January 2003 Macro Comp – Professor Shea's Question

(Sketch of the Solution)

The problem faced by the household is the following

$$\max_{\{C_t\}_{t=1}^3} \sum_{t=1}^{3} U(C_t) \quad \text{subject to} \tag{1}$$

$$C_1 + S_1 = y_1 + S_0 \tag{2}$$

$$C_2 + S_2 = y_2 + S_1 \tag{3}$$

$$C_3 + S_3 = y_3 + S_2 \tag{4}$$

Also $y_1 = y_3 = 1$, y_2 equals either 1-d or 1+d and $S_0 = 0$. Because utility is increasing in consumption we will also have that $S_3 = 0$.

Let's first solve the certainty equivalence case. In this case d=0 and then $y_1 = y_2 = y_3 = 1$. Given that by assumption $R = 1$, the intertemporal budget constraint can be written as

$$\sum_{t=1}^{3} C_t = \sum_{t=1}^{3} y_t = 3 \tag{5}$$

Write the Lagrangian

$$\max_{\{C_t\}_{t=1}^3} L = \sum_{t=1}^{3} U(C_t) + \lambda \left[3 - \sum_{t=1}^{3} C_t \right]$$

FOC

$$C_1 - \lambda = 0 \tag{6}$$

$$C_2 - \lambda = 0 \tag{7}$$

$$C_3 - \lambda = 0 \tag{8}$$

(6), (7) and (8) imply

$$C_1 = C_2 = C_3 \tag{9}$$

Replacing (9) into (5) we get that $C_1 = C_2 = C_3 = 1$. Therefore, $S_1 = S_2 = 0$.

Now, let's solve the case with uncertainty. To do that, we have to work backwards. In period 3, S_2 is given and the agent is going to consume $C_3 = 1 + S_2$. This means that value function for period 3 is

$$V_3(S_2) = U(1 + S_2) \tag{10}$$

Now, period 2 problem can be written as

$$V_2(S_1, y_2) = \max_{S_2}\{U(C_2) + V_3(S_2, 1)\} \tag{11}$$

Replacing (2.3) and (2.10) into (2.11) we get

$$V_2(S_1, y_2) = \max_{S_2}\{U(y_2 + S_1 - S_2) + U(1 + S_2)\} \tag{12}$$

FOC

$$-U'(y_2 + S_1 - S_2) + U'(1 + S_2) = 0$$

Rearranging and taking inverse

$$y_2 + S_1 - S_2 = 1 + S_2$$

$$S_2 = \frac{y_2 + S_1 - 1}{2} \tag{13}$$

The actual savings in period 2 will depend on the shock we get that period. If we get a positive shock, saving will be

$$S_2^{Pos} = \frac{1 + d + S_1 - 1}{2} = \frac{S_1 + d}{2} \tag{14}$$

Now, the value function in period 2 with a good shock

$$V_2^{Pos}(S_1, 1 + d) = U(y_2^{Pos} + S_1 - S_2^{Pos}) + U(1 + S_2^{Pos})$$
$$V_2^{Pos}(S_1, 1 + d) = U(1 + d + S_1 - \frac{S_1 + d}{2}) + U(1 + \frac{S_1 + d}{2})$$
$$V_2^{Pos}(S_1, 1 + d) = U(\frac{2 + S_1 + d}{2}) + U(\frac{2 + S_1 + d}{2})$$
$$V_2^{Pos}(S_1, 1 + d) = 2U(\frac{2 + S_1 + d}{2}) \tag{15}$$

Now, let's consider what would happen if the household gets a bad shock in the second period. Equation (13) implies that if $S_1 < d$, $S_2 < 0$ which violates the no-borrowing constraint. Then, if $S_1 < d$, S_2 should be 0. Note that in a setting with quadratic utility and $\beta = R = 1$, S_1 will always be less than d. If the household were to save more than d, it would be shifting too many resources to the future. That is, the marginal utility of consumption today will be less that the expected value of the marginal utility of consumption tomorrow which would violate the Euler Equation. To see this more clearly, let's take, for example, the case of $S_1 = d$. In this case, $U_1'(C_1) = A - C_1 = A - 1 + d$. After a bad shock,

$$U'_2{}^{Bad}(C_1) = A - (y_2 + S_1) = A - 1 + d - d = A - 1$$

After a good shock,

$$U'_2{}^{Good}(C_1) = A - 1 - d .$$

(note that I am writing $-d$ and not $-2d$ because after a good shock some resources will be optimally put away to increase period 3 consumption).

Therefore, for savings $S_1 \geq d$, $U_1'(C_1) > E_2 U_2'(C_2)$ imply that the Euler Equation is being violated, which can't be optimal. The household will always choose $S_1 < d$ which implies S_2 will optimally be set equal to 0 after a bad shock.

$$V_2^{Neg}(S_1, 1-d) = U(y_2^{Neg} + S_1 - S_2^{Neg}) + U(1 + S_2^{Neg})$$

$$V_2^{Neg}(S_1, 1-d) = U(1 - d + S_1) + U(1) \tag{16}$$

Then, the value function in period one is

$$V_1(1) = \max_{S_1}\left\{U(1-S_1) + \frac{1}{2}V_2^{Pos}(S_1, 1+d) + \frac{1}{2}V_2^{Neg}(S_1, 1-d)\right\} \tag{17}$$

Replacing (15) and (16) into (17) we get

$$V_1(1) = \max_{S_1}\left\{U(1-S_1) + U(\frac{2 + S_1 + d}{2}) + \frac{1}{2}U(1 - d + S_1) + \frac{1}{2}U(1)\right\}$$

FOC

$$-U'(1-S_1) + \frac{1}{2}U'(\frac{2 + S_1 + d}{2}) + \frac{1}{2}U'(1 - d + S_1) = 0 \tag{18}$$

Now, the agent's utility function is $U(C_t) = AC_t - \frac{1}{2}C_t^2$. Then, $U'(C_t) = A - C_t$. Replacing the marginal utility into (18) we get

$$-A + 1 - S_1 + \frac{1}{2}A - \frac{1}{2}(\frac{2 + S_1 + d}{2}) + \frac{1}{2}A - \frac{1}{2}(1 - d + S_1) = 0$$

$$-A + 1 - S_1 + \frac{1}{2}A - \frac{1}{2} - \frac{S_1}{4} - \frac{d}{4} + \frac{1}{2}A - \frac{1}{2} - \frac{S_1}{2} + \frac{d}{2} = 0$$

$$-S_1 - \frac{S_1}{4} - \frac{d}{4} - \frac{S_1}{2} + \frac{d}{2} = 0$$

Rearranging,

$$-\frac{7S_1}{4}+\frac{d}{4}=0$$

$$S_1=\frac{d}{7} \tag{19}$$

The precautionary savings is the difference between (19) and the savings in the certainty equivalence case. The later were 0 in period 1 and 2. Then

$$PS=\frac{d}{7}.$$

With borrowing constraints, the consumption history in the case of the positive and negative shocks will then be

$$\left\{1-\frac{d}{7},1+\frac{4d}{7},1+\frac{4d}{7}\right\} \text{ and } \left\{1-\frac{d}{7},1-\frac{6d}{7},1\right\} \text{ respectively.}$$

Note that after the shock is realized in the second period, there is no more uncertainty. Given the assumption of quadratic utility and $R=\beta=1$ the Euler equation without borrowing constraints implies that $C_2=C_3$. Looking at the histories above we observe that the Euler Equation will hold in the case of a positive shock and not in the case of a negative shock. This was expected given that borrowing constrains prevents households from borrowing but not from saving.

b)

The quick answer is that there won't be any precautionary savings in the case without borrowing constraints. There reason is that the agent has quadratic utility which implies that the third derivative of the utility function equals zero. The certainty equivalence property will then quick in and we will observe zero precautionary saving.

Let's check that. Without borrowing constraints, equation (13) implies

$$S_2^{Neg}=\frac{1-d+S_1-1}{2}=\frac{S_1-d}{2}$$

Then, the value function becomes

$$V_2^{Neg}(S_1,1-d)=U(y_2^{Neg}+S_1-S_2^{Neg})+U(1+S_2^{Neg})$$

$$V_2^{Neg}(S_1,1-d)=U(1-d+S_1-\frac{S_1-d}{2})+U(1+\frac{S_1-d}{2})$$

$$V_2^{Neg}(S_1, 1-d) = U(\frac{2+S_1-d}{2}) + U(\frac{2+S_1-d}{2})$$
$$V_2^{Neg}(S_1, 1-d) = 2U(\frac{2+S_1-d}{2}) \qquad (20)$$

Replacing (15) and (20) into (17) we get

$$V_1(1) = \max_{S_1}\left\{U(1-S_1) + U(\frac{2+S_1+d}{2}) + U(\frac{2+S_1-d}{2})\right\}$$

FOC

$$-U'(1-S_1) + \frac{1}{2}U'(\frac{2+S_1+d}{2}) + \frac{1}{2}U(\frac{2+S_1-d}{2}) = 0$$

Using the fact that $U'(C_t) = A - C_t$

$$-A + (1-S_1) + \frac{A}{2} - (\frac{2+S_1+d}{4}) + \frac{A}{2} - (\frac{2+S_1-d}{4}) = 0$$
$$-A + (1-S_1) + \frac{A}{2} - (\frac{2+S_1+d}{4}) + \frac{A}{2} - (\frac{2+S_1-d}{4}) = 0$$
$$(1-S_1) - (\frac{2+S_1+d}{4}) - (\frac{2+S_1-d}{4}) = 0$$
$$-S_1 - \frac{S_1}{4} - \frac{S_1}{4} = 0$$
$$S_1 = 0$$

As expected, we get the same solution as the certainty equivalence case, which implies no precautionary savings.

Note that in this case in the second period after a bad shock the household will borrow $S_2^{Neg} = \frac{-d}{2}$, and it will lend in the case of a good shock $S_2^{Pos} = \frac{d}{2}$ in order to smooth consumption. Then, with no borrowing constraints, the consumption history in this case of the negative and positive shock will be $\left\{1, 1-\frac{d}{2}, 1-\frac{d}{2}\right\}$ and $\left\{1, 1+\frac{d}{2}, 1+\frac{d}{2}\right\}$.

c)

Period three problem is going to be the same for both agents. For lucky households, the period 2 value function will be $V_2 = V_2^{Pos}(S_1, 1+d) = 2U(\frac{2+S_1+d}{2})$. Their period 1 value function will be given by

$$V_1(1) = \max_{S_1}\left\{U(1-S_1) + 2U(\frac{2+S_1+d}{2})\right\}$$

FOC

$$-U'(1-S_1) + U'(\frac{2+S_1+d}{2}) = 0$$

Using the fact that $U'(C_t) = A - C_t$

$$-A + 1 - S_1 + A - \frac{2+S_1+d}{2} = 0$$

$$-A + 1 - S_1 + A - \frac{2+S_1+d}{2} = 0$$

$$-A + 1 - S_1 + A - \frac{2+S_1+d}{2} = 0$$

$$S_1 = -\frac{d}{3} \qquad (21)$$

Replacing (21) into Equation (14) we get

$$S_2 = \frac{d}{3}$$

Then, the consumption history for the lucky household will be $\left\{1+\frac{d}{3}, 1+\frac{d}{3}, 1+\frac{d}{3}\right\}$.

For the unlucky households, the period 2 value function will be

$$V_2 = V_2^{Neg}(S_1, 1-d) = U(1-d+S_1) + U(1).$$

Their period 1 value function will be given by

$$V_1(1) = \max_{S_1}\{U(1-S_1) + U(1-d+S_1) + U(1)\}$$

FOC

$$-U'(1-S_1) + U'(1-d+S_1) = 0$$

Using the fact that $U'(C_t) = A - C_t$

$$-A + (1-S_1) + A - (1-d+S_1) = 0$$

$$-S_1 + d - S_1 = 0$$

$$S_1 = \frac{d}{2}$$

$$S_2 = 0$$

Then, the consumption history for the unlucky household will be $\left\{1-\frac{d}{2}, 1-\frac{d}{2}, 1\right\}$.

Then, the economy-wide savings in period one will be

$$S_1^{\ E} = \tfrac{1}{2} S_1^{\ Lucky} + \tfrac{1}{2} S_1^{\ Unlucky} = \tfrac{1}{2}(-\frac{d}{3}) + \tfrac{1}{2}(\frac{d}{2}) = \frac{d}{12}$$

d)

The economy-wide savings:

- Homogeneous agents & borrowing constraints: $S_1^{\ E} = \frac{d}{7}$
- Homogeneous agents & no borrowing constraints: $S_1^{\ E} = 0$
- Heterogeneous agents & borrowing constraints in the second period: $S_1^{\ E} = \frac{d}{12}$