this economy to be in equilibrium $\sum_{i=1}^{N} c_{i1} = Y_1 \equiv \sum_{i=1}^{N} y_{i1}$

$$Y_1 = \frac{\beta R}{1+\beta R^2}\left[E_t Y_2 + R Y_1 + \frac{1-\beta R}{\beta R d} bN\right]$$

$$\left(1+\beta R^2\right) Y_1 = \beta R\left[E_t Y_2 + R Y_1 + \frac{1-\beta R}{\beta R d} bN\right]$$

$$Y_1 = \beta R\left[E_t Y_2 + \frac{1-\beta R}{\beta R d} bN\right]$$

$$\frac{Y_1}{\beta R} = E_t Y_2 + \frac{bN}{\beta R d} - \frac{bN}{d}$$

$$\frac{dY_1 - bN}{\beta R d} = E_t Y_2 - \frac{bN}{d}$$

$$\beta R d = \frac{dY_1 - bN}{E_t Y_2 - \frac{bN}{d}}$$

$$R = \frac{1}{\beta d}\left[\frac{dY_1 - bN}{E_t Y_2 - \frac{bN}{d}}\right]$$

Given that $E_t Y_2 = 0$

$$R = \frac{1}{\beta d}\left[\frac{dY_1 - bN}{-\frac{bN}{d}}\right]$$

$$R = \frac{1}{\beta}\left[1 - \frac{dY_1}{bN}\right]$$

Note that $\frac{dR}{dY_1} = -\frac{dY_1}{b} < 0$. The interest rate is countercyclical because when current aggregate endowment is high most workers in the economy are getting an endowment that is higher than their expected endowment next period. This creates incentives to save more, which reduces the interest rate in equilibrium.