Macro Comp Long Question 1

Consider the following problem, set in continuous time. Households have infinite horizons and the following objective function:

$$\text{Max } E_0 \int_0^{\infty} \exp(-rt)U(c_t)dt$$

where c is consumption and r is a time discount factor. In periods where consumption does not adjust, the household faces the usual continuous-time dynamic budget constraint:

$$da_t/dt = ra_t + y_t - c_t$$

where a_t is financial wealth at t after the payment of any discrete adjustment costs (defined below), r is the flow interest rate (set equal to the discount factor for simplicity), and y_t is flow labor income. In the absence of adjustment costs, the dynamic budget constraint implies that for any two dates t and t+T (where T > 0),

$$a_{t+T} = \exp(rT)a_t + \int_0^T \exp(r(T-i))(y_{t+i} - c_{t+i})\, di$$

Assume that labor income at any instant is given by

$$y_t = y + e_t$$

where e is normally distributed with mean zero and variance σ^2 and is i.i.d. over time. In addition to the dynamic budget constraint, households face the usual no-ponzi game condition on wealth, which plays no role in the analysis.

Unlike a standard problem, households face a discrete cost to change c. The adjustment technology is as follows: if the household adjusts at some date t, it pays a fixed cost K and chooses a level of consumption c* and a time interval T. The household commits to keep consumption equal to c* for the next T periods, and commits to adjust again in period t+T. In period t + T, the household must again pay K and is allowed to choose a possibly new c* and T.

(a) Show that if the household adjusts in period t, then financial wealth in period t + T is (conditional on the household's choices and information at t) a normally distributed random variable satisfying

(1.1) $$E_t(a_{t+T}) = -K + a_t\exp(rT) + (y-c^*)(\exp(rT)-1)/r$$

(1.2) $$Var_t(a_{t+T}) = (\exp(2rT)-1)^*(\sigma^2/r)$$

[NOTE: recall that a is measured AFTER the payment of any adjustment costs]

Let $V(a_t)$ be the household's value function at date t, assuming that the household adjusts at t. The Bellman's equation for the problem can be written as

$$V(a_t) = \underset{(c^*,T)}{\text{Max}} \quad \int_0^T \exp(-ri)U(c^*)di \ + \exp(-rT)E_tV(a_{t+T})$$

where the value function V depends only on a_t because y is i.i.d. and because the horizon is infinite.

(b) Write down the first order and envelope conditions for this Bellman's Equation.

Now assume that $U(c) = -(1/\gamma)\exp(-\gamma c)$. A reasonable (and true) conjecture that the value function is of the form $V(a_t) = -A\exp(-Ba_t)$, where A and B are functions of underlying parameters to be determined.

(c) Given this conjecture on V, use (1.1), (1.2) and the envelope condition to show:

(3.1) $\quad c^* = y + ra_t - (\exp(rT)+1)*(B\sigma^2/4) \ - (rK/(\exp(rT)-1))$

HINT: recall that if X is normal, $E(\exp(X)) = \exp[E(X) + (Var(X)/2)]$

(d) Show that $B = r\gamma$, combining the first order and envelope conditions from part (b) and using (3.1). You do not need to solve for A.

(e) Establish the following closed-form solution for optimal T:

$$T^* = (1/r) \ln(1 + \text{sqrt}[4K/\gamma\sigma^2])$$

How do changes in K, γ and σ^2 affect optimal T? (No math needed, just tell me the signs of the derivative). Provide intuition for these results.

(f) Write down a closed-form solution for optimal c*.