The country of Upper Slobovia (U.S.) is described by the standard two-period overlapping generations model with two types of people, “Democrats” and “Republicans”, denoted by $D$ and $R$. A share $\phi^D$ of each generation are “Democrats” and a share $\phi^R = 1 - \phi^D$ are “Republicans”. Democrats and Republicans differ in their discount factors, where $1 > \beta^R > \beta^D > 0$. The utility of an individual of type $j$ born in period $t$ may be written

$$U^j_t(c^j_t, c^j_{t+1}) = \ln c^j_t + \beta^j \ln c^j_{t+1} \quad j = D, R$$

where $c^j_t$ is consumption by type $j$ individuals in period $t$ at age $a$ (where $y = \text{young}$ and $o = \text{old}$). Population of both types grows at rate $n > 0$.

Output is given by a Cobb-Douglas Production function

$$Y_t = K_t^\alpha L_t^{1-\alpha} \quad 1 > \alpha > 0$$

where $K_t$ is capital and $L_t$ is labor. Assume that capital depreciates fully on use. Individuals of both types are endowed with one unit of labor when young and to zero units when old.

(a) Write down the optimization problem faced by each type of individual in the economy and solve for optimal behavior.

**SOLUTION:** For each type (where $R_{t+1} = 1 + r_{t+1}$):

$$\max_{s^j_t} \ln \left( w^j_t - s^j_t \right) + \beta^j \ln \left( R_{t+1} s^j_t \right) \quad j = D, R$$

yielding

$$s^j_t = \frac{\beta^j}{1 + \beta^j} w^j_t \quad c^j_t = \frac{1}{1 + \beta^j} w^j_t \quad c^j_{t+1} = \frac{\beta^j}{1 + \beta^j} R_{t+1} w^j$$

(b) Write down and solve the firm’s optimization problem to obtain the wage $w_t$ and the interest rate $r_t$ as a function of the average capital-labor ratio $k_t$.

**SOLUTION:** Derivation obvious from lecture.

$$r_t + \delta = \alpha k_t^{\alpha - 1} \quad w_t = (1 - \alpha) k_t^\alpha$$

(c) Write down the market clearing condition for capital (in terms of the savings functions) and solve for the steady state equilibrium capital-labor ratio $k_{SS}$.

**SOLUTION:** Market clearing:

$$(1 + n) k_{t+1} = \phi^D s^D \left[ w \left( k_t \right), r \left( k_{t+1} \right) \right] + \phi^R s^R \left[ w \left( k_t \right), r \left( k_{t+1} \right) \right]$$

which becomes

$$(1 + n) k_{t+1} = \left( \phi^D \frac{\beta^D}{1 + \beta^D} + (1 - \phi^D) \frac{\beta^R}{1 + \beta^R} \right) (1 - \alpha) k_t^\alpha$$
Steady state given by $k_{t+1} = k_t = k_{SS}$, so that the above equation may be solved for

$$k_{SS} = \left[ \frac{1 - \alpha}{1 + n} \left( \phi^D \frac{\beta^D}{1 + \beta^D} + (1 - \phi^D) \frac{\beta^R}{1 + \beta^R} \right) \right]^{\frac{1}{1 - \alpha}} \tag{1}$$

(d) Write down the social planner’s (infinite horizon) problem, assuming the planner treats everyone equally, and the first-order conditions for the optimal solution. If you can, solve for the optimal steady state for both consumption and capital allocations.

**SOLUTION:** The infinite horizon social planner’s problem may be written as a Lagrangian (ignoring the initial old):

$$\mathcal{L} = \sum_{j \in \{D, R\}} \left\{ \sum_{t=0}^{\infty} \phi^j \left[ \ln c^o_{t} + \beta^j \ln c^{o}_{t+1} \right] \right\}$$

$$+ \sum_{t=0}^{\infty} \lambda_t \left[ f(k_t) - (1 + n) k_{t+1} - \sum_{j \in \{D, R\}} \phi^j \left( c^o_{t} + \frac{c^{o}_{t+1}}{1 + n} \right) \right]$$

yielding FOC’s (where “~” represents the optimum).

- $c^o_j: \quad \frac{\phi^j}{c^{o}_{t}} - \lambda_t \phi^j = 0 \quad \Rightarrow \quad \frac{c^{o}_{t}}{c^{o}_{t}} = \frac{\phi^D}{\phi^R}$
- $c^{o}_{t+1}: \quad \frac{\phi^j \beta^j}{c^{o}_{t+1}} - \lambda_{t+1} \frac{\phi^j}{1 + n} = 0 \quad \Rightarrow \quad \frac{c^{o}_{t+1}}{c^{o}_{t+1}} = \frac{\beta^D}{\beta^R} \frac{\phi^D}{\phi^R}$
- $k_{t+1}: \quad \lambda_{t+1} f'(k_{t+1}) - \lambda_t (1 + n) = 0$

In (optimal) steady state $\lambda_t = \lambda_{t+1} = \lambda_{SS}$, $k_t = k_{t+1} = k_{SS}$, etc. The optimality condition for the Planner’s allocation for both types is

$$\frac{1}{c^{o}_{SS}} = \beta^j (1 + n) \frac{1}{c^{o}_{SS}} \quad j = D, R$$

while the associated capital stock solves

$$f'(k_{SS}) = 1 + n$$

(e) Does the steady state in this market economy represent a Pareto optimum? (Ignore any initial generation.) Why or why not? Comment on the differences in this respect between this economy and one with a homogeneous population.

**SOLUTION:** **Note the difference between Pareto Optimum and the optimal solution $k_{SS}$.** The market equilibrium in (1) will be optimal (and Pareto optimal) if $k_{SS} = \tilde{k}_{SS}$. If $k_{SS} \neq \tilde{k}_{SS}$, the market equilibrium is not the social planner’s optimum. If, $f'(k_{SS}) < 1 + n$, then $k_{SS} > \tilde{k}_{SS}$, so that the capital stock could be reduced to the optimum $\tilde{k}_{SS}$ by increasing someone’s consumption. In this case, the market equilibrium is not Pareto optimal. If, however, $f'(k_{SS}) > 1 + n$, then $k_{SS} < \tilde{k}_{SS}$. Increasing $k_{SS} < \tilde{k}_{SS}$ would require taking income from some generation and thus making them worse off. The market equilibrium $k_{SS}$ is Pareto optimal, in that the change could not leave everyone weakly better off.
Introducing two types of individuals makes no difference to the efficiency properties of the steady state competitive equilibrium. More patient households, that is, those with the higher \( \beta \), will consume more in old age than less patient households, but each type chooses its consumption optimally given the common interest rate they both face.

Now suppose that policy is chosen not by a social planner but by the U.S. President. Suppose the economy reached steady state under President George W. Shrub, who believed that the government should have no role in the economy and therefore set taxes equal to zero.

Suppose a new Democratic President takes office. The new Secretary of the Treasury, Larry Winters, proposes taxing consumption of the young at rate \( \tau^c \) to encourage saving and to return the tax revenues to the young as a lump sum payment.

(f) What is the effect of this fiscal policy? What tax rate maximizes social welfare of U.S. residents in the steady state? Explain.

**SOLUTION:** Note: It is **not** correct to say that since this is a distortionary tax, the social-welfare maximizing level is therefore \( \tau^C = 0 \). A distortionary tax can correct the distortion caused by consumers having a finite horizon and hence underaccumulating capital. That is, if \( k_{SS} < \bar{k}_{SS} \), the social planner would want to raise saving and capital accumulation by means of \( \tau^C > 0 \).

The consumer’s budget constraints become

\[
w_t + T_t = (1 + \tau^C) c_t^j + s_t^j \quad (1 + r_{t+1}) s_t^j = c_{t+1}^j \quad j = D, R
\]

so that the maximization problem can be written (where \( \hat{w} = w_t + T_t \))

\[
\max_{s_t^j} \left( \frac{\hat{w}_t - s_t^j}{1 + \tau^C} \right) + \beta^j \ln \left( R_{t+1} s_t^j \right) \quad j = D, R
\]

yielding

\[
s_t^j = \frac{\beta^j}{1 + \beta^j} \hat{w}_t
\]

\[
c_t^j = \frac{1}{(1 + \beta^j)(1 + \tau^C)} \hat{w}_t
\]

\[
c_{t+1}^j = \frac{\beta^j R_{t+1}}{1 + \beta^j} \hat{w}_t
\]

Since \( \hat{w}_t > w_t \) (see below), the income effect unambiguously raises \( s_t^j \) and hence capital accumulation.

To calculate welfare and the optimal tax rate, one would maximize the indirect utility function (over the whole population)

\[
V(\tau^C) = \phi^D (\ln c^D(\tau^C) + \beta^D \ln c^D(\tau^C)) + \phi^R (\ln c^R(\tau^C) + \beta^R \ln c^R(\tau^C))
\]

(2)

(where variables without a time subscript represent steady-state values) over \( \tau^C \). To calculate the social-welfare-maximizing \( \tau^C \), one needs to calculate the \( c^j(\tau^C) \), which depend on the equilibrium level of \( T_t \). To calculate \( T_t \), one would use aggregate consumption of young, namely \( C^y_t = L_t (\phi^D c^yD + (1 - \phi^D) c^yR) (w_t + \)
\( T_t \), to calculate \( T_t = \tau^C C_t^y / L_t \) and hence \( w_t + T_t \) in terms of \( w_t \), then calculate \( c_t^y \) and \( c_{t+1}^o \) substitute into (2) and maximize over \( \tau^C \).

In more detail this would work as follow (though the following computations are far more detailed than I would expect on an exam):

\[
C_t^y = L_t \left( \frac{\phi^D}{1 + \beta^D} \frac{1}{1 + \tau^C} + (1 - \phi^D) \frac{1}{1 + \beta^D} \right) (w_t + T_t)
\]

\[
= L_t \left( \frac{\phi^D}{1 + \beta^D} + \frac{1 - \phi^D}{1 + \beta^D} \right) \frac{w_t + T_t}{1 + \tau^C}
\]

\[
= L_t \Phi \frac{w_t + T_t}{1 + \tau^C}
\]

where \( \Phi \equiv \left( \frac{\phi^D}{1 + \beta^D} + \frac{1 - \phi^D}{1 + \beta^D} \right) < 1 \). We may write \( T_t = \tau^C C_t^y / L_t \) so we may solve for

\[
w_t + T_t = \frac{1 + \tau^C}{1 + \tau^C (1 - \Phi)} w_t > w_t
\]

so that

\[
c^y (\tau^C) = \frac{1}{(1 + \beta^D) (1 + \tau^C (1 - \Phi))} w (\tau^C)
\]

\[
c^o (\tau^C) = \frac{\beta^D R_{t+1}}{(1 + \beta^D) 1 + \tau^C (1 - \Phi)} w (\tau^C)
\]

Aggregate saving is

\[
\phi^D s^D + \phi^R s^R = \left( \phi^D \frac{\beta^D}{1 + \beta^D} + (1 - \phi^D) \frac{\beta^R}{1 + \beta^R} \right) \frac{1 + \tau^C}{1 + \tau^C (1 - \Phi)} w_t
\]

The steady state capital-labor ratio as a function of \( \tau^C \) may be written

\[
k_{SS} (\tau^C) = \left[ 1 - \alpha \left( \phi^D \frac{\beta^D}{1 + \beta^D} + (1 - \phi^D) \frac{\beta^R}{1 + \beta^R} \right) \frac{1 + \tau^C}{1 + \tau^C (1 - \Phi)} \right]^{\frac{1}{\alpha - 1}}
\]

(3)

Note that \( k_{SS} (\tau^C) \) is unambiguously increasing in \( \tau^C \). This allows one to calculate \( w_{SS} (\tau^C) = (1 - \alpha) (k_{SS} (\tau^C))^\alpha \) and \( r_{SS} (\tau^C) = \alpha (k_{SS} (\tau^C))^{\alpha - 1} \). One may maximize the indirect utility function given above.

(g) What tax rate (on all young consumers) maximizes welfare of Democrats? Explain.

SOLUTION: Here one would maximize the indirect utility function over Democrats

\[
V^D (\tau^C) = \ln c^y (\tau^C) + \beta^D \ln c^o (\tau^C)
\]

but the functional relations \( c^y (\tau^C) \) and \( c^o (\tau^C) \) would remain exactly as above. As Democrats prefer a lower \( k_{SS} \), they prefer lower \( \tau^C \).
(h) Suppose that instead of returning the tax revenues to the young, they’re simply given to the president of Halfstanistan with no direct welfare effects on U.S. residents. Briefly, what would be the total effect on welfare relative to your answer in part (f)? Under what conditions on $\beta^D$ and/or $\beta^R$, if any, might such a fiscal policy increase welfare?

SOLUTION: The key point here is that income is $w_t$ rather than $w_t + T_t$, and saving depends only on income and **not directly** on $\tau^C$. $s^I_t = \frac{\beta^I}{1 + \beta^I} w_t \Rightarrow$ saving in steady state same as if $\tau^C = 0 \Rightarrow k_{SS}$ in (1) is unaffected $\Rightarrow$ steady state welfare is unaffected via $k$. Hence, the only effect is via the transfer of tax revenues (lowering $e^y$) and welfare unambiguously falls independent of the relation between $k_{SS}$ and $\bar{k}_{SS}$ (which depends on $\beta^D$ and $\beta^R$). Hence, under **no** conditions on $\beta^D$ and $\beta^R$ would this policy raise welfare (but this does suggest why in part (f) welfare might rise.)