Solution to Shea's January 2007 Comp Question
(Sketch of Solutions)

Part a

$$\underset{R,\{c_t\}_{t=0}^T}{Max}\left\{\sum_{t=0}^{T}\left[\frac{c_t^{1-\sigma}}{1-\sigma}\right]+\sum_{t=0}^{R-1}\log(1-L)+\sum_{t=R}^{T}\log(1)\right\} \tag{1}$$

subject to

$$\sum_{t=0}^{T} c_t \leq \sum_{t=0}^{R-1} wL$$

or

$$\sum_{t=0}^{T} c_t \leq RwL + A_0 \tag{2}$$

The Lagrangian for the problem above is given by

$$L=\sum_{t=0}^{T}\left[\frac{c_t^{1-\sigma}}{1-\sigma}\right]+R\log(1-L)+\lambda\left[A_0+RwL-\sum_{t=0}^{T}c_t\right] \tag{3}$$

Taking FOC with respect to c_t

$c_t\rfloor$

$$c_t^{-\sigma}-\lambda=0 \tag{4}$$

Forwarding the equation above one period and equating the λ we get the Euler Equation

$$c_t^{-\sigma}=c_{t+1}^{-\sigma} \tag{5}$$

which implies that $c_t=c_{t+1}=c$ for all $t=0,1,..,T$

Replacing into the budget constraint and noting that at and optimum the budget constraint holds as equality

$$\sum_{t=0}^{T} c = A_0 + RwL$$

$$(T+1)c = A_0 + RwL$$

$$c=\frac{A_0+RwL}{(T+1)} \tag{6}$$

Replacing (6) into the lifetime utility function (1)

$$V(T) = \frac{(T+1)}{1-\sigma}\left(\frac{A_0 + RwL}{T+1}\right)^{1-\sigma} + R\log(1-L)$$

To obtain to optimal retirement age we need to maximize $V(T)$ with respect to R

<u>FOC</u>

$R\rfloor$

$$(T+1)\frac{wL}{T+1}\left(\frac{A_0 + RwL}{T+1}\right)^{-\sigma} + \log(1-L) = 0 \tag{7}$$

$$R = -\frac{1}{wL}\left[(T+1)\left(\frac{wL}{\log(1-L)}\right)^{1/\sigma} + A_0\right]$$

<u>Part b</u>

If we set $A_0 = 0$ the optimal retirement age becomes

$$R = -\left(T+1\right)\left(wL\right)^{\frac{1-\sigma}{\sigma}}\left[\log(1-L)\right]^{-1/\sigma}$$

If T is increasing, for R to go down we need $(wL)^{\frac{1-\sigma}{\sigma}}$ to go down. Given that w is increasing , we need $\frac{1-\sigma}{\sigma}$ to be negative. Then, $\sigma > 1$.