Do All Questions. Point values in parentheses. First, read the whole exam. Read an entire question before starting to work on it. PLEASE WRITE CLEARLY! Note that each question gets harder as you go along, so, if you are time constrained, concentrate on earlier parts of a question.

1. (25%) Consider the standard two-period OLG model, where individuals are endowed with one unit of labor when young and zero units when old. Population grows at rate $n$. The individual’s utility function is:

$$u(c_t^y, c_{t+1}^o) = (1 - \beta) \ln c_t^y + \beta \ln c_{t+1}^o$$

where $1 > \beta > 0$. The production function is

$$F(K_t, L_t) = (K_t)^\alpha (L_t)^{1-\alpha}$$

for $1 > \alpha > 0$. The depreciation rate of capital is $1 > \delta > 0$ and denote the capital-labor ratio at $t$ by $k_t$.

(a) Set up and solve the representative consumer’s maximization problem. Set up and solve the representative firm’s maximization problem.

(b) Solve for the steady-state capital-labor ratio.

(c) Show how the introduction of a pay-as-you-go social security system, in which the government collects an amount $d$ from each young person and gives $(1 + n) d$ to each old person affects the steady state capital-labor ratio and steady state utility.

(d) Given your results in part (c), would such a system be adopted under majority voting? Why or why not?

2. (40%) Consider a discrete, infinite horizon growth model with representative consumers (households), where the population size is normalized to 1 (and there is no population growth). Output per capita is $y_t = f(k_t)$, where $k$ is capital input per unit of labor and production displays constant returns to scale. The household’s instantaneous utility function over consumption, $c$, is $u(c)$, which is strictly increasing and strictly concave. The discount factor is $1 > \beta > 0$. The households’ initial capital stock is $k_0$ and each household is endowed with 1 unit of time. The government has fixed expenditures of $g$ (in per capita terms). Assume $f(k_0) + (1 - \delta) k_0 > g$. (Required to get the economy off the ground.) There are a a large number of profit maximizing firms.

(a) Derive the system of equations that specify the dynamics of the Planner’s solution for this economy.

(b) Now consider the decentralized economy. If the government’s only means of collecting revenue is a proportional tax $\tau_t$ on capital $k_t$ held by the household (Note: tax is on capital, not on income from capital) and the budget is balanced every period, write down the households’ and firms’ optimization problems.

(c) Representing the marginal utility of consumption by a new variable $\eta_t \equiv u'(c_t)$, solve for the system of two dynamic equations in $\eta_t$ and $k_t$ for given parameters $\beta, \delta,$ and $g$ that specify equilibrium dynamics.
(d) Show the dynamics of the system in a (discrete time) phase diagram with $\eta$ and $k$ on the two axes for a given value of $g$. (HINT: What restriction on the relation between $f''(k)$ and $g$ is required for saddlepath stability?)

(e) Explain why the response to a change in $g$ in the equilibrium allocation is not the same as that in the Planner’s allocation.

(f) Show diagrammatically how an increase in $g$ affects the saddle path and the steady state. Explain.

3. (35%) Consider a representative-agent, non-stochastic growth model in which investment does not produce capital immediately. Let $n = 0$. Assume that agents have preferences given by

$$\sum_{t=0}^{\infty} \beta^t (\ln c_t + \theta (1 - h_t))$$

where $c_t$ is consumption and $h_t$ is time spent in work activity. Aggregate output is produced according to

$$y_t = k_t^\alpha h_t^{1-\alpha}$$

where $y_t$ is output and $k_t$ denotes capital. (Note that there is no technological progress in the economy.) In each period, agents choose consumption, work effort and investment in order to maximize lifetime utility. In this economy, an investment project started at time $t$ does not produce capital until period $t + 2$. The costs associated with this project are spread out over the two-period horizon. Let $s_{it}$ denote an investment project that is finished after $i$ periods ($i = 1, 2$) where households pay a fraction $\omega_i$ of the total costs in period $i$ of the project. Then total investment expenditures are given by

$$i_t = \omega_1 s_{1t} + \omega_2 s_{2t}$$

where, of course, $\omega_1 + \omega_2 = 1$. The law of motion for the capital stock is given by

$$k_{t+1} = (1 - \delta) k_t + s_{1t}$$

(a) Express the associated social planner problem for this economy as a dynamic programming problem, using $k_t$ and $s_{1t}$ as state variables and Lagrange multipliers for your constraints. Be explicit in identifying the control variables in each period (along with the laws of motion for the state variables).

(b) Using the set-up in part (a), derive a set of three necessary conditions associated with an optimum and interpret each one.

(c) Solve for the steady-state output-capital ratio, the investment-capital ratio, and the ratio of time spent in work activity to time spent in leisure as a function of the exogenous parameters.

(d) Discuss briefly the relation between this model and the “installation function” model of investment.