Question 1

Consider the standard two-period OLG model, where individuals are endowed with one unit of labor when young and zero units when old. Population grows at rate \( n \). The individual’s utility function is:

\[ u(c^y_t, c^o_{t+1}) = (1 - \beta) \ln c^y_t + \beta \ln c^o_{t+1} \]

where \( 1 > \beta > 0 \). The production function is

\[ F(K_t, L_t) = (K_t)^\alpha (L_t)^{1-\alpha} \]

for \( 1 > \alpha > 0 \). The depreciation rate of capital is \( 1 > \delta > 0 \) and denote the capital-labor ratio at \( t \) by \( k_t \).

a) Set up and solve the representative consumer’s maximization problem. Set up and solve the representative firm’s maximization problem.

The representative consumer’s problem is

\[ \max_{c^y_t, c^o_{t+1}} u(c^y_t, c^o_{t+1}) = \max_{c^y_t, c^o_{t+1}} (1 - \beta) \ln c^y_t + \beta \ln c^o_{t+1} \]

such that \( c^y_t + s_t = w_t \) and \( c^o_{t+1} = R_{t+1} s_t \) where \( R_{t+1} = 1 + r_{t+1} \). Using the LBC \( c^y_t + \frac{c^o_{t+1}}{R_{t+1}} = w_t \), we may write the Lagrangian as

\[ \max_{c^y_t, c^o_{t+1}, \lambda} L(c^y_t, c^o_{t+1}, \lambda) = \max_{c^y_t, c^o_{t+1}, \lambda} (1 - \beta) \ln c^y_t + \beta \ln c^o_{t+1} + \lambda[w_t - c^y_t - \frac{c^o_{t+1}}{R_{t+1}}] \]

which yields the FOCs

\[ \frac{1 - \beta}{c^y_t} = \lambda \quad \text{and} \quad \frac{\beta}{c^o_{t+1}} = \frac{\lambda}{R_{t+1}}. \]

Thus, merging the FOCs we get the Euler equation

\[ c^o_{t+1} = \frac{\beta}{1 - \beta} R_{t+1} c^y_t \]

and using the Euler and the LBC we get \( c^y_t = (1 - \beta) w_t \), \( s_t = \beta w_t \) and \( c^o_{t+1} = \beta R_{t+1} w_t \).
The representative firm’s problem is

$$\max_{K_t, L_t} K_t^\alpha L_t^{1-\alpha} - w_t L_t - (r_t + \delta) K_t$$

which yields the FOCs

$$\alpha \left( \frac{K_t}{L_t} \right)^{\alpha-1} = r_t + \delta$$

$$(1 - \alpha) \left( \frac{K_t}{L_t} \right)^\alpha = w_t.$$ 

b) Solve for the steady-state capital-labor ratio.

Using the conditions $L_{t+1} = (1+n)L_t$, $K_{t+1} = L_t s_t$ and defining $k_t \equiv \frac{K_t}{L_t}$, we obtain $(1+n)k_{t+1} = s_t$ or $(1+n)k_{t+1} = \beta(1-\alpha)k_t^\alpha$. Setting $k = k_t = k_{t+1}$ in steady state we get $(1+n)k = \beta(1-\alpha)k^\alpha$. Thus, $k = \left( \frac{\beta(1-\alpha)}{1+n} \right)^{1/\alpha}$.

c) Show how the introduction of a pay-as-you-go social security system, in which the government collects an amount $d$ from each young person and gives $(1+n)d$ to each old person affects the steady state capital-labor ratio and steady state utility.

If $d$ is the lump sum tax collected from the young and $v = (1+n)d$ is the lump sum transfer to the old, the period budget constraints of a representative agent become $c_t + s_t = w_t - d$ and $c_{t+1} = R_{t+1} s_{t+1} + (1+n)d$. We may write the new LBC as $c_t^v + \frac{c_{t+1}^v}{R_{t+1}} = w_t - d + \frac{(1+n)d}{R_{t+1}} = w_t - \frac{r_{t+1} - n}{1+r_{t+1}} d$, whereas the Euler equation stays the same as before. Solving simultaneously we get

$$c_t^v = (1-\beta)[w_t - \frac{r_{t+1} - n}{1+r_{t+1}} d]$$

$$s_t = w_t - d - (1-\beta)[w_t - \frac{r_{t+1} - n}{1+r_{t+1}} d] = \beta w_t + [(1-\beta)\frac{r_{t+1} - n}{1+r_{t+1}} - 1] d$$

$$c_{t+1}^o = \beta R_{t+1} w_t + [(1-\beta)(r_{t+1} - n) + (1+r_{t+1}) + (1+n)]d = \beta R_{t+1} [w_t - \frac{r_{t+1} - n}{1+r_{t+1}} d]$$

Using $(1+n)k_{t+1} = s_t$ we may conclude that

$$(1+n)k_{t+1} = \beta w_t + [(1-\beta)\frac{r_{t+1} - n}{1+r_{t+1}} - 1]d < \beta w_t = \beta(1-\alpha)k_t^\alpha$$

if $n \geq r_{t+1}$.

Thus, in steady state we have $(1+n)k < \beta(1-\alpha)k^\alpha$ or $k < \left( \frac{\beta(1-\alpha)}{1+n} \right)^{1/\alpha}$ as $n \geq r$. Note that even when $r > n$, since $(1-\beta)(r - n)/(1+r)$ is likely
to be less than one, we would have $s < \beta w$ and steady state $k$ decreases. Therefore, introduction of a pay-as-you-go social security system decreases the steady state capital-labor ratio as it decreases savings. For the steady state lifetime utility we compare the utility in part a) given by

$$U_1 = (1 - \beta) \ln c^y + \beta \ln c^o = (1 - \beta) \ln (1 - \beta) w + \beta \ln \beta Rw$$

to the new utility

$$U_2 = (1 - \beta) \ln (1 - \beta) [w - \frac{r - n}{1 + r} d] + \beta \ln \beta R [w - \frac{r - n}{1 + r} d].$$

When $n > r$ clearly both $c^y$ and $c^o$ increase for any feasible $d > 0$ and $U_2 > U_1$. Otherwise steady state utility decreases when $r > n$, as government forces individuals to save less and pay the social security tax when young, which yields a lower return than private savings would yield.

d) Given your results in part (c), would such a system be adopted under majority voting? Why or why not?

At any point in time, if the system is voted, old people would be in favor of it since transfers will increase their utility as they are not taxed. For young people there are cases. If we maximize the indirect utility $U_2(w, R, d)$ of the representative young with respect to $d$ to get his most preferred social security tax, we would get $d = 0$ if $n < r$ and $d > 0$ if $n > r$. As all generations are homogeneous and young population is greater than the old, the median voter will be a young voter. If $n > r$ the most preferred social security tax of the median voter (representative young) becomes positive. Thus, such a system would be adopted under majority voting only if $n > r$ in this setup.
When \( g \) increases, 
\( \Delta \eta = 0 \) locus shifts left,  
\( \Delta k = 0 \) locus shifts up.
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SOLUTIONS TO PROBLEMS 2 AND 3

2. (a) The planner solves

\[\max_{\{k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1} + (1 - \delta)k_t - g)\]

leading to FOCs

\[\frac{u'(c_{t+1})}{u'(c_t)} = \frac{1}{\beta(f'(k_{t+1}) + 1 - \delta)}\]
\[c_t = f(k_t) - k_{t+1} + (1 - \delta)k_t - g\]

(b) The household’s problem is

\[\max_{\{k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(w_t + r_t k_t - k_{t+1} + (1 - \delta - \tau_t)k_t)\]

leading to FOCs

\[\frac{u'(c_{t+1})}{u'(c_t)} = \frac{1}{\beta(r_{t+1} + 1 - \delta - \tau_{t+1})}\]
\[c_t = w_t + r_t k_t - k_{t+1} + (1 - \delta - \tau_t)k_t\]

The firm’s problem implies \(r_{t+1} = f'(k_{t+1})\), \(w_t = f(k_t) - k_t f'(k_t)\) etc.

(c) Using firms’ FOC and \(\tau_t k_t = g\), where \(k_t\) is average capital holding, we may write equilibrium equations:

\[\eta_{t+1} = \left(\beta\left(f'(k_{t+1}) + 1 - \delta - \frac{g}{k_{t+1}}\right)\right)^{-1} \eta_t\] (1a)
\[k_{t+1} = f(k_t) - (1 - \delta)k_t - c(\eta_t) - g\] (1b)

where \(c(\eta_t)\) is defined by \(u'(c_t) = \eta_t\) and \(k_t = k_t\) in equilibrium.

(d) From (1a), \(\eta_{t+1} = \eta_t\) at \(k\) s.t. \(f'(k) + 1 - \delta - \frac{g}{k} = 1/\beta\). With \(k\) on horizontal axis, \(\eta_t\) is rising (i.e., \(\eta_{t+1} > \eta_t\)) to the right of locus as long as \(\frac{d f'(k) + 1 - \delta - g/k}{dk} = f''(k) + g/k^2 < 0\), and falling to the left of it. From (1b), \(k_{t+1} = k_t\) at \(f(k) + \delta k = c(\eta) - g\). \(k\) is rising above the \(\Delta k = 0\) locus \((k_{t+1} > k_t)\) and falling below, implying phase diagram and saddlepath stability.

(e) In the market solution the representative household doesn’t take account of how increasing its own \(k_t\) reduces distortionary tax rate \(\tau_t\) (consider discussion of representative agent in TIC notes).
(f) Differentiating the $\eta_{t+1} = \eta_t$ locus with respect to $g$, one obtains

$$\frac{dk}{dg} = \frac{1/k}{f''(k) + g/k^2} < 0$$

where the sign follows the assumption that $f''(k) + g/k^2 < 0$, so that the locus shifts to the left. Analogously, differentiating the $k_{t+1} = k_t$ locus with respect to $g$, one obtains

$$\frac{d\eta}{dg} = -\frac{1}{c'(\eta)} > 0$$

so that the locus shifts up. Hence the saddle path shifts up and to the left and $k^{SS}$ falls.

Note that $k^{SS}$ is fixed by the condition $f'(k^{SS}) + 1 - \delta - g/k^{SS} = 1/\beta$ (where $g/k^{SS} = \tau^{SS}$). Higher $g$ means higher taxes and lower $k^{SS}$.

3. In this economy investment takes two periods to “mature”. Defining $s_{it}$ as an investment project that is finished after $i$ periods, one has $s_{2t} = s_{1t+1}$ which then yields $k_{t+2}$.

(a) With the state variable vector $z_t = (k_t, s_{1t})$ the vector of controls is $x_t \equiv (c_t, h_t, s_{2t}, k_{t+1})$.

$$V(z_t) = \max_{x_t} \left\{ \ln c_t + \theta \ln (1 - h_t) + \beta V(z_{t+1}) \right\}$$

where $\gamma_t = (1 - \delta) k_t + s_{1t} - k_{t+1}$

(b) In evaluating, we use $s_{1t+1} = s_{2t}$ in $z_{t+1}$. Using the FOCs for the four controls and applying the envelope theorem, we obtain

$$\frac{c_t}{1 - h_t} = \frac{1 - \alpha}{\theta} k_t^{1-\alpha} h_t^{-\alpha}$$

$$\gamma_t = \beta \lambda_t + \delta \lambda_{t+1} \alpha \frac{k_t^{1-\alpha} h_t^{-\alpha}}{k_{t+1}^{1-\alpha} h_{t+1}^{-\alpha}} + \beta (1 - \delta) \gamma_{t+1}$$

The first expression is the standard condition associated with the labor/leisure tradeoff. The second condition states that the shadow price associated with capital is equal to the MPK weighted by the shadow price of output and the discounted value of the capital stock remaining after depreciation. The third condition states that the expenditure cost of investment (over the two period horizon) must be equal to the discounted shadow price of a unit of capital.

(c) In steady state conditions (2b) and (2c) imply, respectively:

$$\frac{y^{SS}}{k^{SS}} = \frac{\gamma^{SS}}{\lambda^{SS}} \frac{1 - \beta (1 - \delta)}{\alpha \beta}$$

$$\frac{\gamma^{SS}}{\lambda^{SS}} = \omega_1 + \frac{1}{\beta} \omega_2$$
Combining these conditions (and using $\omega_1 + \omega_2 = 1$) one obtains

$$\frac{y^{SS}}{k^{SS}} = \frac{1 - \beta (1 - \delta) 1 - \omega_1 (1 - \beta)}{\alpha \beta}$$

(3)

In steady-state, $s_1 = s_2$. Combining this with the law of motion of capital yields

$$\frac{i^{SS}}{k^{SS}} = \delta$$

(4)

which is of course consistent with constant $k$, so that investment covers depreciation.

Finally, from (2a) we have

$$\frac{h^{SS}}{1 - h^{SS}} = \frac{1 - \alpha y^{SS}}{\theta}$$

In steady state we have $y^{SS} = c^{SS} + i^{SS}$, so that

$$\frac{y^{SS}}{c^{SS}} = 1 + \frac{i^{SS}}{c^{SS}} = 1 + \frac{i^{SS} k^{SS} y^{SS}}{k^{SS} y^{SS} c^{SS}}$$

or $\frac{y^{SS}}{c^{SS}} = \left(1 - \frac{i^{SS} k^{SS}}{y^{SS} c^{SS}}\right)$, which can be computed from (3) and (4).

(d) Investment doesn’t go into capital one for one after one period, as in “installation function” approach. But, there really are no installation costs here, only waiting, as all expenditure on investment goes unit-for-unit to capital (Note (4)).