Sufficient Decisions in Multi-Sided and Multi-Product Markets[*]

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Abstract

We show that in many applied economic models, it is possible to reduce the dimensionality of the space of actions to what we call "sufficient decisions." We find that for monopoly and oligopoly in multi-sided markets and multi-product markets, the market equilibrium can be transformed into an equivalent market equilibrium in which each firm makes a single decision. Because profit maximization connects a firm's decisions to each other, it is often possible to introduce a constraint linking the firm's decisions. For example, the number of facilitated transactions is a sufficient decision for a monopolist in a two-sided market. We also analyze a related distortion akin to the quality choice distortion by a profit maximizing firm. Our approach is useful for addressing public policy questions using standard intuition and comparative statics developed for one-dimensional economic models.

1   Introduction

Firms typically make multiple decisions, choosing combinations of prices, output quantities, product qualities, input purchases, marketing and sales efforts, R&D, and innovations. Economists have a good understanding of markets in which each firm makes a single decision, including both monopoly and oligopoly markets.[1] Does this understanding break down or continue to apply to markets in which firms make many decisions? We obtain conditions that are sufficient to transform an equilibrium of a market in which each firm makes multiple decisions into an equivalent equilibrium of a market in which each firm makes a single decision. We refer to such a decision

[1]Oligopoly models with single decisions include Cournot (1838), Bertrand (1883), Chamberlin (1933), Hotelling (1929), and Salop (1979).

as a *sufficient decision*. The intuition for our results is that a firm's decisions are connected by profit maximization, so that it is possible to identify a particular decision that serves as a proxy for the firm's other decisions. We show that sufficient decisions apply to a variety of standard models including monopoly and oligopoly in multi-sided markets and multi-product markets. Our approach applies very basic optimization techniques used in neoclassical economics. The analysis of sufficient decisions helps determine when policy prescriptions for markets in which each firm makes a single decision carry over to markets in which each firm makes multiple decisions.

A sufficient decision reduces the dimensionality of the space of actions to what is sufficient to address a particular economic question, such as examining the effects of competition or cost pass through. This transformation allows economists to apply familiar techniques and intuition to problems involving more complicated markets. The analysis of sufficient decisions helps extend standard empirical predictions to markets in which each firm makes multiple decisions.[2] The transformation of a multiple-decision market into a single-decision market is based on standard analysis of profit maximization. To illustrate the basic approach, consider the standard neoclassical profit-maximization problem of a price-taking firm with one output and multiple inputs. Deriving the firm's cost function reduces the multi-dimensional problem to the problem of choosing the profit-maximizing output. We focus on analyzing the sufficient decision of each firm choosing the number of transactions (alternatively, output or quantity) it facilitates. However, in the appendix we derive conditions for a more general sufficient decision.

Consider a firm with a profit function $\Pi(X, Z)$, where X is a vector of decision variables and Z is a vector of parameters. Our main finding is that for the purposes of profit, consumer surplus, and social welfare, one can instead analyze the problem of a firm with a profit function of $\pi(t, Z)$, where t is a sufficient decision (we focus on t as the number of transactions that the firm facilitates in main text). We extend our main finding to Nash equilibrium between n firms, where instead of firms with profit functions $\Pi_i(X_i, Z_i)$ for firm i we can analyze a Nash equilibrium of firms with profit functions $\pi_i(t_i, Z_i)$ instead.

Our finding implies that comparative statics with respect to any components of Z in the simplified problem $\pi(t, Z)$ are equivalent to the ones in the multi-decisional problem $\Pi(X, Z)$, with relevance both to our theoretical understanding of markets with firms that make multiple decisions as well as to empirical estimation. We show that all the first order approaches apply directly, for example the first order conditions (as long as the original $\Pi(X, Z)$ is strictly quasiconcave in X), the profit maximizing t^* is unique and can be found from $p'(t)t + p(t) - c'(t) = 0$, where $p(t)$ is the inverse demand function faced by the firm. Similarly, pass-through rate calculations for tax/subsidy incidence or Upward Pricing Pressure calculations for merger outcomes apply.[3]

Standard limitations encountered in one-decisional models apply to the shape of the inverse demand function $p(t)$ as well. As noted by Leibenstein (1950), demand functions might be upward sloping for products exhibiting non-additivity of demand: aggregate demand is not necessarily the

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[2] Our concept also is roughly analogous to that of sufficient statistics, which is a reduction of the dimensionality of the sample space for the purpose of estimating a parameter of the population distribution, see Chetty (2009).

[3] See, for example Weyl and Fabinger (2013) and Farrell and Shapiro (2010).

sum of all individual consumers' demands, leading to a possibility of an upward sloping demand, for example, in the case of Veblen goods. Given the standard first order condition, the possibility that demand might be upward sloping implies two conlcusions: in this case, optimal price is below marginal cost and monopolist might produce more quantity than would be produced in perfect competition. However, we show that in the example of a two-sided market monopolist discussed in Rochet and Tirole (2006) relatively weak conditions ensure that the demand curve of the simplified problem is indeed downward sloping.

For the purposes of consumer surplus and social welfare, the difference with the standard (one-decisional) setup is that generally there is a nonpecuniary externality imposed by the marginal consumer onto the inframarginal consumers through the firm's choice of components of X. In other words, the change in consumer surplus due to an additional unit of t is not fully internalized by the firm through additional revenue. However, such externalities are frequently present in one-decisional markets as well (for example, network effects for positive externalities or congestion effects for negative externalities), are related to the aforementioned demand non-additivity, and our setup simply highlights the general issues with externalities and distortions akin to the one in Spence (1975) that are frequently not even discussed in relevant economic models. Well-known externality findings, such as the fact that the presence of externalities might result in perfect competition not leading to a Pareto efficient outcome, apply here too.

Empirical estimation of markets where firms make multiple decisions currently presents its own set of challenges over and above estimating, say, cross-elasticities of single-product firms that choose their product's price. Reducing a model to its essential decisions may simplify empirical estimation. Effectively, the assumption of profit maximization allows one to use transactions as a sufficient statistic for all the decisions that a firm is making. To estimate the demand function of transactions we need the average revenue that firm derives per transaction, similarly to needing both price and quantity to estimate the standard demand function. Of course, all the standard demand estimation problems still remain.[4] Our point is only that estimating demand functions and equilibria when firms make multiple decisions is not harder than the single-decision market estimation. Thus, instead of having an equation to estimate for each first order condition of each firm in the market, the econometrician only needs to estimate a single first order condition per firm. The driving assumption is profit maximization. However, that assumption is already made in structural industrial organization papers either explicitly or implicitly, since that is the assumption that allows the econometrician to use the first-order condition based equations for estimating, for example, marginal cost, based on observed price and quantity combinations.

Such a simplification of empirical analysis may be useful if some of the individual product data is unavailable. For example, it might be straightforward for antitrust authorities or a researcher to collect data on the overall revenue of a firm (from the top-line accounting statements) and on the overall volume that is produced. However, detailed data on pricing and volume distributions across

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[4]Similarly, all the theoretical issues present in standard one-decisional markets remain, for example tipping and multiple equilibria. See Alexandrov (2015) and Hagiu and Spulber (2013) who show how one can overcome tipping via product differentiation and content.

dozens of products might be much more difficult to obtain. If the econometrician is already using first-order conditions to estimate a structural model, there is no loss in examining this aggregate data and applying a sufficient decision.

We analyze an example of two-sided markets in detail. Our results apply to other settings as well, for example, markets where firms have multiple products (including aftermarket products), nonlinear price schedules, choose quality in addition to quantity, and markets where firms are intermediaries and/or facilitate matching.

There are limitations to our approach. We use profit maximization to construct a black box that transforms many decisions of a firm into a single one. All the limitations are related to us not being able to go inside this black box. Consider oligopolistic competition in a two-sided market. Our approach allows us to apply any familiar first-order antitrust techniques, such as Upward Pricing Pressure, with ease. However, the effect of, for example in a case of nightclubs or paid dating sites, a government decree disallowing charging men and women different prices is not possible to calculate using our approach, at least without further modifications. Similarly, the effect of a decree requiring firms to stop nonlinear pricing would also be impossible to calculate using our model, at least without further modifications. Both of these exercises put a constraint on profit maximization, and effectively change our black box – something that our approach cannot deal with, at least as presently constructed. While either of these hypotheticals might not be pressing policy concerns right now, there are many markets and settings where constraints like these are indeed relevant, and in these cases it would make more sense to use models that explicitly account for many decisions by each firm. For example, one of the concerns of the two-sided market literature is the price structure: the relative prices on the two sides of the market. Our approach cannot recover the comparative statics with respect to that.

Examples of applications above did not require either empirical estimation or theoretical analysis of the underlying primitives as functions of particular decisions. While our transformation does not allow one to uncover these parameters easily, our argument is that often one does not need to. For example, our sufficient decision of transactions does not tell us whether one of the sides of a two-sided market is more competitive than the other. However, what we need to know to apply, for example, the standard antitrust techniques is the overall competitiveness that we can recover. Aside from other contributions, our sufficient decision presents a straightforward and a theoretically grounded way to gauge a market's overall competitiveness level. An interesting question for further research is going deeper into this aggregation and uncovering whether, for example, competitiveness on one dimension is sufficient to ensure competitiveness in a sufficient decision; however, we argue that this information is not needed for many decisions.

2   Markets as a function of a sufficient decision

Consider a general economic model consisting of a monopoly firm with profit $\Pi(X)$ and a set of consumers with consumer surplus function $B(X)$. The firm's profit can be decomposed into

revenue that the firm derives, $V(X)$, and the cost that the firm has to incur, $C(X)$. Accordingly, $\Pi(X) = V(X) - C(X)$.

Let $X \in \mathbb{R}^m$ be a vector of variables chosen by the firm consisting of prices, outputs, and/or product quality. The firm solves a multi-dimensional optimization problem

$$\max_{X \in \mathbb{R}^m} \Pi(X). \tag{M}$$

Generally, one of the variables in the firm's decision vector either is or can be expressed as the number of transaction that the firm facilitates (analogously, output or quantity). WLOG, denote this variable x_1. By introducing a constraint on the firm's problem, $x_1 = t$, we can reduce the firm's optimization problem to the choice of a scalar t. We will refer to t as the firm's *sufficient decision*. We discuss a more general version of sufficient decisions, $g(X) = t$, in the appendix.

One can think of this as the firm performing a two-step optimization process: first, maximize over $\{x_2...x_m\}$ for a given $x_1 = t$, and second, maximize over t. We effectively subsume the first step into the second step, and focus on the second step only, keeping in mind that the first step occurs as well.

Consider the firm's constrained optimization problem.

$$X^*(t) = \arg \max_{X \in \mathbb{R}^m} \Pi(X) \text{ subject to } x_1 = t. \tag{1}$$

Note that $X^*(t)$ does not have to be a function. Then, define the firm's profit as a function of the sufficient decision t,

$$\pi(t) = \Pi(X^*(t))$$

This allows us to consider the reduced-form problem in which the firm maximizes profit by choosing the scalar t,

$$\max_{t \in \mathbb{R}} \pi(t). \tag{M'}$$

Maximizing $\pi(t)$ over t in problem (M') produces the same answer as the original problem of maximizing (M) over X. Letting X^* be the argmax of (M) and letting $t^* = x_1^*$, it follows that $\pi(t^*) = \Pi(X^*)$. So, for the purposes of the firm's profit, we can focus on (M').

A more general approach to the presentation above is that with an appropriate cover of the domain $\mathbb{R}^m$, one can always decompose problem (M) as

$$\max_t \left(\max_{X \in r(t)} \Pi(X) \right), \tag{2}$$

where $\mathbb{R}^m = \cup_t r(t)$. We analyze the more general approach in the appendix, including the conditions needed to be satisfied.

Showing that the profit maximization problem is the same is not enough if we want to examine consumer surplus and social welfare as well. The concern is that if profit maximization is not unique – different vectors X^* could potentially result in different consumer surplus and social

welfare despite resulting in the same profit.

Assumption 1 (Continuity) *Let $\Pi(X)$ and $B(X)$ be continuous functions.*

Assumption 2 (Profit Strict Quasiconcavity) *The profit function $\Pi(X)$ is strictly quasiconcave.*

We now obtain our main result. The proposition derives cost and demand functions from the underlying profit and consumer surplus functions. Social welfare function is obtained by adding consumer surplus to profit.

Proposition 1 *If assumptions (1) and (2) are satisfied, then one can express profit, consumer surplus, and social welfare stemming from the original problem (M) as, respectively,*

$$\pi(t) = tp(t) - c(t), \tag{3a}$$
$$b(t) = u(t) - p(t)t, \tag{3b}$$
$$sw(t) = \pi(t) + b(t), \tag{3c}$$

where $p(t) = \frac{V(X^*(t))}{t}$ is the average revenue the firm derives per t, $c(t) = C(X^*(t))$ is the total cost of producing t, and $u(t) = B(X^*(t)) + p(t)t$ is the consumers' gross utility from t units. Each of these functions exists and is uniquely defined.

Proof. Given that Π is strictly quasiconcave, it is clear that $X^*(t)$ is unique. By Berge's Maximum Theorem, $X^*(t)$ is continuous. Thus, for example, $b(t) = B(X^*(t))$ is a continuous function of t. Similarly, we can derive functions p and c. Given functions p and b, there is a unique function u. ∎

The proposition above shows that the optimal pricing decision of a monopolist that makes multiple decisions can be characterized by standard cost and demand functions with the monopolist optimally choosing the number of transactions based on these cost and demand functions. Profit can be derived from these standard cost and demand functions and is equivalent to that derived using the full model with multiple decisions. While we do not know what the shape of these demand and cost functions is based on the primitives of the multi-decisional market, we can simply start with the derived functions as the primitives of the model or we can estimate them empirically (which is easier than it would have been to estimate the complete model where each firms makes multiple decisions). Due to this equivalency, we can apply familiar one-decisional results and intuition to markets where firms make multiple decisions. Thus, for example, the tax incidence results of Weyl and Fabinger (2013) extend to markets where firms make multiple decisions, and so do the results on demand rotations of Johnson and Myatt (2006). The overall price level, relevant for both consumer surplus and social welfare, behaves exactly like price does in a standard one-decisional market.

Methodologically, using just one of the variables as sufficient for overall profit maximization is a well-known technique in economics. We are not the first ones to note that analyzing optimal

decisions might simplify the multi-decisional problem. As Weyl (2010) observes, referring to a multi-sided optimal pricing model, "perhaps surprisingly, [adding] optimization simplifies the analysis." Note that this method of analysis is in line with well-developed literature on cost minimization. A price-taking firm's production function, $q = F(X)$ where q is a scalar output and X is a vector of inputs, is equivalent to the firm's cost function $c(q)$, with the production function and the cost function again connected by profit maximization.[5]

We first establish that the standard first-order conditions hold. Suppose we have a firm whose profit function satisfies all the conditions for Proposition 1.

Corollary 1 *There exists a unique t^* that maximizes $\pi(t^*)$. Moreover, this t^* is implicitly defined by*

$$\pi'(t^*) = p'(t^*)t^* + p(t^*) - c'(t^*) = 0. \tag{4}$$

Since the firm's maximization problem is a familiar one, we should explore the properties of the inverse demand function $p(t)$.

Corollary 2 *If revenue and cost functions of the original maximization problem ($V(X)$ and $C(X)$) are continuous and differentiable, then $p(t)$ is also continuous and differentiable.*

While the smoothness properties of the original maximization problem get passed onto $p(t)$, $p(t)$ does not necessarily possess the familiar property of downward sloping demand, in other words it is possible that $p'(t) > 0$. This effect arises due to the fact that the marginal consumer exerts an externality on the inframarginal consumers through the firm's choice of variables in X, for example the price structure in two-sided markets or quality in the model of Spence (1975).

We discuss this externality more extensively below; however, this externality is not due to the multidecisional markets, had been explored in the one-decisional literature, is intricately related to the non-additivity of demand, and predates most of the literature discussed in this paper. In particular, Leibenstein (1950) derived an upward sloping demand function for products exhibiting Veblen effects. With respect to non-additivity of demand in general, he wrote the following: "[b]oth Cunynghame and Pigou pointed out that Marshall's treatment of consumers' surplus did not take into account interpersonal effects on utility. Marshall seemed to feel that this would make the diagrammatical treatment too complex. Recently, Reder and Samuelson noticed that external economies and diseconomies of consumption may vitiate (or, at best, greatly complicate) their "new" welfare analysis, and hence, in true academic fashion, they assume the problem away."[6] However, we discuss further in the paper straight-forward conditions that ensure downward sloping demand in a two-sided model from Rochet and Tirole (2006).

Since the first order condition holds, standard first order approaches apply regardless of the aforementioned externalities and non-additivity. Consider the standard pass-through rate analysis. Instead of analyzing a pass-through matrix, we showed that it is sufficient to analyze the

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[5]See, for example, Mas-Colell, Whinston, and Green (1995).

[6]See also McClure and Kumcu (2008) deriving an upward sloping demand-like curve when consumers' utility depends on the level of quality chosen by the firm. Similarly, see Rohlfs (1974) deriving an upward sloping demand curve for network effects.

pass-through rate of the marginal cost of facilitating one more transaction onto the price of that transaction. Suppose we are analyzing a firm, to which we apply Proposition 1. Suppose the firm has a linear cost (for easier exposition) of facilitating transactions, $c(t) = kt$. To make the comparisons easier, instead of working with the inverse demand function of transactions, $p(t)$, we can work with the corresponding demand function (with a slight abuse of notation) $t(p)$.

Then, using Proposition 1, instead of analyzing the firm's vector of decisions X, and the pass-through matrix of k onto X, we can simply analyze the pass-through of k onto t, resulting in the well-known (see, for example, Weyl and Fabinger (2013))

$$\frac{\partial p^*}{\partial k} = \frac{1}{2 - t''(p)\frac{t(p)}{(t'(p))^2}}. \tag{5}$$

Similarly, consider UPP. Suppose we are analyzing two firms, to which we apply Corollary 4. Again, suppose the firms have linear costs of facilitating transactions: $c_i(t_i) = k_i t_i$ for firm i and, again to ease the comparison, let's work with the corresponding demand functions of $p_1(t_1, t_2)$ and $p_2(t_1, t_2)$. Then, if the firms were to merge, the first round overall tax on firm 1's transactions is

$$\overline{\tau_1} = t_{12}(p_2 - k_2), \tag{6}$$

where p_2 and k_2 are pre-merger per-transaction revenue and cost, and t_{12} is the diversion ratio between transactions of the two firms, defined by $\left\| \frac{dt_2}{dt_1} \right\|$ (the impact on transactions of firm 2 when p_1 falls by enough to facilitate one more transaction for firm 1). This is the exact same definition as in Farrell and Shapiro (2010), see their equation (1).

3 Application: Monopolist in a two-sided market

Consider the profit function equivalent to that in Rochet and Tirole (2006) and the setup used for most of the analysis in Weyl (2010):

$$\pi(p_1, p_2) = (p_1 + p_2 - c)D_1(p_1, p_2)D_2(p_1, p_2).$$

Following Weyl (2009), let the consumer surplus be

$$B(p_1, p_2) = D_2(p_1, p_2) \int_{p_1}^{\infty} D_1(z, p_2)dz + D_1(p_1, p_2) \int_{p_2}^{\infty} D_2(p_1, z)dz. \tag{7}$$

Consider the system of equations

$$q_1 = D_1(p_1, p_2), \tag{8a}$$

$$q_2 = D_2(p_1, p_2). \tag{8b}$$

This system of equations has a unique solution for p_1 and p_2 iff cross-price effects differ from

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own-price effects,[7]

$$\frac{\partial D_1(p_1,p_2)}{\partial p_1}\frac{\partial D_2(p_1,p_2)}{\partial p_2} \neq \frac{\partial D_1(p_1,p_2)}{\partial p_2}\frac{\partial D_2(p_1,p_2)}{\partial p_1}. \tag{9}$$

Assuming that is true, we can re-write the firm's profit as

$$\pi(q_1,q_2) = (p_1(q_1,q_2) + p_2(q_1,q_2) - c)q_1q_2. \tag{10}$$

Let the sufficient decision be the volume of transactions, $t = q_1q_2$, with the vector of firm's decisions being $X = \{t, q_2\}$. This allows us to apply Proposition 1. We know that there is a unique and continuous function $q_2^*(t)$, and therefore $p_1(t, q_2^*(t))$ and $p_2(t, q_2^*(t))$.

Corollary 3 (Two-sided Monopolist) *The firm's profit function, consumer surplus, and social welfare can be written as functions of the volume of transactions*

$$\pi(t) \ = \ t(\rho(t) - c), \tag{11a}$$

$$b(t) \ = \ u(t) - \rho(t)t, \tag{11b}$$

$$sw(t) \ = \ \pi(t) + b(t). \tag{11c}$$

where $\rho(t) = p_1(t) + p_2(t)$, $u(t) = B(p_1(t, q_2^*(t)), p_2(t, q_2^*(t))) + \rho(t)t$.

Instead of starting with the primitives of the market, this time $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$, we can start with $\rho(t)$ and treat this as a standard one-sided market. If lump-sum fees are an option, then $\rho(t)$ becomes average revenue per transaction, so $\rho(t) = \frac{Revenue(t)}{t}$.

We can establish simple conditions that ensure that $\rho(t)$ is downward sloping.

Proposition 2 *The demand function $\rho(t)$ is downward sloping in the setting of Rochet and Tirole (2006) if and only if*

$$\frac{\partial D_1}{\partial p_1}D_2 + \frac{\partial D_2}{\partial p_1}D_1 < 0 \tag{12}$$

at the optimum.

Proof. The proof is in the Appendix. ■

This condition is satisfied in most two-sided settings with positive cross-market externalities. One would expect that in such settings a price change on one of the sides of the market affects both sides in the same qualitative fashion (for example, a price increase on one side lowers demand on both sides). From equation (12), the demand from the second market might even increase in response to a price increase in the first market, and the overall demand $\rho(t)$ is still downward sloping.[8] This is a familiar comparative statics from network effects: many of the equilibirum

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[7]By Lemma 2 in Kellogg (1976), assuming an interior solution, see also Konovalov and Sandor (2010).

[8]Of course, the same condition has to apply from the other side (derivatives w.r.t. p_2): the two conditions are equivalent at the optimum prices.

uniqueness issues from the network effects literature like the ones in Katz and Shapiro (1985) are subdued, if not eliminated, by introducing sufficient relatively inelastic own-price demand by, for example, a sufficient amount of product differentiation.[9]

Our results raise an issue of whether these markets are actually two-sided. Rochet and Tirole (2006) state "The market for interactions between the two sides is one-sided if the volume V of transactions realized on the platform depends only on the aggregate price level $a = a^B + a^S$, i.e., it is insensitive to reallocations of this total price a between the buyer and the seller. If by contrast V varies with a^B while a is kept constant, the market is said to be two-sided." The only reason why we can focus on, for example, the sum of the prices as a sufficient decision is because we are implicitly conditioning on firms maximizing profit. In other words, if a firm maximizes profit, then the firm will not vary a_B when it's already at the optimal a. Rochet and Tirole's two-sided market definition is satisfied in our model: if a firm reallocates some of the total price while keeping the total price the same, the number of transaction changes. We are simply saying that in equilibrium (or at the optimum), the firm has no reason to do that. Weyl (2010) emphasizes the importance of different dimensions of consumer heterogeneity in two-sided markets. In our setup, any consumer heterogeneity will manifest itself in the shape of the derived demand functions faced by the competitors.

Note that our approach does not mean that previous literature is in any way incorrect. For example, Wright (2004) suggests that "using conventional wisdom from one-sided markets in two-sided settings" leads to various "fallacies." We agree that each of Wright (2004)'s eight statements is a fallacy. Our results are a complement to the two-sided markets literature and suggest that careful application of conventional wisdom in one-sided settings can generate further insights into how two-sided markets work.

Other studies have explicitly tried to uncover when two-sided markets effectively collapse into a one-sided analog. Rozanski and Thompson (2011) examine agricultural markets, and use the farm-to-retail spreads to analyze buyer power.[10] Gans and King (2003) outline the conditions for the interchange fees to be neutral in a payment cards market, although neutrality differs from the question of sufficiency of decisions.[11]

4 Strategic interaction

4.1 General formulation

Our results apply to competitive settings as well, where each firm makes multiple decisions. Antitrust analysis becomes straight-forward to perform, by using the standard techniques. Oligopolistic competition implications also become clear, with a straight-forward and theoretically grounded

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[9]See Anderson, De Palma, and Thisse (1992) and Alexandrov (2015).

[10]Also see Marx and Shaffer (2008) for buyer power argument in a similar context.

[11]Also see Prager, Manuszak, Kiser, and Borzekowski (2009) for a discussion of interchange fee neutrality in payment card markets.

ways to define competitiveness, markup, or whether firms' strategies are strategic complements or substitutes.[12]

Corollary 4 (Strategic Interaction) *Consider the game $(A_i, \pi_i; i \in \mathbb{N})$ with n players, where $A_i \in \mathbb{R}^m$, convex and compact, is the (pure) strategy space of player i and $\pi_i : A_1 \times ... \times A_n \to \mathbb{R}$ for each i are profit functions translating actions into payoffs. Let $t_i = A_{1i}$, $t_i \in \mathbb{R}$. If assumptions (1) and (2) are satisfied then there exists a Nash equilibrium and the game is equivalent to the game $(\mathbb{R}, \Pi_i; i \in \mathbb{N})$ with n players, where the (pure) strategy of player i consists of choosing $t_i \in \mathbb{R}$ and $\Pi_i : \mathbb{R}^n \to \mathbb{R}$ for each i are profit functions translating t into payoffs.*

Nash existence follows from Tarski's theorem, see for example Vives (2001). Since this is a normal form game, the corollary also encompasses multi-stage games.

4.2 Oligopolistic competition in two-sided markets

We can extend the monopolist firm in a two-sided market setup to an oligopolistic setup where firm i's profit function is

$$\pi_i(X) = (p_{1i} + p_{2i} - c_i) \times D_{1i}(p_{1i}, p_{2i}, p_{1j}, p_{2j}) \times D_{2i}(p_{1i}, p_{2i}, p_{1j}, p_{2j}), \tag{13}$$

Under a condition similar to (9), Proposition 1 applies in an analogous manner. A corollary similar to Corollary 3 applies in this case as well. The resulting inverse demand function for each firm, $\rho_i(t_i, t_j)$, incorporates the competitiveness of firms on both sides of the market, as well as whether their strategies are strategic complements or strategic substitutes on both sides of the market. It is natural to suppose that if one of the sides is particularly competitive, or in other words if D_{1i}'s are particularly cross-elastic, that will translate into $\rho_i(t_i, t_j)$ also being cross-elastic. It is not clear how the cross-elasticities on both sides interact to generate this derived inverse demand, $\rho_i(t_i, t_j)$, and we leave this question for future research. However, we believe that for many applications this is not an important question. It is easier, and all the standard one-sided intuition applies, if we start our analysis of the market by treating $\rho_i(t_i, t_j)$ as the primitive. As we argue above, $\rho_i(t_i, t_j)$ is also easier to estimate empirically.

Our analysis suggests that standard intuition can guide antitrust policy for two-sided markets. If antitrust authorities are to do an inquiry into a particular practice of an online platform, what would the procedure and the theoretical underpinnings of that inquiry be? For example, the issue might be the online platform favoring its own web pages or products while aggregating information in response to a consumer's query. Just like newspapers, online platforms are in the market of connecting consumers with companies that advertise. Each time that a consumer is exposed to an

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[12]See Werden (1998) and Farrell and Shapiro (2010) on merger review and competitiveness and Bulow, Geanakoplos, and Klemperer (1985) on strategic complements and substitutes. Suppose that the original game is firms choosing prices. We are not transforming the game in question into a strategic substitutes Cournot-like game from a strategic complements Bertrand-like game, despite focusing on transactions. The firms are still deciding on prices, and those could be strategic complements, and that fact will be reflected in the derived demand functions.

ad is a transaction between that consumer and the firm that advertised. The price on the consumer side happens to be zero. However, the overall volume of transactions is a sufficient decision. The overall revenue per transaction corresponds to price in a standard one-decision market. The online platform's overall markup is then the price received from the advertising firm, together with the price received from the consumer (zero), less the marginal cost incurred. It would be hard to examine this market using standard tools otherwise, in particular because many aspects of the online experience are free to consumers, and the online platform then derives no markup from that side of the market. However, using our metric of transactions, one can apply the standard Upward Pricing Pressure and the Critical Loss Analysis tools, and define the market in a familiar way.

However, if one is interested in effects on a particular side of the market, the standard methodology that is already described in the literature applies. See, Filistrucchi (2008) derives SSNIP for two-sided markets and Affeldt, Filistrucchi, and Klein (2013) derive UPP directly from the two-sided model of Rochet and Tirole (2003) and provide an empirical application. Emch and Thompson (2006) argue that in the context of credit cards, the SSNIP test should be applied to the sum of prices on both sides.

4.3   Intermediaries and dealers

Alexandrov, Deltas, and Spulber (2011), building on Stahl (1988) and Spulber (1999), examine intermediaries (dealers) a two-sided Salop model, and lay out a similar intuition for symmetric oligopolistic competition in that particular model, applying it to market definition and critical elasticity concepts from the antitrust literature.[13]

More generally, Proposition 1 holds for the following oligopoly model. Suppose there are n firms (dealers) that produce m_{inputs} interchangeable inputs and resell them to $m_{outputs}$ demand markets. The firms can price (and wage) discriminate between all markets. Each firm's supply and demand functions for each market, $D_{ij}(P_i, W_i, P_{-i}, W_{-i})$, firm i's demand in demand market j, and $S_{ij}(P_i, W_i, P_{-i}, W_{-i})$, firm i's supply from input market j, can depend not only on its own prices and wages, but also on competitors' prices and wages. This model results in firm i maximizing the following profit function:

$$\pi_i(P_i, W_i, P_{-i}, W_{-i}) = \sum_{j=1}^{m_{outputs}} (p_{ij} - c_{ij})D_{ij}(P_i, W_i, P_{-i}, W_{-i}) - \sum_{j=1}^{m_{inputs}} w_{ij}S_{ij}(P_i, W_i, P_{-i}, W_{-i}) \quad (14a)$$

$$\text{s.t.} \quad \sum_{j=1}^{m_{outputs}} D_{ij}(P_i, W_i, P_{-i}, W_{-i}) \leq \sum_{j=1}^{m_{inputs}} S_{ij}(P_i, W_i, P_{-i}, W_{-i}) \quad (14b)$$

where $P_i \in \mathbb{R}^{m_{outputs}}$ and $W_i \in \mathbb{R}^{m_{inputs}}$ for each i and $D_{ji}(\bullet)$ and $S_{ij}(\bullet)$ are monotone, continuous, and differentiable for each i and j. Also, denote the joint surplus of suppliers of the interchangeable

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[13]Alexandrov, Deltas, and Spulber (2011) also show how a similar framework can be applied to yield results akin to competitive bottlenecks in Armstrong (2006). Also see Loertscher (2007) and Reisinger and Schnitzer (2012) for two-sided spatial models.

inputs and of consumers from the demand markets as $B(P, W)$, where P and W are vectors of all prices and wages in the market (for example, vector P is of dimensionality $n \times m_{outputs}$).

Conditional on rivals' strategies, no firm gains from generating excess demand or excess supply, in other words the inequality in the profit function binds.[14] Define $t_i \equiv \sum_{j=1}^{m_{outputs}} D_{ij}(P_i, W_i, P_{-i}, W_{-i})$: the number of transactions that firm i facilitates. As long as demand and supply functions satisfy a condition similar to (9), vectors P_i and W_i can be expressed as functions of T (the vector of each firm's t), and thus Proposition 1 and Corollary 4 apply, and it follows that

Corollary 5 (Oligopoly Dealers) *Firm i's profit function, consumer surplus, and social welfare can be written as functions of the volume of transactions chosen by each firm in equilibrium*

$$\pi_i(T) = t_i \rho_i(T) - C_i(T), \tag{15a}$$

$$b(T) = u(t) - \sum_{k=1}^{n} \rho_k(T) t_k, \tag{15b}$$

$$sw(T) = \pi(T) + b(T), \tag{15c}$$

where $\rho_i(t) = \frac{1}{t} \left(\sum_{k=1}^{m_{outputs}} P_{ik}(T) D_{ik}(T) - \sum_{k=1}^{m_{inputs}} W_{ik}(T) S_{ik}(T) \right)$ and $u(t) = B(P(T), W(T)) + \sum_{k=1}^{n} \rho_k(T) t_k$.

Dealer i's inverse demand function depends on T, the vector of each dealer's transactions, thus yielding the standard differentiated Cournot formula in sufficient decisions.

5 Product quality and Spence's distortion

It is instructive to analyze the example of a firm choosing quality from Spence (1975). Keeping with the notation in that paper, a firm faces an inverse demand curve of $P(x, q)$ and a cost curve $C(x, q)$, where x is quantity and q is quality. Consumer surplus is then $S(x, q) = \int_0^x P(z, q) dz - xP(x, q)$. Let the sufficient decision be $t = x$. Proposition 1 applies, resulting in the following corollary.

Corollary 6 (Spence) *The firm's profit function, consumer surplus, and social welfare can be written as functions of the volume of transactions*

$$\pi(t) = t(p(t)) - c(t), \tag{16a}$$

$$b(t) = u(t) - p(t)t, \tag{16b}$$

$$sw(t) = \pi(t) + b(t). \tag{16c}$$

where $p(t) = P(t, q^*(t))$, $c(t) = C(t, q^*(t))$, and $u(t) = S(t, q^*(t)) + p(t)t$.

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[14]Suppose there is either excess demand or excess supply. Then, due to continuity of the profit function, it is possible to either increase some of the prices or decrease some of the wages and still maintain the same quantity, thus increasing profits.

From the corollary, it appears that the Spence distortion, the fact that the quality is set to reflect the preferences of the marginal consumer instead of the average consumer, is not present. However, that is not the case. The specification in the corollary (and in Proposition 1) allows for $u(t) \neq \int_0^t p(x)dx$ or, in other words, for

$$u'(t) \neq p(t). \tag{17}$$

While we are used to the inverse demand function to be marginal utility in the standard one-product one-characteristic model, this does not have to be the case. It is easy to see this in the case of Spence (1975). In this case, marginal utility also encompasses the fact that an additional unit of output also induces the firm to change the quality of the product, which in turn also influences consumer utility. More generally, this is the case for markets where firms make decisions other than quantity produced, with the exception of knife-edge cases. So, in particular, this is the case for quality as observed in Spence (1975) and in two-sided markets, Weyl (2010).

One of the effects of this distortion is that since $u'(t) \neq p(t)$, perfect competition $p(t) = c'(t)$ does not necessarily result in a Pareto optimal outcome.[15] Even in perfect competition, the firms have an incentive to distort the relative magnitudes of their decisions. Thus, even if the quantity is optimal, the firms have an incentive to distort quality in a suboptimal way from the social welfare perspective.[16] Similarly, even if the total price level is equal to the marginal cost in a two-sided market, the firms generally have a different incentive in terms of price structure (p_1 vs p_2) than what would lead to a Pareto optimal outcome.

There are, effectively, two parts to this distortion. The first part is that for any given quantity (or price) level, the firms have an incentive to pick a suboptimal allocation of resources. Almost needless to say, this is not relevant for single-decisional markets (there is nothing to allocate aside the single decision). The second part is that firms cannot fully internalize the marginal utility ($u'(t) \neq p(t)$), and therefore the second-best (conditional on the distortion described above) price level can be either below or above the marginal cost. The second-best price level is below the marginal cost when an extra unit of output makes inframarginal consumers better off. This is the case, for example, in two-sided markets with positive cross network effects or any positive externality from additional consumption. Conversely, negative externalities results in the second-best price level above the marginal cost. This is consistent with Spence (1975): the key variable in that analysis was the sign of $\frac{\partial^2 P(x,q)}{\partial x \partial q}$ or, in other words, the effect of additional consumption on inframarginal consumers' enjoyment of quality.

Note that firms making decisions aside from quantity produced is neither necessary nor sufficient for this distortion to appear. For example, consider a standard one-sided network effects model, see Katz and Shapiro (1985). The same distortion arises: the firm cannot internalize the marginal utility of the inframarginal consumers from marginal output. Similarly, any externality produces

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[15]Even aside from the excessive entry incentives with sunk costs described in Mankiw and Whinston (1986).

[16]As noted in Spence (1975), "this aspect of market failure has very little to do with monopoly. It is, rather, a result of the fact that price signals carry marginal information, while averages or totals are required in locating the optimum. Any profit-oriented supply side runs into similar difficulties." See also footnote 4 in Spence (1975).

the same result. As in the classic externality models, if consumers had no transaction costs in dealing with each other, Coase theorem would apply, $u'(t) = p(t)$, and perfect competition would result in a Pareto optimal outcome, see Coase (1960).

On the other hand, markets where firms make multiple decisions do not necessarily have these externalities. Consider an intermediary without network effects, such as a retailer: the intermediary transfers units from suppliers (producers) to consumers, while competing for both. The intermediary can internalize all the marginal utility of an additional unit transfered, which is the utility of the marginal consumer less the outside option of the marginal supplier. Therefore, that market is exactly the same as a standard one-decisional market, $u'(t) = p(t)$, and perfect competition leads to a Pareto optimal outcome. The same situation arises in a matching market where the firm facilitates matching, and consumers care only about the quality of their match, not about how many people are on the other side of the market. This is true regardless of whether consumers engage in Nash bargaining upon being matched, see Alexandrov et al. (2011) for more.

Finally, it is worth noting that $u'(t) \neq p(t)$ does not affect the applicability of first-order approaches, such as the upward pricing pressure. These approaches are based on the firms' first-order conditions of profit maximization, and do not attempt to recover the magnitude of the impact on consumer surplus. Of course, the downsides of these approaches remain: they only consider the local first-order incentives; however, see Jaffe and Weyl (2013) showing how to extend the upward pricing pressure approach to derive changes in consumer surplus as well.

Interestingly, one can derive another set of inverse demand and cost-like functions, p_a and c_a such that $u'(t) = p_a(t)$, even if $u'(t) \neq p(t)$.

Proposition 3 *If assumptions (1) and (2) are satisfied, then one can express profit and consumer surplus stemming from the original problem (M) as*

$$\pi(t) = p_a(t)t - c_a(t), \tag{18a}$$

$$b(t) = \int_0^t p_a(z)dz - p_a(t)t, \tag{18b}$$

$$sw(t) = \pi(t) + b(t), \tag{18c}$$

where functions $p_a(\bullet)$ and $c_a(\bullet)$ exist and are uniquely defined up to a constant.

Proof. As in the proof of Proposition 1, by Berge's Maximum Theorem $X^*(t)$ is continuous. Thus, $b(t) = B(X^*(t))$ is a continuous function of t.

Define $y(t) \equiv \int_0^t p_a(x)dx$. Then, $y'(t) = p_a(t)$. We can re-write the consumer surplus equation in the proposition above as

$$y'(t) - \frac{1}{t}y(t) + \frac{1}{t}b(t) = 0. \tag{19}$$

This is a first order linear differential equation and $b(t)$ is continuous, and thus there exists a solution for $y(t)$. However, the only reasonable initial condition is $y(0) = 0$, but $t = 0$ is not in the domain. Therefore, the uniqueness of $y'(t)$ is defined only up to a constant.

This implies that there is a unique (up to a constant) $p_a(t)$ as well. From the profit function, $\pi(t) = tp_a(t) - c_a(t)$, we know $\pi(t)$ from (1) and we have just found $p_a(t)$. Thus, $c'_a(t)$ is uniquely defined (up to the same constant) as well. ■

Note that these functions are defined up to a constant: if one imagines the standard supply-demand graph from Micro 101, it is clear that moving the x-axis either up or down does not affect anything aside from the nominal price level. The upside of this derivation is that the second-best price level is equal to the marginal cost. The downside is that these functions are not inverse demand and cost in the usual sense: for example, in a two-sided market with positive cross-externalities c_a is a negatively-valued function even when there is no production cost and p_a does not correspond to the actual revenue per transaction that the firm is receiving (with any choice of the constant). Of course despite that the proposition shows that the overall profit function is the correct one.

6   Conclusion

We presented conditions sufficient to express a market where each firm makes many decisions as a market where each firm makes just one decision. Examples of markets that can be expressed this way are two-sided markets, intermediaries, matchmakers, firms selling product lines or involved in nonlinear pricing, and firms involved in aftermarkets.

The idea of looking at transactions between the sides as a sufficient statistic in multi-decisional markets is not new. One of the first U.S. Supreme Court cases clarifying the Sherman Antitrust Act, the Chicago Board of Trade v. United States (1918), considered what we would now call a platform (the Chicago Board of Trade, where buyers and sellers would get together to trade commodities).[17] The court ruled that the Board disallowing after hour trading was not anticompetitive. The reasoning was that the Board's action effectively expanded the market, increased the number of transactions, and thus benefited the market welfare. Coming back to more modern time, in the academic two-sided market literature, Rochet and Tirole (2006) discuss 'interactions,' that are effectively the volume of transactions from *Chicago Board of Trade*.[18]

Our analysis of a sufficient decision allows the application of standard intuition and techniques from markets where firms make one decision to consideration of markets where firms make multiple decisions. For example, a merger review in this market becomes straight-forward: transform the market into the single-decision equivalent and proceed as usual with the standard single-decision formulas, such as the small but significant and non-transitory increase in price (SSNIP) test, upward

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[17]246 U.S. 231.

[18]From Rochet and Tirole (2006), "The interaction can be pretty much anything, but must be identified clearly. In the case of videogames, an interaction occurs when a buyer (gamer) buys a game developed by a seller (game publisher), and plays it using the console designed by the platform. Similarly, for an operating system (OS), an interaction occurs when the buyer (user) buys an application built by the seller (developer) on the platform. In the case of payment cards, an interaction occurs when a buyer (cardholder) uses his card to settle a transaction with a seller (merchant). The interaction between a viewer and an advertiser mediated by a newspaper or a TV channel occurs when the viewer reads the ad. The interaction between a caller and a receiver in a telecom network is a phone conversation and that between a website and a web user on the Internet is a data transfer."

pricing pressure (UPP), and diversion ratios.[19] Also, we know that the markup each firm enjoys on its transactions decreases as the market becomes more competitive as long as some conditions on that unique demand function of t are satisfied.[20] Similarly, one can perform any tax incidence exercises in a straight-forward manner. In short, analyzing solely this sufficient decision of each firm, as opposed to the many underlying decisions, allows one to apply all the familiar intuition and techniques from standard one-decision markets to the multi-decisional markets either while doing research (either empirical or theoretical), teaching, or informing the policy makers.

A closely related question left unanswered by our research is what occurs when firms face constraints on maximizing their profit: for example, inability to price discriminate or a price ceiling on the aftermarket product. We believe that this is an important avenue for future research.

Finally, we show that the distortion of a profit-maximizing firm choosing quality described by Spence (1975) is effectively a special case of a distortion that would arise in any setting where the firm cannot fully internalize the consumers' marginal utility from an additional unit of output through price. Firms making many decisions is neither necessary nor sufficient for this distortion to arise. This distortion does not affect the applicability of first-order approaches such as UPP.

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[19]See, for example, Werden (1998) and Farrell and Shapiro (2010).
[20]See Chen and Riordan (2008) and Weyl and Fabinger (2013) for the conditions.

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Appendix A: general sufficient decision

Instead of focusing on some x_1, we can focus on a more general version of a sufficient decision: $g(X)$. Analogously to the main text, by introducing a constraint on the firm's problem, $g(X) = t$, where $g : \mathbb{R}^m \to \mathbb{R}$, we can reduce the firm's optimization problem to the choice of a scalar t. We continue to make the assumptions on continuity and strict quasiconcavity: assumptions (1) and (2).

We now consider the firm's constrained optimization problem.

$$X^*(t) = \arg \max_{X \in \mathbb{R}^m} \Pi(X) \text{ subject to } g(X) = t. \tag{20}$$

Let $t^* = g(X^*)$. It follows that $\pi(t^*) = \Pi(X^*)$. So, for the purposes of the firm's profit, we can focus on the constrained maximization problem. We have to make an additional assumption on the constraint function g.

Assumption 3 (Regularity) *The constraint function can be expressed as $g(X) = k \left[\sum_{i=1}^m h^i(X) \right]$, where k is differentiable and strictly monotone, h^i is differentiable for each i, and $J \equiv \left| \nabla h^i \right| \neq 0$.*

The regularity assumption is related to constraint qualification conditions for a standard Karush-Kuhn-Tucker problem. The regularity assumption holds if the contraint function (g) is linear,

which occurs in many applied economic problems. Also note that if k is linear, in other words $g(X) = \sum_{i=1}^{m} h^i(X)$, then J is a linear approximation of the constraint.

Define correspondence $r : \mathbb{R} \rightrightarrows 2^{\mathbb{R}^m}$, s.t. $X \in r(t)$ iff $g(X) = t$.

Lemma 1 *If assumptions (1), (2), and (3) are satisfied, then the correspondence r is continuous (upper and lower hemi continuous) and for each $t \in \mathbb{R}$, $X^*(t)$ is a function (unique).*

Proof. We will use substitution by trying to express the problem as maximizing Π with respect to a vector D, where $d_i = h^i(X)$ for each i. By Lemma 2 in Kellogg (1976), assuming an interior solution (see also Konovalov and Sandor (2010)), system of equations

$$h^i(X) = d^i, \text{ for } i = 1...m \tag{21}$$

has a unique solution iff $\left| \nabla h^i \right| \neq 0$. That also implies that the system is invertible, and thus function $H(X) = D$ defined by the system is strictly increasing (technically, it is monotone, but could be made increasing with further appropriate substitutions), and it is clear that it is continuous. Thus, the inverse is strictly increasing and continuous as well. A strictly quasiconcave function of a strictly increasing function is strictly quasiconcave, thus $\Pi(X(D))$ is strictly quasiconcave.

Equation $k \left(\sum_{i=1}^{m} h^i(X) \right) = t$ can be transformed into $\sum_{i=1}^{m} d^i = k^{-1}(t)$ since $k(\bullet)$ is strictly monotone. At this point g can be transformed into a linear function, trivially resulting in a continuous correspondence r.

Finally, note that strict quasiconcavity of Π together with linearized g implies unique $X^*(t)$. $\blacksquare$

Note that condition $\left| \nabla h^i \right| \neq 0$ is encountered relatively frequently in economics. For example, if functions h^i are demand functions, then this condition is similar to the standard equilibrium stability condition for differentiated Bertrand price competition, for example see Vives (2001).

Proposition 4 *If assumptions (1), (2), and (3) are satisfied, then one can express profit, consumer surplus, and social welfare stemming from the original problem (M) as, respectively,*

$$\pi(t) = tp(t) - c(t), \tag{22a}$$
$$b(t) = u(t) - p(t)t, \tag{22b}$$
$$sw(t) = \pi(t) + b(t), \tag{22c}$$

where $p(\bullet) = \frac{V(X^*(t))}{t}$ is the average revenue the firm derives per t, $c(t) = C(X^*(t))$ is the total cost of producing t, and $u(t) = B(X^*(t)) + p(t)t$ is the consumers' gross utility from t units. Each of these functions exists and is uniquely defined.

Proof. By Lemma 1 correspondence r is continuous (upper and lower hemi continuous) and for each $t \in \mathbb{R}$, $X^*(t)$ is a function (unique).

Since $X^*(t)$ is unique, by Berge's Maximum Theorem, $X^*(t)$ is continuous (since an upper hemi continuous function with the domain of $\mathbb{R}^n$ is continuous). Thus, $b(t) = B(X^*(t))$ is a continuous function of t.

Similarly, we can derive p and c. Given p and b, there is a unique u. ∎

Define

$$\mathcal{L}(\lambda, x_1, ..., x_m) = \pi(X) + \lambda(g(X) - t), \tag{23}$$

in other words, the standard Lagrangean for the constrained optimization problem that defines equation (1). If the bordered Hessian matrix associated with $\mathcal{L}$ is negative definite, then $X^*(t)$ is a function.

A less strict condition that would be sufficient to ensure that our assumptions are satisfied is a condition on the multivariable equation $g(X) = t$. That condition would ensure that the correspondence between t and the set of solutions $(X^*(t))$ of the equation is both upper and lower hemi-continuous. Unfortunately (and, perhaps, surprisingly), the mathematical literature is still in search of general necessary conditions to ensure this.

More generally (regarding both g and π functions), the problem is the one of aggregation functions (see, for example, Grabisch, Marichal, Mesiar, and Pap (2011)): expressing an n-dimensional function as one-dimensional. The fact that we can outline relatively simple sufficient conditions is because our case is simpler than the general one: we need to aggregate only for the purposes of finding consumer surplus (and thus social welfare).

Note that assumption (3) is also necessary for an oligopoly corollary and a proposition deriving alternate forms of inverse demand and cost-like functions similar to the ones in the main text.

Appendix B: proof of the law of demand conditions in Rochet and Tirole (2006) model

Consider the first order condition w.r.t. p_1 if firm's profit is $\Pi(p_1, p_2) = (p_1+p_2-c)D_1(p_1, p_2)D_2(p_1, p_2)$, dropping unambiguous subscripts and arguments:

$$(p_1 + p_2 - c)\left(\frac{\partial D_1}{\partial p_1}D_2 + \frac{\partial D_2}{\partial p_1}D_1\right) + D_1 D_2 = 0. \tag{24}$$

It is clear that $p_1 + p_2 - c > 0$ iff the condition in Proposition 2 is satisfied. Define $\rho(t) \equiv p_1(t) + p_2(t)$, where $t = D_1 D_2$. In other words, $\rho(t)$ is the inverse demand function of the one-decisional problem $\pi(t)$. Now that we know that $\rho - c > 0$ iff the condition is satisfied, it is clear that the first order condition of the aggregated problem, $\rho'(t)t + \rho(t) - c = 0$ and the fact that $t > 0$ imply that $\rho'(t) < 0$.