Solutions to Shea's January 2008 Comp Question

Households maximize $E_0 \sum_{t=0}^{\infty} \beta^t \log \left(c_t - \frac{l_t^2}{2} \right)$ suject to $c_t = (1-\tau_t) w_t l_t + g_t$

PART A

$\max_{l_t} \log \left[(1-\tau_t) w_t l_t + g_t - \frac{l_t^2}{2} \right]$

FOC

$\frac{1}{(1-\tau_t) w_t l_t + g_t - \frac{l_t^2}{2}} \left[(1-\tau_t) w_t - l_t \right] = 0$

$$l_t = (1-\tau_t) w_t$$

Replacing l_t into the equation for c_t

$$c_t = (1-\tau_t) w_t \left[(1-\tau_t) w_t \right] + g_t$$

$$c_t = (1-\tau_t)^2 w_t^2 + g_t$$

Replacing the optimal choices into the utility function
$V = \log \left((1-\tau_t)^2 w_t^2 + g_t - \frac{[(1-\tau_t) w_t]^2}{2} \right)$
$V = \log \left(\frac{1}{2} (1-\tau_t)^2 w_t^2 + g_t \right)$

The lifetime indirect utility function can then be written as
$V \equiv E_0 \sum_{t=0}^{\infty} \beta^t \log \left(\frac{1}{2} (1-\tau_t)^2 w_t^2 + g_t \right)$

Part B

State variable: b_t
Control variables: τ_t, g_t, b_{t+1}

$b_{t+1} = (1+r)(b_t + \tau_t w_t l_t - g_t)$

Replacing optimal l_t into the transition equation for government assets we get

$b_{t+1} = (1+r)(b_t + \tau_t w_t \left[(1-\tau_t)w_t\right] - g_t)$
$b_{t+1} = (1+r)(b_t + \tau_t w_t^2 - \tau_t^2 w_t^2 - g_t)$

<u>Bellman Equation</u>

$$V(b_t) = \max_{\tau_t, g_t} \left\{ \log\left(\frac{1}{2}(1-\tau_t)^2 w_t^2 + g_t\right) + \beta V\left[(1+r)\left((1+r)(b_t + \tau_t w_t^2 - \tau_t^2 w_t^2 - g_t)\right)\right] + \mu g_t \right\}$$

<u>FOC</u>

$\tau_t \rfloor$

$$\frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2 + g_t}\left[-(1-\tau_t)w_t^2\right] + \beta V'(b_{t+1})(1+r)\left[w_t^2 - 2\tau_t w_t^2\right] = 0$$

$$\frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2 + g_t}(1-\tau_t) = \beta V'(b_{t+1})(1+r)\left[1-2\tau_t\right]$$

$$\beta V'(b_{t+1})(1+r) = \frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2 + g_t}\frac{1-\tau_t}{1-2\tau_t} \tag{1}$$

$g_t \rfloor$

$$\frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2 + g_t} + \beta V'(b_{t+1})(1+r)\left[-1\right] + \mu = 0$$

$$\beta V'(b_{t+1})(1+r) = \frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2 + g_t} + \mu \tag{2}$$

Combining equation (1) and (2)

$$\frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2 + g_t}\frac{1-\tau_t}{1-2\tau_t} = \frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2 + g_t} + \mu$$

$$\mu = \left[\frac{1-\tau_t}{1-2\tau_t} - 1\right]\frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2 + g_t}$$

$$\mu = \left[\frac{\tau_t}{1-2\tau_t}\right]\frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2 + g_t} \tag{3}$$

<u>Envelope</u>

$V'(b_t) = \beta V'(b_{t+1})(1+r)$

Combining the equation above with equation (1)

$$V'(b_t) = \frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2 + g_t}\frac{1-\tau_t}{1-2\tau_t} \tag{4}$$

Forwarding (4) one period and replacing it into (1)

$$\frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2 + g_t}\frac{1-\tau_t}{1-2\tau_t} = \beta(1+r)\frac{1}{\frac{1}{2}(1-\tau_{t+1})^2 w_{t+1}^2 + g_{t+1}}\frac{1-\tau_{t+1}}{1-2\tau_{t+1}}$$

<u>Part C</u>

i)

If $g_t > 0$, then $\mu = \left[\frac{\tau_t}{1-2\tau_t}\right]\frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2 + g_t} = 0$. This implies that $\tau_t = 0$

ii)

If $g_t = 0$, then $\mu \geq 0$. If $\mu = 0$, then $\tau_t = 0$

If $\mu > 0$, then $\left[\frac{\tau_t}{1-2\tau_t}\right]\frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2} > 0$. Given that $\frac{1}{\frac{1}{2}(1-\tau_t)^2 w_t^2} > 0$ then $\frac{\tau_t}{1-2\tau_t} > 0$. This condition is only satisfied if $0 < \tau_t < 1/2$.