1. Complete Markets and Productive Efficiency

There are two agents, A and B, who live for one period. Preferences satisfy

\[ U_i = \log(c_i) + \log(1 - L_i) \]

Where \( c_i \) is consumption and \( L_i \) is hours worked. Agents produce according to

\[ Y_i = w_i L_i, \]

Where \( w \) refers to productivity (or the “wage”). Assume that \( w \) for each agent is a random variable drawn from some finite set with strictly positive support. Wage risk is the only source of uncertainty in the problem. Assume that the realizations of \( w \) are i.i.d. across agents; in general wages will differ across agents.

**PART I: Autarky.** Suppose in this part that there are no markets for trading risk ex ante. Agents have no reason to trade ex post, so that each agent observes their draw of \( w \) and then maximizes (*) subject to (**) and subject to \( c_i = Y_i \).

(a) [15 points] Derive the first order condition relating the wage to the marginal utilities of consumption and labor effort for each agent. Solve for optimal \( L_i \) for each agent. Solve for aggregate \( Y \) and aggregate productivity, defined as \( (Y_A + Y_B) / (L_A + L_B) \).

(b) [10 points] How is optimal \( L_i \) related to \( w_i \)? Explain this relationship (or the lack thereof) intuitively. The first order condition you solved for in (a) should help understand the competing effects underlying this relationship.

**PART II: Complete Markets.** For the rest of the problem assume that there are complete Arrow-Debreu (AD) markets for trading risk ex ante. We adopt the convention that AD securities are traded in period 0. Wages are realized in period 1, followed by production, payment on AD securities, and consumption.

Let \( s = [w_A \ w_B] \) denote the state of nature drawn in period 1. Let \( P(s) \) denote the probability density over \( s \). In general, agents in period 0 will choose optimal state contingent plans for consumption and labor supply, \( c_i(s) \) and \( L_i(s) \). Because of complete markets, they need not set consumption equal to output in each state;
instead, they set \( c_i(s) = Y_i(s) + n_i(s) \), where \( n_i(s) \) is agent i’s net purchases of AD securities for state \( s \). If \( n_i(s) < 0 \), the agent is a net issuer in state \( s \).

Agent i’s choice problem in period 0 is thus

\[
(***) \quad \text{Max} \sum_s P(s) \left[ \log(c_i(s)) + \log(1 - L_i(s)) \right]
\]

subject to \( \sum_s q_0(s) \left[ c_i(s) - w_i(s) L_i(s) \right] = 0 \)

Where \( q_0(s) \) is the price of the AD security for state \( s \). In equilibrium, output is perishable, aggregate consumption must equal aggregate output in each state, and thus \( q \) will adjust so that \( n_A + n_B = 0 \) for each \( s \); you need not solve for \( n \)’s or \( q \)’s.

(c) [15 points] Take the FOCs of (***), and find a FOC relating the wage and the marginal utilities of consumption and labor effort for each agent. Use the FOCs to prove that optimal \( c_i(s) \) is a state-invariant share of aggregate output: \( c_i(s) = \mu_i Y(s) \), where \( Y(s) = Y_A(s) + Y_B(s) = w_A L_A(s) + w_B L_B(s) \).

Note: since the two agents are symmetric ex ante, you can assume that \( \mu_i = \frac{1}{2} \) for each agent for the rest of the problem.

It will be convenient from now on to write \( w_A/w_B = (1 - e_B) \) and \( w_B/w_A = (1 - e_A) \), where \( (1 - e_A) \times (1 - e_B) = 1 \). If both agents draw the same wage, then both \( e \)’s are zero. Otherwise, the \( e \)’s have opposite sign; if \( w_A > w_B \) then \( e_A > 0 \) and \( e_B < 0 \).

(d) [25 points] Solve for optimal \( L_A \) in terms of \( e_A \) and \( L_B \) in terms of \( e_B \). Explain intuitively why the relationship between \( L \) and \( w \) differs under complete markets from that under autarky. The FOC relating the wage and the marginal utilities of consumption and labor effort may be useful.

(e) [25 points] Show that aggregate \( Y \) under complete markets equals that under autarky for each \( s \). Show that aggregate labor supply \( L_A + L_B \) under complete markets equals that under autarky when \( w_A = w_B \), and is strictly lower when \( w_A \) differs from \( w_B \), so that productivity is (weakly) higher under complete markets than under autarky. Explain intuitively why productivity is higher under complete markets.

(f) [10 points] Suppose \( w_A > w_B \). Whose ex-post utility is higher under complete markets? What sort of problems might arise if enforcement were not frictionless?
2. Consumption and Taste Shocks

Consider an agent who maximizes

\[ (*) \quad E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, x_t) \quad \text{where} \quad u(c,x) = [c - (b/2)*(c - x)^2] \quad \text{subject to} \]

\[ (**) \quad a_{t+1} = (1 + r) (a_t - c_t) \]

and a No Ponzi Game condition. Here, c refers to consumption; x refers to a taste shock that affects the marginal utility of consumption; \( a_{t+1} \) refers to financial wealth at the beginning of \( t+1 \); and \( (1+r) \) is the gross return on saving, where we assume that \( \beta(1+r) = 1 \). Notice that there is no labor income.

The taste shock is assumed to be an AR(1) process with persistence p in [0,1]:

\[ (***) \quad x_{t+1} = px_t + e_{t+1}, \quad \text{where} \quad E_t e_{t+1} = 0. \]

(a) [15 points] Derive the intertemporal budget constraint for this problem. [If you can’t derive it, guess it and move on].

(b) [15 points] Identify state and control variables and write down the Bellman Equation for this problem; take first order conditions and derive the Euler Equation relating consumption and taste shocks over time.

(c) [35 points] Find a closed form solution for \( c_0 \) in terms of \( a_0, x_0 \) and model parameters. (Note that a similar expression will hold for all t)

(d) [20 points] Suppose there is a one-time positive shock to x at time t. In other words, assume that x is zero prior to t; is positive at t; and then obeys the AR(1) process (***) with no further shocks after t.

Describe the impulse response function for consumption. How does consumption respond initially and in subsequent periods? How does the long run level of c compare to its level before the shock? How do the magnitude of the initial response and the subsequent behavior of c vary with p? [Consider the cases p = 0, 0 < p < 1 and p = 1]. Explain these findings intuitively.

(e) [15 points] Explain how taste shocks could create Type I error in the Hall (1978) test of the LCH. (Focus on the estimated impact of Z.)