Solution to Drazen's January 2006 Comp Question
(Sketch of Solutions)

Part a

$$\underset{\{c_t,k_{t+1},l_t\}_{t=0}^{\infty}}{Max} \beta^t \left[\frac{c_t^{1-a}}{1-a} - b\frac{l_t^{1+v}}{1+v}\right] \tag{1}$$

subject to

$$k_{t+1} = w_t l_t + (1+r_t)k_t - c_t \tag{2}$$

The Bellman equation for the problem above is given by

$$V(k_t) = \underset{k_{t+1},l_t}{Max} \left[\frac{(w_t l_t + (1+r_t)k_t - k_{t+1})^{1-a}}{1-a} - b\frac{l_t^{1+v}}{1+v} + \beta V(k_{t+1})\right] \tag{3}$$

Taking FOC we get

$k_{t+1}\rfloor$

$$-(w_t l_t + (1+r_t)k_t - k_{t+1})^{-a} + \beta V'(k_{t+1}) = 0 \tag{4}$$

$l_t\rfloor$

$$w_t(w_t l_t + (1+r_t)k_t - k_{t+1})^{-a} - b l_t^v = 0 \tag{5}$$

Envelope Condition

$$V'(k_t) = (1+r_t)(w_t l_t + (1+r_t)k_t - k_{t+1})^{-a} \tag{6}$$

Combining eq. (4) and eq. (15) updated one period we get

$$(w_t l_t + (1+r_t)k_t - k_{t+1})^{-a} = \beta(1+r_{t+1})(w_{t+1}l_{t+1} + (1+r_{t+1})k_{t+1} - k_{t+2})^{-a}$$

The equation above can be rewritten as

$$(c_t)^{-a} = \beta(1+r_{t+1})(c_{t+1})^{-a} \tag{7}$$

Eq. (5) can also be written as

$$w_t(c_t)^{-a} = bl_t^v \tag{8}$$

Or

$$l_t = \left[\frac{w_t(c_t)^{-a}}{b}\right]^{1/v} \tag{9}$$

Under one of three conditions l_t will be constant through time. Looking at eq (9) one can notice that l_t will be constant if either v is ∞ and/or b is ∞. These cases don't seem very realistic. When $v = \infty$ we have a perfectly inelastic labor supply curve at one. When $b = \infty$ we have a perfectly inelastic labor supply curve at zero. To get the final condition take logs of eq (9)

$\ln l_t = \frac{1}{v}\left[\ln w_t - a\ln c_t - \ln b\right]$

Taking time derivatives we get

$$\frac{d\ln l_t}{dt} \equiv \hat{l}_t = \frac{1}{v}\left[\hat{w}_t - a\hat{c}_t\right] \tag{10}$$

Given that we know that in the steady state rate of growth of real wages is the same as the rate of growth of per capita comsumption, $\hat{l}_t = 0$ only if $a = 1$.

Part b

If a=1, equation (9) becomes

$$l_t = \left[\frac{w_t}{c_t b}\right]^{1/v}$$

For the labor supply to be procyclical, changes in real wages have to be bigger in absolute value than changes in consumption. That is, real wages have to be more volatile than consumption. This condition is likely to hold given that agent's incentive to smooth consumption.

Part c

$$\underset{\{c_t,k_{t+1},l_t\}_{t=0}^{\infty}}{Max} \frac{\beta^t}{1-s}\left\{\left[\frac{(\frac{c_t}{X_t})^{1-a}}{1-a} - b\frac{l_t^{1+v}}{1+v}\right]\right\}^{1-s} \tag{11}$$

subject to eq. (2).

The Bellman equation for the problem above is given by

$$V(k_t) = \underset{c_t,l_t}{Max}\left[(1-s)^{-1}\left\{\left[\frac{(\frac{c_t}{X_t})^{1-a}}{1-a} - b\frac{l_t^{1+v}}{1+v}\right]\right\}^{1-s} + \beta V(w_t l_t + (1+r_t)k_t - c_t)\right] \tag{12}$$

Taking FOC we get

$c_t\rfloor$

$$\left\{\left[\frac{(\frac{c_t}{X_t})^{1-a}}{1-a} - b\frac{l_t^{1+v}}{1+v}\right]\right\}^{-s}(\frac{c_t}{X_t})^{-a} - \beta V'(k_{t+1}) = 0 \tag{13}$$

$l_t\rfloor$

$$-\left\{\left[\frac{(\frac{c_t}{X_t})^{1-a}}{1-a} - b\frac{l_t^{1+v}}{1+v}\right]\right\}^{-s} b l_t^v + \beta w_t V'(k_{t+1}) = 0 \tag{14}$$

Envelope Condition

$$V'(k_t) = \beta(1+r_t)V'(k_{t+1}) \tag{15}$$

Combining eq. (13) and eq. (15) updated one period we get

$$V'(k_t) = (1+r_t)\left\{\left[\frac{(\frac{c_t}{X_t})^{1-a}}{1-a} - b\frac{l_t^{1+v}}{1+v}\right]\right\}^{-s}(\frac{c_t}{X_t})^{-a} \tag{16}$$

Replacing eq. (16) into eq. (13) we get

$$\left\{\left[\frac{(\frac{c_t}{X_t})^{1-a}}{1-a} - b\frac{l_t^{1+v}}{1+v}\right]\right\}^{-s}(\frac{c_t}{X_t})^{-a} = \beta(1+r_{t+1})\left\{\left[\frac{(\frac{c_{t+1}}{X_{t+1}})^{1-a}}{1-a} - b\frac{l_{t+1}^{1+v}}{1+v}\right]\right\}^{-s}(\frac{c_{t+1}}{X_{t+1}})^{-a} \tag{17}$$

Or more simply

$$\frac{\partial U(c_t, l_t)}{\partial c_t} = \beta(1 + r_{t+1})\frac{\partial U(c_{t+1}, l_{t+1})}{\partial c_{t+1}}$$

To obtain the intratemporal labor supply, combine equations (13) and (14) to get

$$\left\{ \left[\frac{(\frac{c_t}{X_t})^{1-a}}{1-a} - b\frac{l_t^{1+v}}{1+v} \right] \right\}^{-s} bl_t^v = w_t \left\{ \left[\frac{(\frac{c_t}{X_t})^{1-a}}{1-a} - b\frac{l_t^{1+v}}{1+v} \right] \right\}^{-s} (\frac{c_t}{X_t})^{-a}$$

$$bl_t^v = w_t(\frac{c_t}{X_t})^{-a} \tag{18}$$

Or

$$l_t = \left[\frac{w_t(\frac{c_t}{X_t})^{-a}}{b} \right]^{1/v} \tag{19}$$

Under one of two conditions l_t will be constant through time. Looking at eq (19) we have again that when either $v = \infty$ and/or $b = \infty$, l_t will be constant. To analyze all other possible case take logs to both side of eq (9)

$\ln l_t = \frac{1}{v} \left[\ln w_t - a \ln c_t + a \ln X_t - \ln b \right]$

Taking time derivatives we get

$$\frac{d \ln l_t}{dt} \equiv \hat{l}_t = \frac{1}{v} \left[\hat{w}_t - a\hat{c}_t + a\hat{X}_t \right] \tag{20}$$

If c_t and X_t grow at the same rate $\hat{l}_t = \frac{1}{v}\hat{w}_t$. Then, if real wages grow so will labor supply (assuming v is finite). For $\hat{l}_t = 0$ we need $\hat{w}_t = a(\widehat{\frac{c_t}{X_t}})$. If, as in the previous case $a = 1$, the condition needed for the labor supply not the grow is that the real wage and the relative consumption $(\frac{c_t}{X_t})$ grow at the same rate.

If a=1, equation (19) becomes

$$l_t = \left[\frac{w_t}{(\frac{c_t}{X_t})b} \right]^{1/v}$$

For the labor supply to be procyclical, changes in real wages have to be bigger in absolute value than changes in relative consumption. That is, real wages have to be more volatile than relative consumption.

<u>Part d</u>

In the first case, if changes in real wages are the same as changes in consumption then eq. (10) implies that $\hat{l}_t = 0$ which is at odds with the empirical observation.

In the second case analyzed here, assuming again that $a = 1$, $\hat{l}_t = \frac{1}{v}\left[\hat{w}_t - (\widehat{\frac{c_t}{X_t}})\right]$. $\hat{l}_t > 0$ only if $\hat{w}_t > (\widehat{\frac{c_t}{X_t}})$. In the case that $\hat{w}_t = \hat{c}_t$, $\hat{l}_t > 0$ only if $\hat{X}_t > 0$.