RELAX!! Do All 3 Questions. First, read the whole exam; Read an entire question before starting to work on it. WRITE CLEARLY! Note that each question gets harder as you go along, so, if you are time-constrained, concentrate on earlier parts of a question. Answer each question in a DIFFERENT booklet. TIME: 2 hours, 15 minutes

1. (36%) Consider the basic Diamond overlapping generations model with two countries \( h = A, B \). Country \( A \) is like the standard model where individuals work in the first period and are retired in the second. The production function is \( y_t = f(k_t) \) (in per worker terms) with all the “nice” properties. Capital depreciates fully on use, that is, \( \delta = 1 \) and population is constant, that is, population growth rate \( n = 0 \). (For simplicity, let population of each generation = 1.) The utility function of the representative individual in \( A \) is

\[
u^A(c^y_t, c^o_{t+1}) = \ln c^y_t + \beta^A \ln c^o_{t+1}
\]

Country \( B \) has the same (constant) population as country \( A \), the same production function and the same depreciation rate \( \delta = 1 \). Country \( B \) differs from country \( A \) in 4 ways:

1) The utility function of the representative individual in country \( B \) has the same form as (U) in \( A \), but a different weight of young versus old utility of consumption. That is, \( \beta^A \neq \beta^B \);

2) Only a fraction \( \lambda \) (0 < \( \lambda < 1 \)) of young people in \( B \) are endowed with labor when young and work. A fraction \( 1 - \lambda \) have no labor endowment and do not work;

3) There is a tax-transfer system in \( B \) such that there is a transfer \( v \) to those don’t work \((1 - \lambda)\) of population, not to everyone) financed by a lump-sum tax \( \tau \) on the young who work, that is, \( \lambda \) of young. There is no internal tax transfer system in country \( A \);

4) Each young worker in country \( A \) has a labor endowment of one unit, while each young worker in \( B \) (fraction \( \lambda \) of young) has an endowment of \( \ell \) units, where \( \ell > 1 \).

a) Derive the saving rate of each type of young person in each country.

Now suppose that \( B \) each period gives a fraction \( \phi \) of \( \tau \) as a lump-sum payment to the young in \( A \), with the fraction \( 1 - \phi \) still being used to finance internal transfers from those who can work to those who can’t in \( B \). (Remember that in each country the size of a generation is 1.)

b) Derive the effect of permanent foreign aid of this sort on the growth path in each country if factors of production are immobile, that is, cannot move between countries. What happens to per-worker output and per-worker income in each country in steady state? How do your results change as the parameters \( \lambda \) and \( \phi \) change?

c) Discuss briefly how your results would change if the transfer went to retired households in \( A \).

d) Suppose that in \( B \) the choice of \( \phi \) was determined by voting. What would be the preferred \( \phi \) of the two types of young people in \( B \)?

e) Suppose that \( 1 - \lambda = \phi = 0 \) in \( B \) (that is, all young work in \( B \) and there is no internal transfer system). Individuals still have preferences over \( \tau \). Could it be the case that a young person in \( B \) would prefer a positive to a zero \( \tau \)? Why? Given \( \tau \), could workers be better off in \( B \) when \( 1 - \lambda \) and \( \phi \) are positive rather than 0? Explain.
2. (28%) Consider a representative agent optimal growth model with cookies $m_t$ in the production function. In every period an individual can divide his time between hours of work $l_t$ and time $b_t$ spent in baking cookies, so that $l_t + b_t = 1$. Assume population size equals 1 and no population growth, so that all variables can be interpreted as per capita values.

Output is a function of physical capital, hours worked, and cookies, namely
\[ y_t = f(k_t, l_t, m_t) \]  
(2a)

Cookies evolve according to a stochastic baking technology (NOTE THE TIMING!):
\[ m_t = b_{t-1} + \eta_t \]  
(2b)
where $\eta_t$ is a mean zero i.i.d. random variable not known as of time $t - 1$. Note that cookies are an intermediate input rather than a consumption good.

Instantaneous utility is a function of current consumption $u(c_t)$, where current consumption is $c_t = y_t - k_{t+1}$. The social planner’s objective is to maximize the present discounted value of expected utility over an infinite horizon, where the discount factor is $0 < \beta < 1$.

a) Set this problem up as a dynamic programming problem with two state variables and two control variables. Find the first-order conditions and interpret them.

b) For this part of the problem only, assume no uncertainty, so that (2b) is simply $m_t = b_{t-1}$. Derive the relation between the two inputs $k_t$ and $m_t$ along the growth path. Derive the conditions that characterize the steady state and interpret.

c) Now, return to the case of uncertainty in (2b) (i.e., as in the original equation) and suppose the production function (2a) is of the form:
\[ y_t = k_t^\alpha + m_t^\gamma l_t^\delta \]

What does this imply about the expected return to physical capital? And, what can one say about the expected return to baking relative to regular labor supply or relative to physical capital?

3. (36%) Consider the following infinite horizon economy. Single-period individual utility is
\[ u(c_t) = \frac{c_t^{1-1/\theta}}{1-1/\theta} \]

Individuals may save either via physical capital or government bonds, where the two types of saving are denoted $i_t^k$ and $i_t^b$. The discount factor is $\beta$, where $0 < \beta < 1$.

Output is produced according to a production function
\[ y_t = K_t^\alpha (h_t L_t)^{1-\alpha} \]  
(3a)
where physical labor $L_t = L$ is constant and inelastically supplied. (You may assume that $L = 1$, so that $K_t = k_t$, capital per worker.) Physical capital $k_t$ evolves according to
\[ k_{t+1} = (1 - \delta) k_t + i_t^k \]  
(3b)
Human capital $h_t$ per worker reflects learning-by-doing, so that the relation between human capital of the representative worker and the aggregate capital stock is given by

$$h_t = \gamma K_t \text{ where } \gamma > 0$$

(3c)

and $h_t L_t$ in (3a) represents effective labor units.

a) Show that the resource constraint in this economy in per capita terms is of the form

$$c_t + k_{t+1} = Ak_t$$

(4)

and find $A$.

b) i) Write out and solve the social planner’s problem in this economy.

ii) What is the optimal growth rate of consumption in the social planner’s solution?

iii) What is the optimal policy rule $k_{t+1} = g(k_t)$ in terms of the parameters $\alpha, \beta, \gamma, \delta, \text{ and } \theta$?

(HINT: (4) implies that in the planner’s solution $c_t$ and $k_{t+1}$ are constant fractions of $Ak_t$)

c) Now consider the competitive equilibrium in this economy

i) Write down the consumer’s budget constraint as a function of the wage rate $w_t$ per effective labor unit, the rate of return for physical capital $k_t$, and the rate of return $\rho_t$ on government bonds $b_t$, as well as $h_t, k_t, \text{ and } b_t$. (NOTE: It is linear in $h_t, k_t, \text{ and } b_t$)

ii) Using the fact that in competitive equilibrium, the rates of return for physical capital and government bonds must be equal, use the households first order conditions to find the Euler equation. (HINT: Consider a composite asset $a_t = b_t + k_t$)

iii) Using the firm’s profit maximizing behavior, what is the equilibrium interest rate for physical capital and the equilibrium wage rate? (Remember, the firm takes human capital per worker as exogenous, but chooses physical inputs).

iv) What are the growth rates of consumption and of physical capital in the competitive equilibrium?

d) Compare the growth rate of the economy in the social planner’s equilibrium from Part b and the decentralized equilibrium in Part c. How are they different, and why? What is it in the nature of human capital accumulation that might explain the difference?

e) Now suppose that the government subsidizes the private cost of capital for firms. In particular, assume that the private cost of capital for firms is now given by $(1 - \sigma) r_t$ and the government pays for this subsidy via lump sum taxes $T$ on individuals. What is the growth rate in the competitive equilibrium in this case?

f) Why would the government want to subsidize investment in this model? What subsidy rate $\sigma$ should the government set so that the competitive equilibrium growth rate of consumption coincides with the social planner’s outcome in Part b?

RELAX!! Use the development of a question and remember what you know from different parts of the course in thinking about how to answer.