I. Consumption, labor supply and investment

Suppose there is a household with standard preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(C_t),$$

where $C$ is consumption, $E_0$ refers to expectations at time zero, and $u(C)$ is increasing and concave.

This household holds financial assets, which pay a gross return $(1+r)$, constant over time. In addition, this household has two sources of non-interest income. First, the household can supply labor to the market and earn wage $w_t$, where $w_t$ is positive and known in period $t$, but stochastic as of period $t-1$. Second, the household has access to a production technology that produces output $y_t$ using the household’s own capital and labor. The household’s production function is

$$y_t = X_t k_t^\alpha n_t^{1-\alpha}$$

where $0 < \alpha < 1$; $X_t$ is a technology shock, positive and known in period $t$, but stochastic as of period $t-1$; $k_t$ is the capital stock owned by the household; and $n_t$ is labor supplied by the household. The household has one unit of labor each period to allocate between the production technology (*) and market labor. Thus, wage earnings from the market in period $t$ are $w_t (1-n_t)$.

The household’s capital stock depreciates at rate $\delta$ each period. The household can augment next period’s capital stock by investing $i_t$ units of goods this period, according to a standard capital accumulation equation:

$$k_{t+1} = (1-\delta) k_t + i_t$$

Capital is subject to a convex adjustment cost, so the total amount of resources required today to augment next period’s capital stock by $i_t$ units equals $i_t + (c/2)i_t^2$, where $c > 0$.

Any income not consumed or invested this period goes to financial assets. Putting all of these assumptions together, the household’s dynamic budget constraint is given by:

$$a_{t+1} = (1+r)[a_t + y_t + (1-n_t) w_t - c_t - i_t - (c/2)i_t^2]$$
(a) [10 points] Identify the state and choice variables of this problem. Distinguish exogenous from endogenous state variables. Write down the Bellman Equation for this problem, making substitutions as needed to narrow the list of choice and state variables to the minimum necessary to describe the problem.

(b) [30 points] Take first order and envelope conditions for this problem.

(c) [15 points] Using one or more of the conditions from (b), derive an expression for optimal \( n_t \) (that is, optimal labor supply to the household’s production technology) in terms of state variables and model parameters. (This expression does not necessarily include all state variables). Explain intuitively the sign of the relationship between optimal \( n \) and each state variable (for instance, is optimal \( n \) increasing or decreasing in \( k \), and why?)

(d) [15 points] Define \( MPK_t \), the marginal product of capital at time \( t \), as the derivative of \( y_t \) with respect to \( k_t \). Using your result in (c), derive an expression for \( MPK_t \) in terms of exogenous state variables and model parameters. This expression should not include any endogenous state variables or choice variables. Explain intuitively the sign of the relationship between \( MPK \) and each exogenous state variable.

(e) [30 points] Using one or more of the conditions from (b) and the definition of \( MPK \), derive two Euler Equations for this problem. The first should involve optimal choices of \( C_t \) and \( C_{t+1} \), the market interest rate \( r \) and model parameters. The second should involve optimal choices of \( C_t \), \( C_{t+1} \), \( i \), and \( i_{t+1} \), along with \( MPK_{t+1} \) and model parameters.
II. Inventories and Smoothing of Marginal Cost

Consider a monopolist firm that maximizes the expected discounted sum of profits:

$$\max_{\pi_t} E_0 \sum_{t=0}^{\infty} \beta^t \pi_t$$

Profits equal revenues minus costs. Revenues equal the firm’s price ($p_t$) times its sales ($s_t$), while costs are given by the labor hired by the firm ($h_t$) times its wage ($w_t$). Thus, we have

$$\pi_t = p_t s_t - w_t h_t$$

The firm produces output $y_t$ using labor, subject to decreasing returns to scale:

$$y_t = (h_t)^\gamma$$

where $0 < \gamma < 1$ indexes returns to scale.

The firm takes its wage $w_t$ as given. However, the firm is a monopolist in its product market. The firm sets its price each period to maximize profits, subject to a downward sloping demand curve relating sales to the firm’s price and an exogenous demand shift $d_t$:

$$s_t = d_t p_t^\varepsilon$$

where $\varepsilon > 1$ is the elasticity of demand.

Assume that the wage $w_t$ and the demand shift $d_t$ are known at time $t$, but unknown and in general stochastic from the point of view of $t-1$. These are the two shocks in the economy.

The firm does not have to set current production ($y_t$) equal to current sales ($s_t$). Rather, it can build up inventory stocks of the good over time by setting output higher than sales in some periods, and it can draw inventory stocks down by producing less than sales in other periods. Let $I_t$ denote the firm’s stock of inventories at the start of $t$. Inventories evolve according to:

$$I_{t+1} = I_t + y_t - s_t$$

In other words, a firm starts period $t$ with $I_t$ units of output available for sale. It then produces $y_t$ additional units of output in period $t$, which are also immediately available for sale. Any inventory or new output that is not sold in period $t$ is carried forward as beginning of period inventories next period ($I_{t+1}$). $I_{t+1}$ is thus known as of time $t$.

Importantly, firms are not allowed have negative inventories at the start of any period. Put differently, firms are not allowed to sell more than they have available for sale in period $t$, including beginning-of-period inventories plus new production:

$$s_t \leq I_t + y_t \quad \text{or equivalently} \quad I_{t+1} = I_t + y_t - s_t \geq 0.$$  

(a) [10 points] Identify the state and choice variables for the problem, distinguishing exogenous from endogenous states. Write down the Bellman Equation for this problem, making substitutions as needed to narrow the list of state and choice variables to the minimum necessary to fully describe the problem.
HINT: your Bellman equation needs to account for the inequality constraint on inventories. To do this, add a term to the first part of the right hand side of the Bellman equation of the form $\lambda_t f_t$, where $f_t \geq 0$ represents the inequality constraint on inventories at the start of period $t+1$, and where $\lambda_t$ is a nonnegative multiplier representing the current benefit of relaxing the inequality constraint at time $t$.

Your choice of $f_t$ depends on what variables you use as choice variables; for instance, if you set $I_{t+1}$ as a choice variable, then set $f_t = I_{t+1}$; if you don’t use $I_{t+1}$ as a choice variable but do use $y$ and $s$, then set $f_t = I_t + y_t - s_t$; if you use something other than $y$ or $s$ as a choice variable, make appropriate substitutions for $y$ or $s$.

[An aside: note that the time $t$ Bellman equation should include an $f(x_t)$ that represents the inequality constraint on $I_{t+1}$ rather than the inequality constraint on $I_t$, since it is the inequality constraint on $I_{t+1}$ that potentially constrains choices at time $t$. The inequality constraint on $I_t$ may have constrained choices at $t-1$.]

(b) [20 points] Take first order and envelope conditions for the problem. Identify and explain the complementary slackness condition for this problem.

(c) [20 points] Derive an expression for the firm’s current marginal cost of production ($MC_t$) as a function of current wages and output. (You don’t need to use any of the results from part (b) to derive this expression). Using this expression and your results from (b), establish a static condition describing the firm’s optimal current price as a function of its current marginal cost and model parameters. Does the firm’s optimal price increase or decrease with its current output? Explain intuitively.

(d) [20 points] Using results from (b) and your expression for the firm’s marginal cost, establish an Euler equation relating $MC_t$ and discounted expected $MC_{t+1}$. This equation will be a weak inequality. Identify the conditions under which the Euler equation will be a strict inequality, and the conditions under which the Euler equation will hold with equality.

(e) [15 points] Suppose that there is a temporary decline in wages so that $w_t$ is less than $E_tw_{t+1}$. By itself this will tend to make current marginal cost ($MC_t$) lower than expected discounted future marginal cost ($\beta E_t MC_{t+1}$). Would this outcome violate the Euler equation you established in (d)? If so, explain how the firm will change its production and inventory accumulation at time $t$, and how these changes will restore the Euler Equation relating marginal cost in periods $t$ and $t+1$. If not, explain how the non-negativity constraint on inventories limits the firm’s options for smoothing $MC$ over time in this case.

(f) [15 points] Now suppose that there is a temporary INCREASE in wages so that $w_t$ is higher than $E_tw_{t+1}$. By itself this will tend to make current marginal cost ($MC_t$) higher than expected discounted future marginal cost ($\beta E_t MC_{t+1}$). Would this outcome violate the Euler equation you found in (d)? If so, explain how the firm will change its production and inventory accumulation at time $t$, and how these changes will restore the Euler Equation relating marginal cost in periods $t$ and $t+1$. If not, explain how the non-negativity constraint on inventories limits the firm’s options for smoothing $MC$ over time in this case.