1. **Financial Markets and Consumption Smoothing/Completeness of Markets.** Consider a two-period model (the two periods are period zero and period one), in which an individual consumer lives for the entire two periods. All analysis in this problem is to be conducted from the perspective of the beginning of period zero, meaning that the entire two periods exist in the consumer utility maximization horizon. (Note: you will consider only standard expected utility in this problem.)

The consumer’s endowment income in period zero, denoted by \( y_0 > 0 \), is known. There is risk regarding period-one income. There are \( J \) possible realizations of \( y_1^j > 0 \), with each realization strictly positive. The probability of income state \( j \) occurring is \( p^j \). The expected value of period-one income is \( E[y_1] \).

There is also risk regarding the private-market real interest rate between period zero and period one on the single asset that exists in financial markets. The consumer knows in period zero all statistical moments of the risky private-market real interest rate, including the expectation \( E[r_{MBKT}] \) and the standard deviation \( \sigma^2 \). These moments are conditional on all information known in period zero or chosen in period zero.

There are \( M \) possible realizations of the private-market real interest rate, and the probability of real interest rate \( i \) occurring is \( p^i \). There is independence in the realizations of the real interest rate and the period-one income level. For now, we have no idea how the numerical value of \( M \) compares to the numerical value of \( J \).

The consumer’s asset position at the beginning of period zero is zero, and, as usual, \( c_0 \) denotes consumption during period zero and \( c_{1}^{i,j} \) denotes consumption in state \((i, j)\) of period one. The expectation of period-one consumption is thus denoted \( E(c_1) \).
a. State as simply/briefly as possible the end-of-planning horizon condition(s) required for the consumer problem to be well-defined. Include a brief explanation.

Solution: For every possible state \((i, j)\) in period one, the asset position \(a_{i,j}^1\) must be equal to zero. Otherwise the consumer is either not consuming enough during his lifetime (if \(a_{i,j}^1 > 0\)), or he is running a Ponzi scheme (if \(a_{i,j}^1 < 0\)).

b. Using the notion of the end-of-planning horizon condition(s) from part a, construct the period-zero budget constraint(s), the period-one budget constraint(s), and, based on them, one single lifetime budget constraint (LBC). Define any new notation you introduce, and clearly explain/show any non-trivial steps.

Solution: The period-zero budget constraint is \(c_0 + a_0 = y_0\) because there is no risk regarding outcomes during period zero and initial assets \(a_{-1}\) are zero. The period-one budget constraint in state \((i, j)\) is \(c_{i,j}^1 = y_i^1 + (1 + r^i) a_0\). Solving this for \(a_0\), \(a_0 = \frac{c_{i,j}^1 - y_i^1}{1 + r^i}\) and inserting this in the period-zero budget constraint, \(c_0 + \frac{c_{i,j}^1}{1 + r^i} = y_0 + \frac{y_i^1}{1 + r^i}\) if state \((i, j)\) occurs. Multiplying by the probability \(p_i p_j\) (which is permitted because the realization of the real interest rate and of real income are independent of each other) and summing over all states,

\[
\sum_{i=1}^{M} \sum_{j=1}^{J} p^i p^j c_{i,j}^1 = \sum_{i=1}^{M} p^i (1 + r^i) \frac{y_i^1}{1 + r^i},
\]

which is the present-value expectational budget constraint. To write things in terms of mathematical expectations and make clear that the summations are computed with respect to different probabilities, replace the summation over states indexed by \(i\) (the real interest rate) with the expectation operator \(E^i(.)\) and the summation over states indexed by \(j\) (period-one income) with the expectation operator \(E^j(.)\) to arrive at

\[
c_0 + \frac{E^i E^j c_{i,j}}{1 + E^i \left[ r^\text{MRKT} \right]} = y_0 + \frac{E^j y_i^1}{1 + E^i \left[ r^\text{MRKT} \right]},
\]

in which, for the remainder of the analysis, we replace \(E^i E^j c_{i,j}\) with \(Ec_1\).

c. Sketch qualitatively (but accurately) in a diagram with \(c_0\) on the horizontal axis and \(Ec_1\) on the vertical axis the single LBC for the consumer (note that it is the expected value of
consumption in period one that appears on the vertical axis). In the sketch, clearly mark the following: i) the slope of the LBC; ii) the point $y_0$; iii) the expected value of $y_1$.

**Solution:** Solving the present-value budget constraint above for $E c_1$,

$$E c_1 = - \left(1 + E \left[ r^{\text{MRKT}} \right] \right) c_0 + \frac{y_0}{1 + E \left[ r^{\text{MRKT}} \right]} + E y_1 .$$

Plotting this downward-sloping straight line, the slope is $- \left(1 + E \left[ r^{\text{MRKT}} \right] \right)$, and the points $y_0$ and $E y_1$ lie on the budget line.

d. Set up a single lifetime Lagrangian (you may choose either a sequential analysis or a lifetime analysis, but make it clear which approach you have chosen) that fully describes the consumer’s utility maximization problem. Define any new notation you introduce, and clearly explain/show any non-trivial steps.

**Solution:** Choosing a sequential analysis (and assuming the lifetime utility function is additively separable over time), the lifetime Lagrangian is

$$u(c_0) + Eu(c_1) + \lambda_0 \left[ y_0 - c_0 - a_0 \right] + E \left\{ \lambda_1 \left[ y_1 + (1 + r) a_0 - c_1 \right] \right\} ,$$

with Lagrange multipliers $\lambda_0$ and $\lambda_1$ on the period-zero and period-one budget constraints (with the latter technically being a series of state-by-state budget constraints and hence $\lambda_1$ being state-by-state multipliers $\lambda^{i,j}_{i} )$.

e. Based on the lifetime Lagrangian above, compute first-order conditions (FOCs) with respect to $c_0$ and $E c_1$ (note the latter argument), and construct the Euler equation between $c_0$ and $E c_1$ (aka, the consumption-savings optimality condition). Define any new notation you introduce, and clearly explain/show any non-trivial steps.

**Solution:** The first-order condition on $c_0$ is $u'(c_0) - \lambda_0 = 0$. The first-order condition on the period-one state $(i, j)$ level of consumption is $p' i' u'(c^{i,j}_1) - p' i' \lambda^{i,j}_{i} c^{i,j}_1 = 0$, in which the probabilities clearly can be canceled so that we can write $u'(c^{i,j}_1) - \lambda^{i,j}_{i} = 0$. Having adopted the sequential (rather than the recursive) formulation, we also need to compute the first-order condition with respect to $a_0$, which is $-\lambda_0 + E \left[ \lambda_1 (1 + r) \right] = 0$. Important to note here is that because there is only the single asset $a_0$, this condition holds for all $(i, j)$ states – that is, $\lambda_0 = \lambda^{i,j}_{i} (1 + r^i)$ for all $(i, j)$. In turn, this clearly means that $\lambda^{i,j}_{i} (1 + r^i) = \lambda^{i',j}_{i'} (1 + r^{i'})$ for all $i \neq i'$, $j \neq j'$, an observation that will be important for the parts of the problem below.

Constructing the Euler equation, we have $u'(c_0) = E \left[ u'(c_1) (1 + r) \right]$, which cannot be broken apart and written in a simpler form.
Now consider the number of possible realizations of interest rates \((M)\), the number of possible realizations of period-one income \((J)\), and the issue of whether or not markets are complete. Define \textbf{completeness of markets} in the following way: if a consumer is optimally able to equate marginal utility across all possible states, then markets are \textbf{complete}. If this is not true, then markets are \textbf{incomplete}.

An important observation was implied by the solution to part e above: given that there is only a single asset, it will be the case that equating marginal utilities across periods/states can only occur in special circumstances. More broadly, equalization of realized marginal utilities is possible only if there is an Arrow-Debreu set of state-contingent securities; here, this is generically not true.

f. Suppose the consumer utility function is \(U(c_0, c_1) = \alpha_0 c_0 + \alpha_1 E[c_1]\), with \(\alpha_0\) and \(\alpha_1\) known constants. Does completeness of markets require \(M = J\)? Or is it impossible to determine? Show/argue based on the analysis above, and/or construct new/related analysis if needed.

\textbf{Solution: } The marginal utility of period-zero consumption is \(\alpha_0\), and the marginal utility of period-one consumption is \(\alpha_1\), the latter in any state of period one. Given these marginal utility functions, it is a measure-zero event for them to be equated \textbf{no matter the nature of financial markets}.

g. Suppose the consumer utility function is \(U(c_0, c_1) = \gamma c_0 - \frac{\alpha c_0^2}{2} + E\left[\gamma c_1 - \frac{\alpha c_1^2}{2}\right]\), with \(\gamma\) and \(\alpha\) known constants. Does completeness of markets require \(M = J\)? Or is it impossible to determine? Show/argue based on the analysis above, and/or construct new/related analysis if needed.

\textbf{Solution: } The marginal utility of period-zero consumption is \(\gamma - \alpha c_0\), and the marginal utility of period-one consumption is \(\gamma - \alpha Ec_1\). The latter clearly depends on which state is realized in period one. Given these marginal utility functions, the only way for them to be equated at the optimum is for \(M = J = 1\) to occur. But this of course means that there is no uncertainty at all. Although we studied the quadratic utility case in detail, it is the case that while no precautionary savings occurs, the marginal utilities are not lined up in terms of realizations (though they are in expectation).

h. Suppose the consumer utility function is \(U(c_0, c_1) = \ln c_0 + E[\ln c_1]\). Does completeness of markets require \(M = J\)? Or is it impossible to determine? Show/argue based on the analysis above, and/or construct new/related analysis if needed.
**Solution:** The marginal utility of period-zero consumption is \( \frac{1}{c_0} \), and the marginal utility of period-one consumption is \( E\left(\frac{1}{c_1}\right) \). The latter clearly depends on which state is realized in period one. Given these marginal utility functions, the only way for them to be equated at the optimum is for \( M = J = 1 \) to occur, just like in part g above. Once again, this means that there is no uncertainty at all.