2. Pension systems in the Diamond OLG model. In this question you will consider the effect of a fully funded versus a pay-as-you-go pension system in the Diamond two-period OLG model in order to compare the effects on aggregate saving and capital accumulation. Use notation we used in class, where, in addition, denote the pension received by an old person in period \( t + 1 \) by \( p_{t+1} \) and denote the mandatory (i.e., required by law) contribution of a young person in period \( t \) by \( x_t \).

The pension arrangements are as follows. In a fully funded pension system ("FF" system) contributions \( x_t \) are invested by the government in capital, so that \( p_{t+1} \) is given by

\[
p_{t+1} = (1 + r_{t+1}) x_t
\]

(1)

(where \( r_{t+1} \) is the real interest rate between \( t \) and \( t + 1 \)) for the representative individual. In a pay-as-you-go pension system (i.e., an unfunded system, denoted "PAYG" system) contributions of the young are simply paid out to the old, so that \( p_{t+1} \) is given by

\[
p_{t+1} = (1 + n) x_{t+1}
\]

(2)

where \( n \) is the rate of population growth.

Each individual inelastically supplies one unit of labor when young and zero units of labor when old. Let the utility function of an individual born at \( t \) be

\[
u (c^y_t, c^o_{t+1}) = \ln c^y_t + \beta \ln c^o_{t+1}
\]

where \( c^y_t \) is his consumption when young, \( c^o_{t+1} \) is his consumption when old, and \( 0 < \beta < 1 \) is the discount factor.

Let the production function (in per worker terms) be

\[
f(k_t) = k_t^\alpha
\]

where \( k_t \) is the capital-labor ratio and \( 0 < \alpha < 1 \). Assume there is no technological progress and that factors are paid their marginal product, where the rate of depreciation is \( 0 \leq \delta \leq 1 \). Assume for now there is no government debt (what we called \( b_t \) in the Diamond model with government debt) independent of the social security program, nor any other type of government expenditure.

(a) Set up the individual’s choice problem and solve for the young person’s saving function under the two pension systems conditional on the mandatory \( x_t \). (HINT: Be specific about the relation of the exogenous values of \( x_t \) and \( x_{t+1} \) to saving in deriving your results.)

(b) Set up the market equilibrium for FF system and solve for \( k_{t+1} \) as a function of \( k_t, x_t \), and parameters.

(c) Perform a similar analysis for the PAYG system, where you assume for simplicity that \( x_t = x_{t+1} = x \) for all \( t \), where \( x \) is a positive constant. You may write \( k_{t+1} \) as an implicit function of \( k_t, x \), and parameters as long as you are clear about where this function comes from.
(d) Taking \( x \) as a positive constant in both cases, what is the effect of an increase in \( x \) on \( k_{t+1} \) at each \( k_t \) in the two cases? Compare and explain briefly. Illustrate the effect in a “phase diagram” in \( k_{t+1} - k_t \) space.

(e) Consider a stable (non-trivial) steady state. Derive the steady state effect on \( k^{SS} \) of an increase in \( x \) under the two systems (either graphically or algebraically).

The rest of the question tests your ability to relate your results to real world issues.

(f) What is the relationship between \( n \) and the dependency ratio (the number of retired people as a proportion of the number of people in the working age population)? How might your results on PAYG system then relate to the problems of financing social security in an “ageing society” (where the old are living longer)?

(g) Consider the “rate of return” to pension contributions in the FF and PAYG systems. How is this related to the problem of possible dynamic inefficiency in the Diamond model \textit{without} pensions? What does this imply about the welfare implications of the choice of a pension system, including whether or not to have a system? (NOTE: When considering the “choice of a pension system”, be sure to consider who exactly is choosing it.)

(h) Suppose a society is in a steady state with a PAYG system. In order to increase aggregate saving and decrease vulnerability of the system to a change in \( n \), the government wants to shift to a FF system. Why is this a problematic change? If you can, show how the government could choose a \textit{non-constant} time path of debt \( b_t \) and contributions \( x_t \) to alleviate the problem.