Solution to Shea's question in January 2005 Comp

PART A

State variables: Endogenous: D_{t-1}, A_t

Exogenous: y_t

Control variables: C_t, F_t, D_t, X_t

The problem faced by the household is given by

$$\max_{C_t, F_t, X_t} E_t \sum_t \beta^t U(C_t, S_t)$$

subject to $S_t = Q(D_t , X_t)$

$D_t = (1-\delta)D_{t-1} + F_t$

$A_{t+1} = (1+r)(A_t+y_t-C_t-pX_t-qF_t)$

Then,

Bellman's equation

$$V(A_t, D_{t-1}) = \max_{C_t,F_t,X_t} \{U[C_t ; Q ((1-\delta)D_{t-1}+ F_t , X_t)]+ \ldots$$
$$\ldots + \beta E_t V [(1+r)(A_t+y_t-C_t-pX_t-qF_t) ; (1-\delta)D_{t-1}+F_t]\}$$

First Order Conditions

$$U_{Ct} (C_t, S_t) - \beta(1+r) E_t V_{At+1}(A_{t+1} , D_t) = 0 \quad (1)$$

$$U_{St} (C_t, S_t) Q_{Dt}(D_t , X_t) - \beta(1+r)q E_t V_{At+1}(A_{t+1} , D_t) + \beta E_t V_{Dt} (A_{t+1} , D_t) = 0 \quad (2)$$

$$U_{St} (C_t, S_t) Q_{Xt}(D_t , X_t) - \beta(1+r)p E_t V_{At+1}(A_{t+1} , D_t) = 0 \quad (3)$$

Envelope Conditions

$$V_{At}(A_t , D_{t-1}) = \beta(1+r) E_t V_{At+1}(A_{t+1} ;D_t) \quad (4)$$

$$V_{Dt-1}(A_t , D_{t-1}) = (1-\delta)U_{St} (C_t, S_t) Q_{Dt}(D_t , X_t) + \beta(1-\delta) E_t V_{Dt} (A_{t+1} , D_t) \quad (5)$$

PART B

From (2) we have that

$$\beta\, E_t\, V_{Dt}(A_{t+1}, D_t) = \beta(1+r)q\, E_t\, V_{At+1}(A_{t+1}, D_t) - U_{St}(C_t, S_t)\, Q_{Dt}(D_t, X_t) \qquad (6)$$

replacing (6) in (5), we get

$$V_{Dt-1}(A_t; D_{t-1}) = (1-\delta)U_{St}(C_t, S_t)\, Q_{Dt}(D_t, X_t) + \beta\,(1-\delta)\,(1+r)q\, E_t\, V_{At+1}(A_{t+1}, D_t) - (1-\delta)\, U_{St}(C_t, S_t)\, Q_{Dt}(D_t, X_t)$$

the first and the last term of the right hand side cancel out. Therefore we get that

$$V_{Dt-1}(A_t; D_{t-1}) = \beta\,(1-\delta)\,(1+r)q\, E_t\, V_{At+1}(A_{t+1}, D_t) \qquad (7)$$

Replacing (4) in (7), we obtain that

$$V_{Dt-1}(A_t; D_{t-1}) = (1-\delta)\, q\, V_{At}(A_t, D_{t-1}) \qquad (8)$$

PART C

From (1) and (4) we get that

$$U_{Ct}(C_t, S_t) = V_{At}(A_t; D_{t-1}) \qquad (9)$$

Forwarding (9) one period

$$U_{Ct+1}(C_{t+1}, S_{t+1}) = V_{At+1}(A_{t+1}, D_t) \qquad (10)$$

Replacing (10) in (1)

$$U_{Ct}(C_t, S_t) = \beta(1+r)\, E_t\, U_{Ct+1}(C_{t+1}, S_{t+1}) \qquad (11)$$

For the functional forms given

$U_{Ct}(C_t, S_t) = 1/C_t$

Then, (11) becomes

$$1/C_t = \beta(1+r)\, E_t\{1/C_{t+1}\} \qquad (12)$$

This is the Euler Equation describing the evolution of nondurable consumption over time.

From equation (8) and (9), we get that

$$V_{Dt-1}(A_t, D_{t-1}) = (1-\delta)\, q\, U_{Ct}(C_t, S_t) \qquad (13)$$

Replacing (1) and (13) forwarded one period in (2),

$$U_{St}(C_t, S_t)\, Q_{Dt}(D_t, X_t) - q\, U_{Ct}(C_t, S_t) + \beta(1-\delta)q\, E_t\, U_{Ct+1}(C_{t+1}, S_{t+1}) = 0 \qquad (14)$$

Replacing equation (11) in (14)

$$(1+r)U_{St}(C_t, S_t)\, Q_{Dt}(D_t, X_t) - (1+r)q\, U_{Ct}(C_t, S_t) + (1-\delta)q\, U_{Ct}(C_t, S_t) = 0 \qquad (15)$$

which implies

$$(1+r)U_{St}(C_t, S_t)\, Q_{Dt}(D_t, X_t) = (r+\delta)q\, U_{Ct}(C_t, S_t) \qquad (16)$$

For the functional forms given

$U_{St}(C_t, S_t) = \gamma/S_t$

$Q_{Dt}(D_t, X_t) = \alpha (X_t/D_t)^{(1-\alpha)}$

$U_{Ct}(C_t, S_t) = 1/C_t$

Then, (16) becomes

$$(1+r)\,\gamma\,\alpha\,(X_t/D_t)^{(1-\alpha)} = (r+\delta)q\, S_t/C_t \qquad (17)$$

remember that $S_t = Q(D_t, X_t) = D_t^{\alpha} X_t^{1-\alpha}$. Replacing S_t in (17)

$$(1+r)\,\gamma\,\alpha\,X_t^{1-\alpha} D_t^{-(1-\alpha)} = (r+\delta)q\, D_t^{\alpha} X_t^{1-\alpha}/C_t \qquad (18)$$

simplifying we get

$$D_t = (1+r)\,\gamma\,\alpha\,C_t / (r+\delta)q \qquad (19)$$

This equation relates today's optimal nondurable consumption to today's optimal durable stock.

Equation (1) and (3) imply

$$U_{St}(C_t, S_t)\, Q_{Xt}(D_t, X_t) = p\, U_{Ct}(C_t, S_t) \qquad (20)$$

For the functional forms given

$U_{St}(C_t, S_t) = \gamma/S_t$

$Q_{Xt}(D_t, X_t) = (1-\alpha)(D_t/X_t)^{\alpha}$

$U_{Ct}(C_t, S_t) = 1/C_t$

Then, (20) becomes

$$\gamma\,(1-\alpha)\,(D_t/X_t)^{\alpha} = p\, S_t/C_t \qquad (21)$$

remember that $S_t = Q(D_t, X_t) = D_t^{\alpha} X_t^{1-\alpha}$. Replacing S_t in (14)

$$\gamma\,(1-\alpha)\,D_t^{\alpha} X_t^{-\alpha} = p\, D_t^{\alpha} X_t^{1} X_t^{-\alpha}/C_t$$

simplifying we get

$$X_t = \gamma\,(1-\alpha)\,C_t / p \qquad (22)$$

This equation relates today's optimal nondurable consumption to today's optimal purchase of the complementary input.

Impacts holding nondurable consumption fixed,

$\partial D_t/\partial r = -(1-\delta)\,\gamma\,\alpha\,qC_t / [(r+\delta)q]^2 \qquad \partial D_t/\partial q = -(1+r)\,\gamma\,\alpha\,C_t / (r+\delta)q^2 \qquad \partial D_t/\partial p = 0$

$\partial X_t/\partial r = \partial X_t/\partial q = 0 \qquad \partial X_t/\partial p = -\gamma\,(1-\alpha)\,C_t / p^2$

Comment: Equation (20) can be rewritten as

$$MRS_{C_tX_t} = U_{S_t} (C_t, S_t)\, Q_{X_t}(D_t , X_t) / U_{C_t} (C_t, S_t) = p$$

Note that this is the usual intratemporal maximization condition for nondurable goods stating that the marginal rate of substitution must equal relative prices (remember that the price of the nondurable good is the numeraire).

Equation (16) also gives us an intratemporal condition but this one doesn't look like the usual one because D is a durable good.

So, in this problem the agent is maximizing along two margins. He is deciding how much to consume today and tomorrow. This is governed by the Euler Equation (11). He is also deciding how much of each good to consume in each period. This is governed by the intratemporal conditions (20) and (16).