Chapter 1
The Basic Intertemporal Model*

Carlos A. Végh
University of Maryland and NBER
E-mail: vegh@econ.umd.edu

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1 Introduction

What is the fundamental difference between a closed and an open economy? Consider first a pure exchange economy (i.e., an economy with no production). A closed economy (i.e., an economy that does not trade in goods and assets with the rest of the world) is forced to consume its own endowment. Consumption will thus be high in times of plenty and low in times of distress. Even though it would be socially desirable, the closed economy has no means of saving in "sunny days" to support higher consumption in "rainy days." In contrast, an open economy can borrow from the rest of the world during bad times and repay in good times. In other words, an open economy can engage in *intertemporal trade*. By borrowing and lending from the rest of the world, an open economy can completely sever the link between today's consumption and today's endowment. In particular, it may choose to fully smooth consumption in spite of a fluctuating endowment path. The ability of an open economy to engage in intertemporal trade is thus at the core of modern open economy macroeconomics.

Consider now a production economy. The ability to invest does give a closed economy some ability for transferring resources over time.[1] However, the closed

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[1] Though this ability is limited compared to the pure exchange open economy described above. Think about the closed economy having a linear technology. While this allows the economy to transfer resources over time in good times (i.e., it lends to itself), it cannot borrow from itself in bad times (assuming, of course, that investment is irreversible). Even

economy is constrained to invest only what it saves. Hence, to take advantage of a profitable investment opportunity, the closed economy is forced to save by reducing consumption. In contrast, by running current account imbalances (i.e., by borrowing from or lending to the rest of the world), an open economy may choose to invest more or less than what it saves. This allows the open economy to, once again, sever all ties between today's consumption (i.e., saving decisions) and investment. An open economy can thus borrow from abroad to finance a profitable investment opportunity without the need to reduce consumption to generate domestic saving.

Given the central role of the current account in open economy macroeconomics, it comes as no surprise that the analysis of the determinants of the current account constitutes the core of open economy macroeconomics. In the early days of open economy macroeconomics, current account determination was essentially viewed as a static (i.e., one period) problem and limited to the determination of the trade balance. The emphasis was thus put on relative prices as the key determinant of a country's trade balance.[2] However, as Sachs (1981, p. 212) aptly put it, "[a] one period theory of the current account that describes a static balance of import and exports makes as much sense as a one-period theory of saving and investment." Saving is, by definition, an intertemporal decision whereby an agent is willing to sacrifice consumption today for greater future consumption. Investment is, of course, just the other side of the coin. A static analysis is thus fundamentally flawed.[3]

Since saving and investment are *intertemporal* choices, the natural conceptual framework to analyze them is in the context of an intertemporal model (i.e., a multi-period model). As a result, intertemporal models of the current account (pioneered by Sachs (1981)) constitute the foundation of modern open economy macroeconomics. Further, intertemporal models have proved to be much more than a natural and elegant apparatus and have delivered results that critically depend on the intertemporal dimension of the analysis. The more relevant example is the very different reaction of the current account to permanent and temporary shocks, which a static model would completely miss.

If anything, current account determination is even more critical for developing countries because they typically face larger shocks, which implies that they should rely more on current account imbalances to smooth consumption over time. It is only natural, therefore, that our journey into the macroeconomic world of developing countries should start by a detailed analysis of the modern intertemporal approach to the current account. Methodologically, as well, this model provides the foundation for the rest of the book.

To isolate the basic ideas, Section 2 abstracts from investment and develops

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with irreversible investment, the fact that the capital stock cannot become negative would impose a constraint on "borrowing" in bad times.

[2] Obstfeld (1987) offers an insightful account of the evolution of ideas in international finance.

[3] Interestingly, while the analysis of an individual's saving and investment decisions in a closed economy was rightly casted in an intertemporal context by Irving Fisher in 1930, it took the economics profession roughly 50 years to take the seemingly obvious step of extending Fisher's analysis to an open economy.

the basic intertemporal model in the context of an endowment economy. Section 3 then derives the central result of this chapter: consumers choose to keep consumption flat over time regardless of the path of output. To achieve perfect consumption smoothing, consumers borrow in bad times and lend in good times. The trade balance thus acts a shock-absorber, improving in good times and worsening in bad times. The critical assumptions behind this result are perfect capital mobility and no intertemporal distortions, both of which will be relaxed in subsequent chapters. Section 4 then turns to the economy's response to unanticipated and negative output shocks. If the output shock is permanent, the economy adjusts immediately by reducing consumption. If the shock is temporary, however, the economy runs a current account deficit (i.e., borrows from abroad) during bad times to keep consumption constant over time. In other words, *the economy adjusts to permanent shocks but finances temporary ones*. Since there are no distortions in this economy, such response is socially optimal. Hence, from a normative point of view, the model yields the key policy dictum that an open economy should adjust to a permanent shock (i.e., reduce consumption) but finance a temporary one (i.e., borrow from abroad). This provides a theoretical rationale for efforts aimed at ensuring that developing countries have access to external finance during bad times (either through well-functioning international capital markets or through multilateral financial organizations such as the International Monetary Fund).

A key prediction of the basic model is thus that the trade balance and the current account are procyclical; in other words, they improve in good times and worsen in bad times. In practice, however, the trade balance behaves countercyclically (it worsens in good times and improves in bad times). This stylized fact provides the main motivation for introducing investment into the model in Section 5. After all, we expect investment to increase in good times and fall in bad times. Since the current account is the difference between saving and investment, changes in investment should, all else equal, lead to countercyclical changes in the trade balance and current account. Hence, the cyclical behavior of the external accounts should depend on the relative strength of the saving and investment effects. The results in Section 5 make clear this intuition by showing that the response of the current account to a, say, positive productivity shock depends on the duration of the shock. The longer the shock, the smaller the saving effect and the more likely that the investment effect will dominate, leading to a deterioration in the current account (countercyclical external accounts). The model can thus be made consistent with the stylized facts.

Section 6 contains concluding remarks. Several appendices contains proofs or extensions of material in the main text. The chapter closes with several exercises that expand on points made in the main text and should be viewed as an integral part of the chapter.

2 The model

Consider a small open economy inhabited by a large number of identical, infinitely-lived consumers. There is no uncertainty. Consumers are blessed with perfect foresight. There is only one (tradable and non-storable) good. Since the economy is small in good markets, it takes the price of the tradable good as given by the rest of the world. The economy is endowed with a flow of the good (i.e., there is no production). There is no government. Capital mobility is perfect in the sense that consumers can borrow/lend in international capital markets as much as they wish (subject to a solvency constraint to be discussed below) at an exogenously-given real interest rate, r_t. We will assume that r_t is constant over time at the value r.[4]

2.1 Consumer's problem

2.1.1 Preferences

The consumer's lifetime utility is represented by a utility functional, U, which assigns utility $U(c_t)$ to each consumption path c_t according to:

$$U(c_t) = \int_0^T u(c_t)e^{-\beta t}dt, \tag{1}$$

where $\beta(> 0)$ is the subjective discount rate, T is a positive parameter, and $u(.)$ is the instantaneous utility function, which is continuously differentiable, strictly increasing, and strictly concave.[5] In addition – and to ensure an interior optimum – we assume that $u(.)$ satisfies:

$$\lim_{c_t \to 0} u'(c_t) = \infty. \tag{2}$$

Some important features of expression (1) merit discussion. First, $U(.)$ is a *functional* and is not to be confused with a *function*.[6] Second, lifetime utility is additively separable over time and has exponential discounting. Such a formulation of lifetime utility is not only more tractable than alternative specifications with non time-separability and/or other type of discounting but also exhibits a key property known as time consistency. Time consistency implies that the consumer's tastes remain unchanged over time and, hence, that plans formulated at time 0 (which are the plans that will be characterized below) would

[4] It should be stressed that capital mobility would still be perfect even if r_t fluctuated over time.

[5] To keep notation as compact as possible, we will use the notation x_t to indicate that variable x is a function of time. In other words, x_t should be read as $x(t)$.

[6] Throughout the book – and to ensure notational consistency – we will use capital letters to denote functionals and small letters to denote functions. As Appendix (7.1) explains in detail, a functional maps a function into a real number, whereas a function maps a number (or, more generally, a vector) into a real number. In other words, $U(.)$ maps a consumption path, c_t, into a number, $U(c_t)$. Note that in a discrete time formulation, lifetime utility is a function and not a functional, as it maps a consumption vector, $\{c_1, c_2, ...\}$ into a number.

be chosen again if the consumer reoptimized later on.[7] Finally, we should note that we have assumed a finite horizon to highlight some subtle issues involved in the switch to the infinite horizon formulation, which will be our main set up throughout this book.

2.1.2 Flow constraint

Let b_t denote net foreign assets denominated in terms of the tradable good held by the consumer at time t.[8] The consumer's flow constraint is thus given by:

$$\dot{b}_t = rb_t + y_t - c_t, \tag{3}$$

where y_t denotes the endowment flow received by the consumer at time t. Equation (3) says that the consumer will accumulate net foreign assets to the extent that his/her total income (interest receipts and endowment) exceeds his/her consumption. Notice that the stock of net foreign assets, b, is a predetermined variable in the sense that, barring some exogenous change, it is a continuous function of time.[9] The consumer is born with some net foreign assets (i.e., b_0).

2.1.3 Basic formulation

The consumer's maximization problem can be formally stated as:

$$\begin{aligned}
&\max_{\{c_t\}_{t=0}^{T}} U(c_t) = \int_0^T u(c_t)e^{-\beta t}dt, \\
&\text{subject to} \\
&\dot{b}_t = rb_t + y_t - c_t, \\
&b_T \geq 0, \\
&b_0 \ \text{ given.}
\end{aligned} \tag{4}$$

Condition 4 is typically referred to as a "no-Ponzi games condition" and requires that the consumer not "die" with positive debt.[10] Since the instantaneous utility function is strictly increasing in consumption (i.e., there is no satiation point),

[7] Exercise 1 and 2 at the end of this chapter asks the reader to verify, in the context of a discrete version of this model, that the preferences in the text are indeed time consistent and shows how alternative discounting methods (Exercise 1) and non-time separability (Exercise 2) may generate time inconsistent preferences. (See Calvo (1996) for a proof that, in the continous time setting of this chapter, preferences are time consistent.)

[8] In our framework with one agent and one asset, there is really no difference between net and gross assets.

[9] By exogenous change, we mean, for instance, a foreign grant. A foreign grant would discretely change the consumer's *stock* of net foreign assets. Barring such events, the stock of net foreign assets must be a continuous function of time because it can only change as a result of saving or dissaving (which are *flows*). This is the reason why the evolution over time of predetermined variables is usually characterized by some accumulation equation such as (3).

[10] After Charles Ponzi (1882-1949), an Italian swindler who run well-known scams in Boston whereby he would paid early investors with money coming from later investors.

it will never be optimal for the individual to "die" with assets since increasing consumption at some point in time would always lead to higher lifetime utility. In other words, at an optimum, $b_T \leq 0$. Combining this condition with the constraint, given by (4), that the consumer not "die" with assets yields the condition

$$b_T = 0. \tag{5}$$

Hence, in solving the basic problem, we can use condition (5) instead of (4).

As it stands, this problem cannot be solved using standard calculus techniques. Standard calculus techniques deal with problems aimed at finding maxima or minima of functions of a real variable. Instead, this problem requires maximizing a functional; that is, a quantity that depends on a function rather than on real numbers. The branches of mathematics that study the maximization or minimization of quantities that depend on functions are referred to as the calculus of variations and optimal control (see, for example, Reed (1998)).[11]

The way we will proceed here is to transform this problem into one that can be solved with standard Lagrange multiplier techniques. To this end, we will reformulate this maximization problem subject to an uncountable number of constraints (i.e., flow constraints in each point of time) into a maximization problem subject to one constraint (i.e., an intertemporal constraint). In fact, throughout this book, we will proceed in this way whenever it is straightforward to use the flow constraint to derive the intertemporal constraint.

2.1.4 Intertemporal budget constraint

To derive the consumer's intertemporal budget constraint, rewrite equation (3) as

$$\left(\dot{b}_t - rb_t\right) e^{-rt} = (y_t - c_t)\, e^{-rt}. \tag{6}$$

Integrating forward:

$$\int_0^T \left(\dot{b}_t - rb_t\right) e^{-rt} dt = \int_0^T (y_t - c_t)\, e^{-rt} dt. \tag{7}$$

The left hand side of equation (7) can be solved to yield

$$\int_0^T \left(\dot{b}_t - rb_t\right) e^{-rt} dt = \int_0^T \frac{d\left(b_t e^{-rt}\right)}{dt} dt = e^{-rT} b_T - b_0. \tag{8}$$

Substituting (8) into (7), we obtain

$$e^{-rT} b_T - b_0 = \int_0^T (y_t - c_t) e^{-rt} dt. \tag{9}$$

[11] Optimal control is a generalization of calculus of variations that enables us to deal with corner solutions. However, for the problems that are analyzed in this book one can consider them as "perfect substitutes." We will use optimal control techniques starting in Chapter 6.

Taking into account condition (5), we can rewrite the consumer's intertemporal budget constraint (9) as

$$b_0 + \int_0^T y_t e^{-rt} dt = \int_0^T c_t e^{-rt} dt. \tag{10}$$

This intertemporal constraint is highly intuitive as it simply says that the present discounted value of consumption (right-hand side) must be equal to the consumer's wealth (left-hand side), given by the initial stock of net foreign assets plus the present discounted value of his/her endowment.

2.1.5 Alternative formulation

The consumer's problem can then be restated as choosing $\{c_t\}_{t=0}^T$ to maximize (1) subject to (10). We can study this problem by means of standard Lagrange-multiplier techniques. The Lagrangean is given by

$$\mathcal{L} = \int_0^T u(c_t) e^{-\beta t} dt + \lambda \left(b_0 + \int_0^T y_t e^{-rt} dt - \int_0^T c_t e^{-rt} dt \right),$$

where λ is the Lagrange multiplier.

It can be shown that the first order condition with respect to c_t is given by[12]

$$u'(c_t) e^{-\beta t} = \lambda e^{-rt}. \tag{11}$$

As usual, the derivative of the Lagrangean with respect to the Lagrange multiplier yields back the intertemporal budget constraint (10).[13]

Assuming $\beta = r$, first-order condition (11) implies that

$$u'(c_t) = \lambda. \tag{12}$$

At an optimum, the consumer equates his/her marginal utility of consumption to the marginal utility of wealth (the Lagrange multiplier). This simple condition constitutes the whole foundation of the modern intertemporal approach to open economy macroeconomics. Since λ is some fixed number and we have not imposed any restrictions on the path of output, condition (12) delivers the strong implication that the path of consumption will be flat over time regardless of the path of output. In other words, anticipated output fluctuations have no effect on consumption (see Figure 1). As discussed in more detail below, this perfect consumption smoothing outcome is the result of a preference for consumption

[12] Appendix 7.2 shows how to obtain this first order condition using (i) a discrete time approximation referred to as "pointwise optimization" or (ii) "perturbation" methods. The term "pointwise optimization" captures the idea that, intuitively, the consumer is choosing optimal consumption at each point in time. In practice, "pointwise optimization" allows us to "ignore" the integral when differentiating. Notice also that we are choosing consumption at each (uncountable) point of time, so this first-order condition holds for any $c_t \in [0, T]$.

[13] It should be remarked that (11) cannot be obtained by taking the partial derivative of $\mathcal{L}$ with respect to c_t and making it equal to zero. In fact, the result of doing that would be $\int_0^T [u'(c_t)e^{-\beta t} - \lambda e^{-rt}] dt = 0$.

smoothing (as a consequence of a strictly concave utility function) combined with perfect access to capital markets that allows consumers to borrow in bad times and lend in good times at constant real interest rate.[14]

We should make two important remarks about the assumption $\beta = r$ (which are, of course, valid for the infinite horizon formulation below). First, if r were still constant over time but different from β (i.e., $\beta \neq r$), there would be consumption tilting, in the sense that the path of consumption would be either increasing or decreasing over time (see Exercise 4 at the end of this chapter). From a conceptual point of view, however, such consumption dynamics are rather uninteresting because they are unrelated to the behavior of any driving force in the model. In other words, there is little we can learn from these cases. In addition – as also illustrated in Exercise 4 – in an infinite horizon setting, an optimal consumption plan may not even exist if $\beta \neq r$.

Second, we could have a fluctuating world real interest rate over time. As illustrated in Exercise 5 at the end of this chapter, whenever $r_t > \beta$, consumption would be increasing reflecting the fact that households discount utility at a lower rate than the market discounts resources. The converse is true whenever $r_t < \beta$. These fluctuations in consumption are optimal since they reflect fluctuations in the world real interest rate. Quantitatively, however, fluctuations in consumption as a response to fluctuating real interest rates seem to be small. [15] For real interest rates to matters quantitatively, one would need to introduce a different channel through which they can have real effect such as affecting the effective cost of labor for firms (Neumeyer and Perri (2005)).

2.2 Infinite horizon formulation

The most attractive formulation of our basic problem relies on an infinite horizon since it avoids an arbitrary cut-off point at T. There are, however, some mathematical subtleties involved in the infinite horizon formulation. Consider first the extension of preferences to an infinite horizon. The functional (1) becomes

$$V(c_t) = \int_0^{\infty} u(c_t) e^{-\beta t} dt. \tag{13}$$

Notice that (13) involves an improper integral that might or might not converge. Clearly, if the integral did not converge for some consumption paths, we would not be able to rank them.[16]

[14] Appendix 7.3 verifies that the consumption path identified by first-order condition is indeed a maximum.

[15] See, for instance, Mendoza (1991) and Correia, Neves, and Rebelo (1995)).

[16] The simplest way of ensuring that the functional (13) converges is to assume that the instantaneous utility function is bounded from above (i.e., that there is satiation). In such a case, however, we would need to ensure that the parameters of the problem are such that the satiation point is not reached so that we can apply the arguments below regarding the transversality condition. In general, we would need to check whether this integral converges to ensure that the problem is well-defined. Exercise 4 at the end of the chapter shows that for a CES utility function, this integral always converges in the case of $r = \beta$ and derives sufficient conditions for convergence in the case of $r \neq \beta$. See Chakravarty (1962) and Chiang (1992, Chapter 5) for a detailed discussion.

As in the finite-horizon case, we can formally state the maximization problem as

$$\begin{aligned}
&\max_{\{c_t\}_{t=0}^{T}} V(c_t) = \int_0^\infty u(c_t)e^{-\beta t}dt, \\
&\text{subject to} \\
&\dot{b}_t = rb_t + y_t - c_t, \\
&\lim_{t\longrightarrow\infty} e^{-rt}b_t \geq 0, \\
&b_0 \text{ given.}
\end{aligned} \tag{14}$$

With infinite horizon, the no-Ponzi games condition takes the form specified in (14). In other words, the present discounted value of debt "at the end of the consumer's life" should be non-negative, which makes sense intuitively. We should make three related observations on this point. First, to establish a complete analogy between the finite and infinite horizon cases, we can think of the no-Ponzi games condition for the finite horizon case as $e^{-rT}b_T \geq 0$. Since $e^{-rT} > 0$, this condition reduces to $b_T \geq 0$, as stated in (4).Second, stating the no-Ponzi games condition in the infinite horizon case as

$$\lim_{t\longrightarrow\infty} b_t \geq 0 \tag{15}$$

would be incorrect. In particular, note that asymptotic debt (i.e., $\lim_{t\longrightarrow\infty} b_t < 0$) is consistent with (14).[17] Equation (14) only imposes a constraint on the growth rate of debt (i.e., debt could grow at a rate which is less than r).

As in the finite horizon, the non-satiation assumption ensures that the consumer will never want to "die" with a positive level of assets. Formally, optimality requires that

$$\lim_{t\longrightarrow\infty} e^{-rt}b_t \leq 0. \tag{16}$$

The combination of the no-Ponzi games condition – given by (14) – and optimality condition (16) implies the condition

$$\lim_{t\longrightarrow\infty} e^{-rt}b_t = 0, \tag{17}$$

typically referred to as the transversality condition.

How is the derivation of the intertemporal constraint affected by the switch to an infinite horizon? By following the same series of steps as above and using the transversality condition (17), we obtain[18]

[17] It might be counterintuitive that the consumer is allowed to "die" with debt. However, the key is to take into account that servicing debt up to infinity is like paying the principal. The example discussed in footnote 24 below may clarify your ideas as well.

[18] It should also be remarked that (18) is a meaningful budget constraint only if the present discounted value of output is finite, which implies that output cannot grow at a rate equal or higher than r.

$$b_0 + \int_0^\infty y_t e^{-rt} dt = \int_0^\infty c_t e^{-rt} dt, \tag{18}$$

which is, of course, the infinite horizon counterpart of the intertemporal constraint for the finite case, given by (10).

As in the finite horizon case, we can re-state this maximization problem in terms of a familiar Lagrangean:

$$\mathcal{L} = \int_0^\infty u(c_t) e^{-\beta t} dt + \lambda \left(b_0 + \int_0^\infty y_t e^{-rt} dt - \int_0^\infty c_t e^{-rt} dt \right).$$

The first order condition in an infinite horizon framework continues to be given by (12). From now on we will work with the infinite horizon model.

2.3 Interpretation of the Lagrange multiplier

Since first-order condition (12) is the foundation of small open macroeconomics, it is worth taking a brief detour to offer an insightful interpretation of the Lagrange multiplier λ. In basic microeconomic theory, λ is thought of as the shadow price of wealth. As will become clear below, this interpretation remains valid in this chapter since intertemporal prices play no role. In general, however, such an interpretation will not be valid because, as shown in Chapter 3, the multiplier will depend not only on wealth but also on the path of intertemporal relative prices.

An alternative – and insightful interpretation for our purposes – is to think of λ/r as an asset price and of λ as a dividend. To this effect, multiply both sides of equation (12) by e^{-rt} and integrate forward to obtain

$$\frac{\lambda}{r} = \int_0^\infty u'(c_t) e^{-rt} dt.$$

The term λ/r can then be interpreted as the (relative) price of an asset that would offer the consumer a "dividend" of $u'(c_t)$ at every point in time. The variable λ would correspond to the annuity value of the present discounted value of dividends. If c_t were constant over time and equal to $\bar{c}$, then λ would correspond to the dividend itself (i.e., $\lambda = u'(\bar{c})$). This is completely analogous to, say, a stock price that prices the present discounted value of a stream of dividends. As we know from basic finance theory, a stock price will not change in response to anticipated changes in dividends, since all the information is already incorporated in the price. But a stock price may change in response to "news," that is, unanticipated events. In the same vein, the multiplier λ will *not* change in response to anticipated fluctuations in output, since the path of consumption – and hence the path of marginal utilities – already incorporates such information. In contrast, the multiplier may change in response to *unanticipated* changes in output because those changes represent "news." We should keep this interpretation in mind as we solve this basic model under different scenarios.

2.4 Equilibrium conditions

Since the consumer is the only agent in this economy, the flow constraint (3) and the intertemporal constraint (18) are also the economy's aggregate constraints. We now take a brief detour into balance of payments accounting.

Let the trade balance, TB, be given by net exports of goods and services and the income balance, IB, by net factor payments from abroad. Then, by definition,

$$TB_t \equiv y_t - c_t, \tag{19}$$

$$IB_t \equiv rb_t. \tag{20}$$

Also by definition, the current account is given by:

$$CA_t \equiv TB_t + IB_t. \tag{21}$$

Substituting (21) into (3), it follows that

$$\dot{b}_t = CA_t. \tag{22}$$

The economy's accumulation of net foreign assets is thus given by the current account balance. In other words, if the economy is running a current account surplus, double-entry bookkeeping ensures that it must be accumulating claims against the rest of the world. If we define the capital account balance, KA, in the standard way (whereby acquiring claims against the rest of the world implies a negative capital account balance), we have:

$$KA_t \equiv -\dot{b}_t. \tag{23}$$

Hence,

$$CA_t + KA_t = 0,$$

which is the fundamental identity of balance of payments accounting. A, say, deficit in current account must be necessarily financed by a capital account surplus (i.e., by borrowing from the rest of the world).

Alternatively, we can express the current account balance as being equal to saving. To this effect, define saving, S, as:

$$S_t \equiv rb_t + y_t - c_t. \tag{24}$$

Then, from the definition of the current account balance (equation (21)) and using (19), (20), it follows that

$$CA_t = S_t. \tag{25}$$

Finally, notice that using the definition of the trade balance, given by (19), we can rewrite the intertemporal constraint (18) as:

$$\int_0^{\infty} TB_t e^{-rt} dt = -b_0,$$

which says that the present discounted value of the trade balances must equal the economy's initial net *foreign debt*. We thus see that basic accounting requires that an indebted economy run, on average, trade surpluses in order to repay the debt over time.

3 Solution of the model

3.1 General solution

We have already mentioned that first-order condition (12) implies that consumption will be constant along a perfect foresight path (at a level denoted by $\bar{c}$). From (18), it follows that:

$$\bar{c} = r \left(b_0 + \int_0^{\infty} y_t e^{-rt} dt \right). \tag{26}$$

Consumption equals "permanent income", defined as the annuity value of the present discounted value of available resources. The best way to think about permanent income is as that level of constant consumption that can be maintained forever. This idea is, of course, hardly new and dates back to Friedman's (1957) seminal contribution on the permanent income hypothesis. In essence, Friedman argued that current consumption should not depend on *current* income as Keynes had argued but rather on long-term expected income (which he called "permanent income").[19] Hence, as a model of consumption behavior, the modern approach to open economy macroeconomics can be viewed as just an elegant and rigorous exposition of Friedman's pioneering work. As in the finite horizon case, Figure 1 illustrates the fact that consumption remains flat over time even if the endowment fluctuates over time.

What will be the path of the trade balance along a perfect foresight equilibrium path? Taking into account (26), the trade balance (19) is given by:

$$TB_t \equiv y_t - \bar{c}.$$

It follows that in "good times" (i.e., when y is high), the economy will run trade surpluses, while in "bad times" (i.e., when y is low) the economy will run trade deficits (Figure 1). Intuitively, the trade balance acts as a shock absorber and allows the economy to sustain a flat path of consumption when faced with a fluctuating path of output.

The current account path along a perfect foresight equilibrium path is given by

[19] Of course, if liquidity and/or borrowing contrainsts were present, then consumption would depend (at least partly) on current income. The same is true for a small open economy, as analyzed in detail in Chapter 2.

$$CA_t \equiv rb_t + y_t - \bar{c}.$$

It is clear that when output switches from high to low, the current account balance worsens since consumption is flat and b_t is a predetermined variable. Hence, in bad times, the economy increases its borrowing from the rest of the world. The opposite is true in good times.

It should be pointed out that the perfect consumption smoothing result captured in equation (26) is just a useful conceptual benchmark and should not be viewed as the key empirical prediction of the modern intertemporal approach to the current account. Clearly – as just a casual glance at the data indicates – consumption is not flat over time (or, more generally, constant along a trend). However, far from being an empirical rejection of this approach, such observation simply suggests that the many frictions examined in subsequent chapters that imply departures from perfect consumption smoothing are pervasive in practice. Depending on the type of friction, deviations from consumption smoothing may be optimal or not:

- Non-optimal deviations. The most prominent frictions that lead to a non-constant path of consumption are imperfections in capital markets (Chapter 2) and intertemporal distortions in intertemporal prices stemming from policy actions or non-credible policies (Chapter 3). The first friction makes it impossible for consumers to borrow as much as they would like during bad times, while the second induces consumers *not* to choose a constant path of consumption to begin with. In both cases, this is a sub-optimal outcome because a planner would choose a constant path of consumption.

- Optimal deviations. The main sources of optimal deviations from consumption smoothing are a fluctuating real interest rate, the introduction of a second argument in the utility function (for instance, labor/leisure or non-tradable goods), and fluctuations in the terms of trade. As examined in Exercise 5 at the end of the chapter, a fluctuating real interest rate will induce consumption tilting. Shocks to labor productivity will induce a non-constant path of consumption in a model in which the marginal utility of consumption depends on labor, as analyzed in Exercise 6 at the end of this chapter. The same is true of shocks to the endowment of non-tradable goods in a model in which the marginal utility of consumption of tradables depends on consumption of non-tradables (Chapter 4). Fluctuations in the terms of trade will affect intertemporal relative prices and lead to a non-constant consumption path (see Chapter 3).

3.2 Stationary equilibrium

Finally – and for further reference – let us characterize a stationary equilibrium.[20] Typically, though not always, if exogenous variables are constant, so

[20] We will purposely use the expression "stationary equilibrium," as opposed to "steady-state," to refer to an equilibrium along which both exogenous and endogenous variables are

will endogenous variables.[21] In any event, stationarity of an equilibrium is a feature that we need to prove; it would be wrong to simply assume it.

In this case, suppose that the endowment path is given by

$$y_t = y^H, \qquad t \geq 0. \tag{27}$$

Since we have already derived a general solution for the model (i.e., a solution valid for any path of output), we can rewrite (26), taking into account (27), as

$$\bar{c} = rb_0 + y^H. \tag{28}$$

Substituting (28) into (19), and taking into account (27), yields

$$TB_t = -rb_0. \tag{29}$$

The stationary trade balance can therefore be positive, zero, or negative depending on the initial level of net foreign assets.[22]

We now show that, in a stationary equilibrium, the current account will always be zero. Substituting (27) and (28) into (3), we obtain:

$$\dot{b}_t = rb_t - rb_0.$$

Evaluate this equation at $t = 0$ to obtain $\dot{b}_0 = 0$. It follows that

$$b_t = b_0$$

for all $t \geq 0$. Since net foreign assets are constant over time, the current account is always zero along a stationary equilibrium.

Finally, notice that the value of the Lagrange multiplier in this stationary equilibrium follows from (12) and (28):

$$\lambda = u'(rb_0 + y^H).$$

The value of the multiplier is thus determined by the level of permanent income.

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constant over time. We will reserve the expression "steady-state" for models that have intrinsic dynamics and, hence, in which endogenous variables converge to the steady-state independently of initial conditions.

[21] There are cases, however, in which this will not be the case (the models of balance of payments crises studied in Chapter 9 being an excellent example).

[22] A stationary equilibrium offers an interesting example of the no-Ponzi game condition $\lim_{t\to\infty} b_t e^{-rt} \geq 0$. Suppose that $b_0 < 0$. Then, the trade balance will be constant over time and equal to $-rb_0 > 0$. Clearly, in the limit the initial debt (b_0) is paid in full because the present discounted value of the trade surpluses is $-b_0$. The no-Ponzi games condition is clearly satisfied. Interestingly, however, the current value of the debt is $b_0 < 0$ for all $t \in [0, \infty)$. This illustrates how a condition of the form $\lim_{t\to\infty} b_t \geq 0$ would be too restrictive.

4 Unanticipated shocks

To gain further insights into how a small open economy reacts to output shocks, we will study the effects of unexpected changes in the path of output that take place at time 0.[23]

4.1 Permanent fall in output

Suppose that an instant before time 0 the economy is in the stationary equilibrium characterized in Subsection 3.2. At time 0, there is an unanticipated and permanent fall in output from y^H to y^L, where $y^L < y^H$ (see Figure 2, Panel A). Since there has been an unanticipated shock, the individual will reoptimize immediately. The first order condition will now be given by

$$u'(c_t) = \tilde{\lambda},$$

where $\tilde{\lambda}$ is the multiplier corresponding to the new intertemporal constraint (recall that the Lagrange multiplier may change in response to unanticipated shocks as any asset price would). Since consumption will be constant along the new perfect foresight path, we can use (18) with $y_t = y^L$ to conclude that the new level of consumption is given by (see Figure 2, Panel B)

$$\bar{c} = rb_0 + y^L.$$

Since consumption falls one to one with output, the trade balance and the current account do not change. Hence an unanticipated and permanent fall in output leads to a *pari passu* fall in consumption and has no impact on the current account. The economy fully adjusts to the lower output level.

Finally, notice that the Lagrange multiplier is now higher (reflecting the lower wealth) as its value is given by

$$\tilde{\lambda} = u'(rb_0 + y^L).$$

We conclude that the economy adjusts immediately to a permanent negative shock. Since there are no distortions in this economy, this response is socially optimal. This result underlies the standard policy prescription that says that, however painful, an economy has no choice but to adjust to any long-lasting negative shock.

[23] A legitimate question that the reader may ask is how to think about *unanticipated* shocks in *perfect foresight* models since, taken at face value, it would seem a contradiction in terms. Strictly speaking, this is true and a "purist" would probably reject such an intellectual experiment. However, it should be viewed as an approximation to a stochastic world in which the shock had such a small probability of occurring that, for all intents and purposes, it did not affect the initial stationary equilibrium.

4.2 Temporary fall in output

Suppose now that an instant before time 0 the economy is in the stationary equilibrium characterized above and that at time $t = 0$ there is an unanticipated and temporary fall in output. Formally, the path of output for $t \geq 0$ is given by

$$\begin{aligned} y_t &= y^L, \qquad 0 \leq t < T, \\ y_t &= y^H, \qquad t \geq T, \end{aligned}$$

for some $T > 0$ (see Figure 3, Panel A).

Since there has been an unanticipated shock, the consumer immediately reoptimizes at $t = 0$. The problem he/she faces is formally the same as before, with his/her intertemporal constraint now given by:

$$b_0 + \int_0^T y^L e^{-rt} dt + \int_T^\infty y^H e^{-rt} dt = \int_0^\infty c_t e^{-rt} dt. \tag{30}$$

The consumer thus maximizes (1) subject to the intertemporal constraint (30). As before, the corresponding first-order condition implies that consumption will be flat along the new perfect foresight equilibrium path. Hence, from (30), it follows that:

$$\bar{c} = rb_0 + y^L(1 - e^{-rT}) + y^H e^{-rT}. \tag{31}$$

Consumption thus falls by the same amount as permanent income (see Figure 3, Panel B). Permanent income has fallen from $rb_0 + y^H$ before the shock to $rb_0 + y^L(1 - e^{-rT}) + y^H e^{-rT}$ after the shock. The larger is T, the bigger will be the fall in permanent income and hence the bigger the fall in consumption. For a small value of T, consumption will fall by very little. For a very large value of T (i.e., $T \longrightarrow \infty$), consumption will fall by almost the same amount as it would fall if the shock were permanent.[24]

Using (31), we can derive the path of the trade balance (see Figure 3, Panel C):

$$\begin{aligned} TB_t &\equiv y^L - \bar{c} = -rb_0 - (y^H - y^L)e^{-rT} < -rb_0, \qquad 0 \leq t < T, \\ TB_t &\equiv y^H - \bar{c} = -rb_0 + (y^H - y^L)(1 - e^{-rT}) > -rb_0. \qquad t \geq T, \end{aligned}$$

To fix ideas, suppose that $b_0 = 0$. Then, in order to keep the path of consumption flat after $t = 0$, the economy runs a trade deficit while output is low and a trade surplus when output goes back to its initial level (see Figure 3, Panel C).

What about saving at $t = 0$? Recalling (24), we know that $S_0 = rb_0 + y^L - \bar{c}$. Hence, using (31), we obtain

$$S_0 = -(y^H - y^L)e^{-rT} < 0. \tag{32}$$

[24] Since consumption falls, the multiplier rises at $t = 0$, as illustrated in Figure 3, Panel F.

Saving is, of course, negative as households dissave to smooth consumption over time. Furthermore, S_0 is an increasing function of T: the longer is the duration of the shock, the smaller is the dissaving carried out by households on impact.

To derive the current account path, we will first derive the path of net foreign assets. To this effect, rewrite flow constraint (3) as

$$\dot{b}_t = rb_t + y^L - \bar{c}, \qquad t \leq T, \tag{33}$$
$$\dot{b}_t = rb_t + y^H - \bar{c}, \qquad t \geq T. \tag{34}$$

These are first-order differential equations with constant term $y^H - \bar{c}$ and $y^L - \bar{c}$, respectively. Denoting the constant term by X, recall that the general solution for each of these first-order differential equations takes the form, respectively,

$$b_t = b_0 e^{rt} + \frac{X}{r}(e^{rt} - 1), \qquad t \leq T,$$
$$b_t = b_T e^{r(t-T)} + \frac{X}{r}\left[e^{r(t-T)} - 1\right], \qquad t \geq T.$$

Substituting for the respective constant terms (i.e., $X = y^L - \bar{c}$ for the first equation and $X = y^H - \bar{c}$ for the second equation) and taking into account that $\bar{c}$ is given by expression (31), we obtain

$$b_t = b_0 - e^{-r(T-t)}(1 - e^{-rt})\frac{(y^H - y^L)}{r}, \qquad t \leq T, \tag{35}$$
$$b_t = b_0 - (1 - e^{-rT})\frac{(y^H - y^L)}{r}, \qquad t \geq T, \tag{36}$$

where we have evaluated the first expression at $t = T$ to get rid of b_T in the second equation.

Several remarks are in order. First, as a check on our solution, we can evaluate the RHS of the first equation at $t = 0$ to verify that $b_t = b_0$ at that point. Second, we can check that b_t is continuous at $t = T$ (though not differentiable, as becomes clear below) by evaluating both expressions at $t = T$ and noting that the values coincide. Third, the level of net foreign assets for $t \geq T$ is constant since the economy is, once again, in a stationary equilibrium. Figure 3, Panel D, illustrates the path of net foreign assets.

To derive the path of the current account (Panel E), we simply differentiate equations (35) and (36) to obtain (see Figure 3, Panel E):

$$\dot{b}_t = -e^{-r(T-t)}(y^H - y^L) < 0, \quad t \leq T,$$
$$\dot{b}_t = 0, \quad t \geq T.$$

At $t = 0$, the current account jumps into a deficit reflecting the deterioration in the trade balance. Further, notice that

$$\begin{aligned}\ddot{b}_t &= -re^{-r(T-t)}(y^H - y^L) < 0, \\ \dddot{b}_t &= -r^2e^{-r(T-t)}(y^H - y^L) < 0.\end{aligned}$$

Hence, the current account falls over time at an increasing rate (in absolute value), as illustrated in Panel E, reflecting the fact that, while the trade balance is constant over time, interest receipts fall over time (and/or debt repayments increase over time). At time T, the current account deficit disappears as output increases and the trade balance improves.

The key message that follows from the analysis of unanticipated shocks is that the economy "adjusts" to permanent shocks but "finances" temporary shocks. Since there are no distortions and the economy is thus always operating in a first-best equilibrium, such responses are socially optimal. If market frictions prevent international capital markets from providing financing to developing countries in bad times, this result provides a rationale for the existence of multilateral financial organizations, such as the IMF, that may provide financing in bad times.

5 Adding investment to the basic model

The model just examined assumed that the economy was endowed with an exogenous output stream. Since there was no investment, the current account was identically equal to savings. In that context, we analyzed how a temporary fall in the endowment leads to lower saving and thus to a current account deficit. The model thus predicts that the trade balance behaves procyclically (i.e., the trade balance improves in good times and worsens in bad times). The evidence, however, suggests precisely the opposite (see Box 1): the trade balance is countercyclical (i.e., it worsens in good times and improves in bad times). A missing ingredient in our model might be investment. After all, our intuition would tell us that a temporary fall in productivity should lead to both lower saving (based on the consumption smoothing motive studied above) and lower investment (since productivity is temporarily lower). It would thus seem that the effect of a temporary fall in productivity on the current account would depend on the relative strength of both effects. In particular, if the investment effect dominates (i.e., if the fall in investment is larger than the in fall in saving), there may be an improvement in the current account. In the case of a temporary improvement in productivity, if the positive investment effect dominates the positive effect on saving, there would be a worsening in the current account. This section will formalize this intuition by introducing investment into our basic model and showing how the relative strength of both effects depends on the duration of the shock.

While our basic model was formulated in continuous time, it will prove convenient at this point to switch to discrete time. The reason is that sticking with continuous time would require adding adjustment costs to the model (otherwise, investment might be "infinite" at any point in time and the current account would not be well-defined). The introduction of adjustment costs would greatly complicate the analytical solution of the model without adding any additional insights. In contrast, discrete time introduces a natural one-period lag in the adjustment of the capital stock to its new equilibrium, which implies that investment is well-defined without having to add adjustment costs.

5.1 Household's problem

5.1.1 Technology

Denote by k_t the stock of capital in period t. Output in period t is thus given by

$$y_t = A_t f(k_t), \tag{37}$$

where A_t (> 0) is a productivity parameter and $f(k)$ is a strictly increasing and strictly concave function:

$$f'(k) > 0, \tag{38}$$
$$f''(k) < 0. \tag{39}$$

We assume that the capital stock does not depreciate. Hence, by definition, investment (I_t) is given by the increase in the stock of capital:

$$I_t \equiv k_{t+1} - k_t. \tag{40}$$

By assumption, the choice variable in period t is k_{t+1}; that is, households choose in period t the capital stock at the beginning of period $t+1$. There is thus a one period adjustment in the stock of capital. In addition, we assume that investment can be negative (i.e., households can "eat" part of their capital stock if they so desire).

5.1.2 Budget constraints

For simplicity, we assume that the household also carries production activities.[25] Let b_t denote net foreign assets held by households in period t. The household's flow constraint is given by

$$b_{t+1} = (1+r)b_t + y_t - c_t - (k_{t+1} - k_t). \tag{41}$$

Except for the investment term, this flow constraint is the discrete time counterpart of flow constraint (3). In addition to consumption, there is now an additional use of resources (investment).

[25] Exercise 7 at the end of this chapter shows how this economy can be decentralized. In a decentralized set-up, households would own both the capital stock and the firms and would rent the capital stock to firms. Firms would rent capital, produce, and return profits to households.

By iterating forward and imposing the condition

$$\lim_{t\to\infty} \frac{b_t}{(1+r)^{t-1}} = 0,$$

we can derive the intertemporal constraint

$$(1+r)b_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t y_t = \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left[c_t + (k_{t+1} - k_t)\right], \tag{42}$$

where b_0 and k_0 are exogenously given. Again, and except for the investment term, this intertemporal constraint is the discrete time counterpart of equation (18).

5.1.3 Utility maximization

The household's lifetime utility is given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{43}$$

where $\beta(>0)$ is now the discount factor and $u(c)$ is strictly increasing, strictly concave, and satisfies condition (2).[26]

The household thus chooses $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to maximize (43) subject to the intertemporal constraint (42). In terms of the Lagrangean, the maximization problem can be stated as (after using (37) to substitute out for output):

$$\begin{aligned} \underset{\{c_t, k_{t+1}, \lambda\}}{\mathcal{L}} &= \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda \left\{ (1+r)b_0 + \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t A_t f(k_t) \right. \\ &\left. - \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left[c_t + (k_{t+1} - k_t)\right] \right\}. \end{aligned}$$

The first-order conditions are given by

$$\beta^t u'(c_t) = \lambda \left(\frac{1}{1+r}\right)^t, \tag{44}$$

$$A_{t+1} f'(k_{t+1}) = r, \tag{45}$$

for $t = 0, 1, 2, ...$

Assuming, for the same reasons discussed for the continuous time case, that $\beta(1+r) = 1$, we can express first-order condition (44) as

$$u'(c_t) = \lambda. \tag{46}$$

[26]Note that, even though we are using the same notation, β here stands for the discount *factor* (which, by definition, equals 1/(1+discount rate) whereas in the continous-time function, it refers to the discount *rate*.

As in our previous model without investment (recall equation (12)), condition (46) implies that, along a perfect foresight equilibrium path, consumption will be flat regardless of the path of output (and, hence, of the path of A_t). Figure 1 thus remains valid for this model with investment. This result was to be expected since perfect access to international capital markets allows consumption decisions to be independent of production decisions (see Fisher (1930)).

Equation (45) determines the optimal level of the capital stock. This arbitrage condition says that, at the margin, the return on a unit of capital (given by the LHS) should be equal to the return on net foreign assets, r. If this were not true, it would be profitable for households to readjust their portfolio between bonds and capital. It is worth emphasizing that condition (45) does *not* hold for k_0 since the initial capital stock is given and is thus not a choice variable.

5.2 Equilibrium conditions

Since, as before, the household is the only agent in this economy, the constraints discussed above also apply to the economy as a whole. In particular, using (40), we can rewrite equation (41) as follows:

$$b_{t+1} - b_t = rb_t + y_t - c_t - I_t. \tag{47}$$

To write this equation in terms of familiar balance of payments accounting, define the current account balance as the change in net foreign assets:

$$CA_t \equiv b_{t+1} - b_t. \tag{48}$$

Hence, combining (47) and (48),

$$CA_t = rb_t + y_t - c_t - I_t. \tag{49}$$

As before, there are two useful ways of rewriting the current account balance. First, define the trade balance and income balance, respectively, as:

$$TB_t \equiv y_t - c_t - I_t, \tag{50}$$
$$IB_t \equiv rb_t. \tag{51}$$

Using these definitions, we can rewrite the current account balance as the sum of the income and trade balances:

$$CA_t = IB_t + TB_t. \tag{52}$$

Alternatively, we can express the current account as the difference between saving and investment. To this effect, define saving as:

$$S_t \equiv rb_t + y_t - c_t. \tag{53}$$

Then, combining (49) and (53),

$$CA_t = S_t - I_t. \tag{54}$$

5.3 Perfect foresight equilibrium for a stationary economy

Let us characterize a stationary perfect foresight equilibrium. Assume that $A_t = \bar{A}$ for all $t = 0, 1, ..$ The arbitrage condition (45) then determines a constant level of the capital stock, $\bar{k}$, implicitly defined by:[27]

$$\bar{A}f'(\bar{k}) = r. \tag{55}$$

We assume that $k_0 = \bar{k}$. In other words, the economy starts with its desired capital stock and hence does not need to invest. (The case of $k_0 < \bar{k}$ is addressed below.) Investment is thus zero in every period. In light of equation (37), the constant level of output will be given by

$$\bar{y} = \bar{A}f(\bar{k}). \tag{56}$$

We know from condition (46) that consumption will be constant over time. Hence, taking into account that investment is always zero, it follows from the intertemporal constraint (42) that[28]

$$\bar{c} = rb_0 + \bar{A}f(\bar{k}). \tag{57}$$

We have enough information to compute saving in period 0. By definition (recall (53)),

$$S_0 = rb_0 + y_0 - c_0. \tag{58}$$

Using (56) and (57), it follows that $S_0 = 0$. To compute saving in subsequent periods, we will first need to derive the path of net foreign assets.

Using (50), (56), and (57), the trade balance in period 0 is given by:

$$\overline{TB} = -rb_0.$$

Recall from (54) that the current account is the difference between saving and investment. Since both saving and investment are zero in period 0, the current account is also zero in period 0. By (48), this implies that $b_1 = b_0$. We can now compute saving in period 1 and check that it will be zero as well. Hence, the current account will also be zero in period 1 and $b_2 = b_1$. Proceeding in this way, it should be clear that the current account is always zero and hence the path of net foreign assets is constant over time. This implies that the trade balance is also constant over time and equal to $-rb_0$.

[27] As before, we will use upper bars to denote stationary values.

[28] In deriving (57), recall that the sum of an infinite geometric progression is equal to $a/(1-x)$ where a is the first term and x is the common multiplier (equal to $1/(1+r)$) in this case.

5.4 Unanticipated and permanent increase in productivity

Suppose that in $t = -1$ the economy is in the stationary equilibrium described in subsection 5.3 above. In $t = 0$, there is an unanticipated and permanent increase in the productivity parameter from $\bar{A}$ to $\bar{A}^H$, where $\bar{A}^H > \bar{A}$ (Figure 4, Panel A). Since there has been an unanticipated shock, households reoptimize. The first-order conditions are still given by (45) and (46), with the multiplier possibly taking a different value as discussed above for the continuos time case.

Since the capital stock in period $t = 0$ (k_0) was chosen in period $t = -1$, it cannot respond to the change in A. In other words, households take as given k_0 in their reoptimization. From period 1 onwards, however, the capital stock is higher (Figure 4, Panel B) and implicitly given by the condition

$$\bar{A}^H f'(\bar{k}^H) = r, \qquad t = 1, 2, ... \tag{59}$$

where $\bar{k}^H$ denotes the new and higher stationary value of the capital stock.

Given the path of k_t depicted in Figure 4, Panel B, the path of investment is given by (Figure 4, Panel C)

$$\begin{aligned} I_0 &= \bar{k}^H - k_0 > 0, \\ I_t &= 0, \qquad t = 1, 2, ... \end{aligned} \tag{60}$$

What happens to output? (Figure 4, Panel D illustrates the path of output.) In period $t = 0$, output is higher than in period $t = -1$ because, even though the capital stock has yet to change, the productivity of the existing capital stock has increased. In period 1, output increases further because the capital stock has now adjusted to its new and higher level. From period 1 onwards, output is constant again. Formally:

$$\begin{aligned} y_0 &= A^H f(\bar{k}) > y_{-1}, \\ y_t &= A^H f(\bar{k}^H) > y_0, \qquad t = 1, 2, ... \end{aligned} \tag{61}$$

Let us now turn our attention to the consumption path (Figure 4, Panel D). As before, first-order condition (46) indicates that consumption will be constant along the new perfect foresight equilibrium. To find out the level of consumption, we solve from the intertemporal constraint (42), taking into account the paths of the capital stock and output just derived to obtain:

$$\bar{c} = rb_0 + \bar{A}^H f(k_0) + \frac{r}{1+r} \underbrace{\left[\frac{\bar{A}^H [f(\bar{k}^H) - f(k_0)]}{r} - (\bar{k}^H - k_0) \right]}_{\text{Net present value of investment}}. \tag{62}$$

In an economy with a fixed capital stock (equal to k_0), the new (and higher) level of consumption would be given by the first two terms on the RHS. This would correspond to an unanticipated and permanent increase in the endowment in our first model. This economy, however, will have a higher capital stock from period 1 onwards as a result of the investment carried out in period 0. The term in square brackets captures the net present value (as of period 0) of undertaking such an investment. (This term is multiplied by $r/(1+r)$ because households consume the annuity value of this net present value.) What are the costs and benefits of this investment project? The cost – which is incurred in period 0 – is of course the investment undertaken in period 0, given by $\bar{k}-k_0$. The benefits consist of higher output from period 1 onwards. The present discounted value (as of period 0) of this *additional* output is given by $\bar{A}[f(\bar{k})-f(k_0)]/r$. The term in square brackets thus denotes the net benefits (i.e., the net present value) of this investment project.

Clearly, we should expect that the net present value of this investment will be positive (otherwise it would not be undertaken). We thus want to show that:

$$\frac{\bar{A}^H[f(\bar{k}^H)-f(k_0)]}{r} > \bar{k}^H - k_0.$$

Rearranging terms, we can write this inequality as

$$\frac{\bar{A}^H[f(\bar{k}^H)-f(k_0)]}{\bar{k}^H - k_0} > r.$$

Using the marginal condition for the capital stock – given by (59) – we get

$$\underbrace{\frac{\bar{A}^H[f(\bar{k}^H)-f(k_0)]}{\bar{k}^H - k_0}}_{\text{average return}} > \underbrace{\bar{A}^H f'(\bar{k}^H)}_{\text{return on the marginal unit}},$$

which holds given the strict concavity of $f(.)$. Intuitively, the LHS of this inequality captures the "average" return on the investment, whereas the RHS captures the return of the last unit (the marginal unit). By strict concavity, all the units of capital that precede the marginal unit (i.e., the "inframarginal" units) have a higher return than the marginal unit. Figure 5 – which depicts the production function given by (37) – provides a graphical illustration of this idea. Point A represents production in period 0 (i.e., with a capital stock of k_0) whereas point B represents production in period 1 (and subsequent periods, with a capital stock of $\bar{k}^H$). The average return on the new capital ($\bar{k}^H - k_0$) is given by the slope of the line connecting points A and B. The marginal return on the last unit of capital, r, is captured by the slope of the line tangent to the production function at point B. Clearly, all units of capital below k_1 have a marginal return (given by the slope of the production function) higher than r.

Since, as we have just shown, the net present value of investment is positive, we infer from equation (62) that consumption is higher than before the shock. In fact, consumption rises by more than output since output increases only by the direct effect of the higher productivity whereas consumption also rises by the permanent component of the net present value of investment.

Since consumption rises in anticipation of the higher output in period 1, saving will be negative (Figure 4, Panel H). From (53) and (62),

$$S_0 \equiv -\frac{r}{1+r}\underbrace{\left[\frac{\bar{A}^H[f(\bar{k}^H)-f(k_0)]}{r}-(\bar{k}^H-k_0)\right]}_{+} < 0. \tag{63}$$

Not surprisingly, the term in square brackets is the net present value of investment which – as shown above – is positive. The dissaving in period 0 is thus equal to the permanent income component of this net present value of investment.[29]

What will be the trade balance in period 0? As in the basic model without investment, the trade balance will reflect the dissaving of period 0 but, in addition, should also reflect the investment carried out in period 0. Formally, from (50) and (60), the trade balance in period 0 is given by

$$TB_0 = y_0 - c_0 - (\bar{k} - k_0).$$

Using (62) and rearranging terms, we can rewrite this expression as

$$TB_0 \equiv -rb_0 - \frac{r}{1+r}\left[\frac{\bar{A}^H[f(\bar{k}^H)-f(k_0)]}{r}-(\bar{k}^H-k_0)\right]-(\bar{k}^H-k_0) < -rb_0.$$

This expression makes clear our intuition: the second term on the RHS reflects the consumption smoothing motive and the last term the investment effect. Assuming, to fix ideas, that initial net foreign assets are zero (i.e., $b_0 = 0$), the trade balance will be negative in period 0 (Figure 4, Panel E).

If $b_0 = 0$, the present discounted value of the path of the trade balance must add up to zero. Hence, the economy will be running trade surpluses from period 1 onwards to repay the external debt incurred in period 0. Formally, using (50), (60) and (62), it follows that

$$TB_t \equiv -rb_0 + \frac{r}{1+r}\left\{\bar{A}^H[f(\bar{k}^H)-f(k_0)]+(\bar{k}^H-k_0\right\}) > -rb_0,$$

which is positive for $b_0 = 0$ (Figure 4, Panel E).

What will be the path of the current account? Clearly, the current account will be negative in period 0 reflecting negative saving (recall (63)) and positive investment. Formally, from (54),

$$CA_0 = \underset{-}{S_0} - \underset{+}{I_0} < 0. \tag{64}$$

The model then predicts that the economy should run a current account deficit reflecting negative saving (in anticipation of future output) and positive investment.

[29] While it should be intuitively clear that saving from period 1 onwards is zero, a formal proof has to wait until we derive the path of net foreign assets (since we need the value of b_1 to compute saving in period 1 and so forth).

Since in period 1 the economy becomes stationary, the current account should be zero from period 1 onwards. To check this, we first need to derive an explicit expression for the current account in period 0. Substituting (60) and (63) in equation (64) and rearranging terms, we obtain

$$CA_0 = -\frac{1}{1+r}\left\{\bar{A}^H[f(\bar{k}^H) - f(k_0)] + (\bar{k}^H - k_0)\right\} < 0.$$

Since, from (48), $b_1 = b_0 + CA_0$, then

$$b_1 = b_0 + \frac{1}{1+r}\left\{\bar{A}^H[f(\bar{k}^H) - f(k_0)] + (\bar{k}^H - k_0)\right\} < b_0. \quad (65)$$

Net foreign assets thus decline as a result of the current account deficit in period 0 (Figure 4, Panel G). We can now compute saving in period 1. From (53),

$$S_1 = rb_1 + \bar{A}^H f((\bar{k}^H) - c_1.$$

Using (62) and (65), it is easy to verify that $S_1 = 0$. Since investment in period 1 is zero, the current account in period 1 will be zero as well and net foreign assets remain constant (i.e., $b_2 = b_1$). Hence, saving in all subsequent periods will also be zero and so will be the current account.

We thus conclude that, in the presence of investment, *a permanent increase in productivity leads to a trade and current account deficit.* The current account deficit results from both negative saving and positive investment. This contrasts sharply with the case studied in subsection 4.1 in which a permanent increase in output would leave both the trade and current account balances unchanged.

Finally, it is worth commenting further on the behavior of saving. It may come as somewhat of a surprise that saving falls in response to a *permanent* increase in productivity. As stressed, above, however, this happens because the rise in productivity leads to an *anticipated* increase in output from period 1 onwards. Saving would also fall in the basic model if the endowment was expected to be higher from some date T onwards. Interestingly enough, if the increase in productivity were small (i.e., infinitesimal), saving would not change (see Exercise 8 at the end of this chapter) which is consistent with our intuition for the case of a permanent change. In that case, the net return on investment is zero and hence there is no reason to dissave in anticipation of higher future output. The current account still goes into deficit, of course, due to the rise in investment.

5.5 Unanticipated one-period increase in productivity

We have just seen that a permanent rise in productivity leads to a current account deficit by reducing saving and increasing investment (Figure 4). (And even if the increase in productivity were small, the rise in investment would generate a current account deficit.) We will now analyze the case of a temporary (one-period) rise in productivity that yields exactly the opposite result (i.e., a current account surplus). In fact, this exercise turns out to be the exact analog to the temporary change in output studied in Subsection 4.2 above.

Suppose that in $t = -1$ the economy is in the stationary equilibrium described in subsection 5.3 above. In period 0, A rises from $\bar{A}$ to $\bar{A}^H$, where $\bar{A}^H > \bar{A}$, and then goes back to its initial level in period 1 (Figure 6, Panel A). Given that an unanticipated shock has taken place, households reoptimize in $t = 0$, taking k_0 as given. The first-order conditions are still given by (45) and (46). Since the increase in productivity lasts for only one period, the capital stock does not change (Figure 6, Panel B). Hence, investment remains zero (Figure 6, Panel C). Given that productivity is higher in period 0, output is also higher in period 0 and then falls back to its initial level in period 1 (Figure 6, Panel D).

As (46) makes clear, consumers spread out over time the temporary increase in output by choosing a constant level of consumption from period 0 onwards (Figure 6, Panel D). To compute this new level of consumption, solve from (42), taking into account the new path of output, to obtain:

$$\bar{c} = rb_0 + \bar{A}f(\bar{k}) + \underbrace{\frac{r}{1+r}f(\bar{k})(\bar{A}^H - \bar{A})}_{\text{consumption smoothing effect}}. \tag{66}$$

The third term on the RHS captures the consumption smoothing effect. The additional output brought about by this one-period increase in productivity is $f(\bar{k})(\bar{A}^H - \bar{A})$. Households spread out this gain over time by consuming the permanent component.[30] As a result, we expect saving to be positive in period 0. Indeed, from (53) and (66), it follows that

$$S_0 = \frac{f(\bar{k})(A^H - \bar{A})}{1+r} > 0. \tag{67}$$

What happens to the trade balance? Since investment remains zero, we know from (50) that

$$TB_0 \equiv A^H f(\bar{k}) - c_0. \tag{68}$$

Substituting (66) into (68) yields

$$TB_0 \equiv -rb_0 + \frac{f(\bar{k})(A^H - \bar{A})}{1+r} > -rb_0. \tag{69}$$

As expected, the trade balance improves in period 0. (If $b_0 = 0$, then the economy runs a trade surplus in period 0 as illustrated in Figure 6, Panel E.) From period 1 onwards, the trade balance will fall below its pre-shock level. To show this, notice that

$$TB_t \equiv A^L f(\bar{k}) - \bar{c} \qquad t = 1, 2, ... \tag{70}$$

Substituting (66) into the last expression yields:

[30] Notice that in discrete time, the permanent component of some stock is $r/(1+r)$ times the stock.

$$TB_t = -rb_0 + \frac{rf(\bar{k})(A^L - A^H)}{1+r} < -rb_0, \qquad t = 1, 2, ..$$

Hence, if $b_0 = 0$, the economy runs a trade deficit from period 1 onwards..

What about the current account? Clearly, this economy will run a current account surplus in period 0 since, as we have seen, saving is positive and investment is zero (see Figure 6, Panel F). Formally, using (54) and (67),

$$CA_0 = \frac{f(\bar{k})(A^H - \bar{A})}{1+r} > 0. \qquad (71)$$

Since there is a current account surplus in period 0, net foreign assets in period 1 will be higher (Figure 6, Panel G). Using the value obtained for b_1, it can be verified that saving from period 1 onwards is zero, as illustrated in Figure 6, Panel H. Hence, the current account is also zero from period 1 onward.

In sum, an unanticipated one-period rise in productivity leads to a trade and current account surplus. Since in this case there is no change in investment, this experiment is analogous (though with the opposite sign) to the temporary fall in endowment analyzed in Subsection 4.2 and illustrated in Figure 3.

5.6 Unanticipated and temporary rise in productivity

Thus far we have seen two extreme cases, as captured in Figures 4 and 6. In the first case (Figure 4), a permanent rise in productivity leads to a current account deficit (fall in saving and increase in investment). In the second case (Figure 6), a one-period rise in productivity leads to a current account surplus (positive saving and no change in investment). These two extreme examples thus illustrate the proposition that a positive productivity shock may lead to either a current account deficit or a current account surplus. For shocks of more than one period, we would conjecture that there will be both a rise in saving (and, eventually, for a large enough horizon a fall in saving) and a rise in investment and that the relative strength of these two effects will depend on the duration of the shock. The longer is the duration of the shock, the smaller will be the saving effect in period 0 and hence the more likely that the shock will lead to a current account deficit in period 0.

To verify this conjecture, we now study an unanticipated and temporary rise in productivity that lasts for T periods. Since we have already analyzed the case $T = 1$ in Subsection 5.5, we will examine the case in which $T \geq 2$. Once again, suppose that, as of $t = -1$, the economy is in the stationary equilibrium described above, with the capital stock given by $\bar{k}$. In period 0, A increases from $\bar{A}$ to $\bar{A}^H$ for T periods (i.e., A is higher from period 0 up to, and including, period $T - 1$) and then goes back to its initial level in period T (Figure 7, Panel A). In response to the unanticipated shock, household reoptimize taking as given $k_0 (= \bar{k})$. The first-order conditions continue to be given by (45) and (46).

Let us begin by deriving the path of the capital stock, illustrated in Figure 7, Panel B. The capital stock in period 0, k_0, is given. From period 1 onwards, the path of the stock of capital follows from condition (45):

$$\begin{aligned} \bar{A}^H f'(\bar{k}^H) &= r, \qquad t = 1, 2, ., T-1, & (72) \\ \bar{A} f'(\bar{k}) &= r, \qquad t = T, ..., T+1, ... & (73) \end{aligned}$$

Hence, the capital stock is higher from period 1 until and including period $T-1$ (and denoted by $\bar{k}^H$) and then falls back to its pre-shock level. It is worth noting that the rise in the capital stock in period 1 does *not* depend on the duration of the shock, T.

The path of investment, illustrated in Figure 7, Panel C, follows immediately from the path of the capital stock. Investment is positive in period 0 as the economy increases its capital stock and negative in period $T-1$ as the economy disinvests in anticipation of the productivity fall in period T.

Let us now derive the path of output, illustrated in Figure 7, Panel D. From (37) and the path of capital just derived, it follows that

$$\begin{aligned} &y_0 = \bar{A}^H f(\bar{k}) > y_{-1}, \\ &y_t = \bar{A}^H f(\bar{k}^H) > y_0, \quad t = 1, 2, .T-1, \\ &y_t = \bar{A} f(\bar{k}) = y_{-1}, \quad t = T, ... \end{aligned} \tag{74}$$

Output thus rises in period 0, increases further in period $t = 1$, and then remains at that higher level up to, and including, period $T-1$. In period T, output returns to its pre-shock level.

To compute the change in consumption, it will be useful to proceed in steps and first compute the present discounted value of output (denoted by $PDV(y)$) and net output (i.e., output net of investment). Using (74), the present discounted value of output is given by:

$$PDV(y) = \bar{A}^H f(k_0) + \sum_{t=1}^{T-1} \left(\frac{1}{1+r}\right)^t \bar{A}^H f(\bar{k}^H) + \sum_{t=T}^{\infty} \left(\frac{1}{1+r}\right)^t \bar{A} f(\bar{k}). \tag{75}$$

Using the formula for a geometric progression, this expression simplifies to:[31]

$$PDV(y) = \bar{A}^H f(k_0) + \left[1 - \left(\frac{1}{1+r}\right)^{T-1}\right] \frac{\bar{A}^H f(\bar{k}^H)}{r} + \left(\frac{1}{1+r}\right)^{T-1} \frac{\bar{A} f(\bar{k})}{r}. \tag{76}$$

[31] To derive the equation below, recall that the sum for a geometric progression, S_n, is

$$S_n = \frac{a(1-r^n)}{1-r},$$

where a is the first term, n is the number of terms, and r is the common multiplier.

To compute the present discounted value of net output (i.e., output net of investment), we must subtract from the present discounted value of output, given by (76), the investment that takes place at time 0 and the disinvestment that takes place at $T-1$:

$$PDV(\text{net output}) = PDV(y) - (k_1 - k_0) - \left(\frac{1}{1+r}\right)^{T-1}(k_T - k_{T-1}). \quad (77)$$

Using (76) and taking into account that $k_0 = k_T = \bar{k}$ and $k_1 = \bar{k}^H$, we can rewrite this expression as:

$$\begin{aligned} PDV(\text{net output}) &= \underbrace{\left(\frac{1+r}{r}\right)\bar{A}f(\bar{k})}_{\text{PDV of output with no shock}} + \underbrace{\left(\frac{1+r}{r}\right)\left[1-\left(\frac{1}{1+r}\right)^{T}\right]f(\bar{k})(\bar{A}^H - \bar{A})}_{\text{PDV of direct effect}} \\ &+ \underbrace{\left[1-\left(\frac{1}{1+r}\right)^{T-1}\right]\left[\frac{\bar{A}^H f(\bar{k}^H) - \bar{A}^H f(\bar{k})}{r} - (\bar{k}^H - \bar{k})\right]}_{\text{NPV of investment}}. \quad (78) \end{aligned}$$

As indicated, the present discounted value of net output can be broken down into three components. The first one tells us what the PDV of output would have been in the absence of the shock. The second one captures the PDV of the direct effect on output of the increase in productivity. By direct effect, we mean the increase in the level of output in the absence of any new investment. The third component – which is by now familiar to us – captures the net present value of the investment carried out by this economy (now adjusted by the fact that the project only lasts for $T-1$ periods). As expected, the PDV of net output is higher the larger is T since the higher productivity will last for a longer period of time.

Let us now turn to consumption. Once again, consumption will be constant along the new perfect foresight equilibrium path. From (42) and (78), it follows that:

$$\begin{aligned} \bar{c} &= rb_0 + \bar{A}f(\bar{k}) + \left[1-\left(\frac{1}{1+r}\right)^{T}\right]f(\bar{k})(\bar{A}^H - \bar{A}) \\ &+ \frac{r}{1+r}\left[1-\left(\frac{1}{1+r}\right)^{T-1}\right]\left[\frac{\bar{A}^H f(\bar{k}^H) - \bar{A}^H f(\bar{k})}{r} - (\bar{k}^H - \bar{k})\right]. (79) \end{aligned}$$

The new level of consumption is simply interest income plus the permanent component of the PDV value of net output. As expected, consumption is an increasing function of T.

What happens to saving in period 0? From (53) and (79), it follows that

$$S_0 = \underbrace{\left(\frac{1}{1+r}\right)^T f(\bar{k})(\bar{A}^H - \bar{A})}_{+} \underbrace{- \frac{r}{1+r}\left[1 - \left(\frac{1}{1+r}\right)^{T-1}\right]\left[\frac{\bar{A}^H[f(\bar{k}^H) - f(\bar{k})]}{r} - (\bar{k}^H - \bar{k})\right]}_{-} \gtreqqless 0.$$

As we should have expected based on the two extreme cases studied above – illustrated in Figures 4 and 6 – saving in period 0 has two components (which go in opposite directions). The first component (which is positive, as indicated below the equation) reflects the saving induced by the temporarily higher output in period 0 (this is the effect that we saw when we studied a one-period increase in output; recall equation (67)). This effect becomes smaller as T increases because the shock becomes "more permanent." In the extreme case of a permanent shock ($T \longrightarrow \infty$), this effect vanishes. The second component (which is negative, as indicated below the equation) reflects the dissaving induced by the future increase in output (this is the effect that we saw when we studied a permanent increase in productivity; recall equation (63)). This dissaving becomes larger as T increases simply because the net present value of the investment increases (as the higher capital stock is in place for a longer time). Whether saving is positive or negative in period 0 will depend on the strength of these two effects. Whatever saving is in period 0, however, we have established that saving is a decreasing function of T.

The fact that saving at $t = 0$ may be positive or negative is also telling us that consumption at $t = 0$ could be higher or lower than output at $t = 0$. To see this, consider the case in which $b_0 = 0$. Then, as equation (58) indicates, the sign of S_0 tells us that whether $\bar{c}$ is higher than y_0. As illustrated in Figure 7, saving is positive (Panel H) and therefore $\bar{c} < y_0$ (Panel F).

What will happen to the trade balance in period 0? Using (50) and (79),

$$TB_0 \equiv -rb_0 + \underbrace{\left(\frac{1}{1+r}\right)^T f(\bar{k})(\bar{A}^H - \bar{A})}_{A} \underbrace{- \left[1 - \left(\frac{1}{1+r}\right)^{T-1}\right]\frac{r}{1+r}\left[\frac{\bar{A}^H[f(\bar{k}^H) - f(\bar{k})]}{r} - (\bar{k}^H - \bar{k})\right]}_{B} - \underbrace{(k_1 - k_0)}_{\text{investment effect}}.$$

In a model without investment – as the one studied above – we would only have term A which, as explained above, is positive. The trade balance would therefore behave procyclically (i.e., improving in the case of a temporary increase in productivity). Once investment enters into the picture, however, we have two additional effects that go in the opposite direction and could therefore induce the trade balance to behave countercyclically. The second effect – term B – captures the dissaving induced by anticipated future higher output. The third is the investment effect. The behavior of the trade balance in period 0 will thus depend on the strength of these three effects. If the last two effects dominate – as Figure 7, Panel E, assumes – then the trade balance worsens on impact (i.e., it moves countercyclically), which is consistent with the data (see Box 1).[32]

From period 1 up to, and including, period $T-2$, the trade balance will improve relative to its pre-shock level. In period $T-1$, the trade balance improves further reflecting that period's negative investment. From period T onwards, there is a trade deficit.[33]

What about the current account? Using (54),

$$CA_0 = S_0 - I_0 \gtreqless 0. \tag{80}$$

The sign of the current account is ambiguous because saving could be either positive or negative and investment is of course positive. Figure 7, Panel F assumes that the current account goes into deficit in period 0. To compute the entire path of the current account, we need to keep track of the path of net foreign assets, as the latter influences saving through the returns on net foreign assets. While straightforward, the algebra is somewhat tedious and is left for Appendix 7.4. Intuitively, as the trade balance moves into surplus in period 1, so does the current account balance. It then continues to improve over time because while the trade balance remains constant up to, and including, period $T-2$, returns on assets keep accumulating. In period $T-1$, the current account surplus is further fed by the disinvestment. In period T, the current account balance falls to zero as the economy becomes stationary thereafter. Figure 7, Panel G, illustrates the corresponding path of assets.

In sum, we have shown that once investment is brought into the picture, a temporary shock can lead to a countercyclical response of the trade and current account balances. In the experiment just analyzed, a temporary increase in productivity can lead to either a current account surplus or to a current account deficit. It is worth noting that, in the event in which the investment effect dominates, the current account goes into deficit for only one period because it only takes one period for the capital stock to increase. If there were adjustment costs and investment took longer to adjust to its new value, the current account deficit could also last for more than one period.

[32] As exercise 8 at the end of this chapter makes clear, if the change in productivity were small, then the second effect would not be present and the effect on the trade balance (and, hence, current account) would capture the tension between the first (consumption smoothing) and third (investment) effects.

[33] See Appendix 7.4 for the formal derivation of the paths of the trade balance, saving, and current account.

A numerical example To fix ideas, let us compute a numerical example to illustrate the impact response (i.e., the response in $t = 0$) of the current account to a temporary increase in productivity. As equation (80) indicates, the response of the current account will depend on the relative strength of the saving and investment effects. While the investment effect does not depend on the length of the shock, T, the saving effect will be smaller the larger is T. In fact, we know that, for large enough values of T, the investment effect will prevail since saving eventually becomes negative as well. We also know that for $T = 1$ there is no investment effect and therefore the current account goes into surplus. But what happens for intermediate values of T?

Suppose that the production function takes the Cobb-Douglas form:

$$f(k) = k^{\alpha}. \tag{81}$$

Assume the following parameterization:

$$\begin{aligned}\bar{A} &= 0.1,\\ \bar{A}^{H} &= 0.11,\\ \alpha &= 0.1,\\ r &= 0.3.\end{aligned}$$

Figure 8 illustrates the corresponding results using equation (67) for $T = 1$, equation (**??**) for $T \geq 2$, and equation (80). Panel A depicts the impact response of saving and investment as a function of T, while Panel B illustrates the impact response of the current account. For $T = 1$, there is no investment effect and hence the current account must necessarily be in surplus in $t = 0$. As proved analytically, saving is a decreasing function of T. For $T \geq 2$, investment does not depend on T. For this particular parameterization, the saving effect dominates until the duration of the shock becomes $T = 4$. For any shock lasting more than 4 periods, the investment effect dominates and a temporary increase in productivity would lead to a current account deficit.[34]

6 Final remarks

This chapter has shown how, in a world with no frictions and a constant world real interest rate, a small open economy will achieve perfect consumption smoothing even when output fluctuates over time. As a benchmark, this is perhaps the most important result of modern open economy macroeconomics. In practice, however, developing countries in particular face a variety of frictions that will imply substantial (and welfare reducing) departures from this first-best world. In the next chapter, we will analyze how imperfections in international capital markets may critically affect the economy's ability to achieve

[34] Due to the absence of adjustment costs, the value of r must be rather large for the investment effect not to always dominate the saving effect. For $r = 0.1$, for example, the investment effect dominates already for $T = 2$, so the current account is in deficit for any $T \geq 2$.

consumption smoothing over time. In chapter 3, we will see how policy-induced intertemporal distortions lead the private sector to choose non-flat (and socially sub-optimal) consumption paths.

7 Appendices

7.1 Functionals: Definition and example

This appendix provides a brief review of the difference between functions and functionals (see, for example, Reed (1998) for a detailed treatment).

7.1.1 Functions

A function maps a real number (or vector) into a real number. For example, the function

$$f(x) = 4x^2 + 1 \tag{82}$$

assigns to each value of x a real number, $f(x)$. Thus, if $x = 0$, $f(x) = 1$. Similarly, the function

$$c(t) = 1 + 2t \tag{83}$$

describes a consumption path by assigning to each value of t, a real value $c(t)$. Thus, $c(0) = 1$ and $c(1) = 3$.

7.1.2 Composite functions

A composite function, which is defined as a function operating on a function, is still a function, as defined above. For example, if we let

$$g(x) = x^2,$$

then the composite function $f(g(x))$ is given by

$$f(g(x)) = f(x^2) = 4(x^2)^2 + 1.$$

As we can see, $f(g(x)$ continues to map real numbers into real numbers. Thus, if, say, $x = 0$, then $f(g(0)) = 1$.

Similarly, in terms of an economic example, if we let the instantaneous utility function be

$$u(c) = \log(c),$$

then we can define a function $v(t)$ as a composite function:

$$v(t) \equiv u(c(t)) = \log(1 + 2t),$$

which assigns to each value of t a corresponding value $v(t)$.

7.1.3 Functionals

A functional maps a *function* into a real number. For example, let $f(.)$ be given by (82) and consider the functional $V(f(.))$ defined by

$$V(f(x)) = \int_0^1 f(x)dx.$$

Solving the integral, we find that

$$V(f(x)) = \frac{7}{3}.$$

In terms of an economic example, let the consumption path be given by (83) and suppose that the instantaneous utility function is linear; that is, $u(c) = c$. Assuming a finite horizon $T = 1$, the utility functional $U(c(t))$ is defined by

$$U(c(t)) = \int_0^1 u(c(t))dt.$$

Solving the integral, we find that

$$U(c(t)) = 2.$$

7.2 Pointwise optimization

This appendix formally derives the first-order condition (11). We will show this in two alternative ways: (i) partition method, and (ii) perturbation method.

7.2.1 Partition method

For convenience, let us restate the consumer's problem as:

$$\max_{\{c_t\}} \int_0^T u(c_t)e^{-\beta t}dt,$$

subject to

$$\int_0^T c_t e^{-rt}dt = K,$$

where.

$$K \equiv b_0 + \int_0^T y_t e^{-rt}dt.$$

We proceed by dividing the interval $[0, T]$ into N equal subintervals, each of length Δ:

$$\Delta = \frac{T}{N}.$$

Let t_0, t_1, .., t_N denote the endpoints of these intervals:

$$t_0 = 0,\ t_1 = \Delta,\ t_2 = 2\Delta,\ ..\ ,\ t_N = N\Delta = T. \tag{84}$$

The original consumer's problem can then be approximated as:

$$\begin{aligned}
&\max_{\{c_{t_1},\dots,c_{t_N}\}} u(c_{t_1})e^{-\beta t_1}(t_1 - t_0) + u(c_{t_2})e^{-\beta t_2}(t_2 - t_1) + .. +\ u(c_{t_N})e^{-\beta t_N}(t_N - t_{N-1}) \\
&\text{subject to} \\
&c_{t_1}e^{-rt_1}(t_1 - t_0) + c_{t_2}e^{-rt_2}(t_2 - t_1) + .. +\ c_{t_N}e^{-rt_N}(t_N - t_{N-1}) = K.
\end{aligned}$$

Using (84), we can restate this problem more compactly as:

$$\begin{aligned}
&\max_{\{c_{t_1},\dots,c_{t_N}\}} \sum_{i=1}^{N} u(c_{t_i})e^{-\beta t_i}\Delta \\
&\text{subject to} \\
&\sum_{i=1}^{N} c(t_i)e^{-rt_i}\Delta = K.
\end{aligned}$$

The corresponding Lagrangean reads as

$$\mathcal{L} = \sum_{i=1}^{N} u(c_{t_i})e^{-\beta t_i}\Delta + \lambda\left[K - \sum_{i=1}^{N} c(t_i)e^{-rt_i}\Delta\right].$$

The first order conditions for this problems are given by

$$u'(c_{t_i})e^{-\beta t_i} = \lambda e^{-rt_i}, \qquad \text{for all } i = 1, .., N.$$

These first order conditions are valid for any size of the subintervals. In particular, they are valid for an infinitesimal subinterval. Since as Δ tends to zero, the values of $t_i, i = 1, ..., N$ become arbitrarily close to each other, we can write

$$u'(c_t)e^{-\beta t} = \lambda e^{-rt}, \qquad \text{for all } t \in [0, T]$$

as the first-order condition when the problem is formulated in continuous time.

7.2.2 Perturbation method

Let $f(x)$ be a function that maps numbers from the real line to numbers in the real line. Consider the problem of choosing x so as to maximize $f(x)$. We will show that

$$f'(x^*) = 0$$

is a necessary condition for x^* to be a local maximum.

The logic of the proof will be useful to understand how to demonstrate first order necessary conditions when maximizing more complicated mappings.

If x^* is a local interior maximum then, by definition,

$$f(x^*) \geq f(x^* + \Delta x), \tag{85}$$

where $\Delta x (\equiv x - x^*)$ represents a "perturbation."

For a small Δx around x^*,

$$f(x^* + \Delta x) = f(x) \simeq f(x^*) + f'(x^*)\Delta x. \tag{86}$$

Inserting (86) in (85) we obtain the fundamental inequality

$$f'(x^*)\Delta x \leq 0. \tag{87}$$

For condition (87) to hold for any arbitrary Δx, it must be the case that

$$f'(x^*) = 0. \tag{88}$$

7.2.3 Maximizing functionals

Let $\int_0^T f(x(t))dt$ be a functional that maps functions from a set of eligible functions to numbers in the real line.[35] Consider the problem of choosing the function $x(t)$ so as to maximize $\int_0^T f(x(t))dt$. In such a problem, if $x^*(t)$ is a local interior maximum then, by definition,

$$\int_0^T f(x^*(t))dt \geq \int_0^T f(x^*(t) + \varepsilon h(t))dt, \tag{89}$$

where ε is an arbitrary real number and $h(t)$ is any arbitrary function in the opportunity set. The term $\varepsilon h(t)$ represents a "perturbation." We will derive a necessary condition for $x^*(t)$ to be a local maximum of the objective functional.

Note that

$$\int_0^T f(x(t))dt = \int_0^T f(x^*(t) + \varepsilon h(t))dt = F(\varepsilon). \tag{90}$$

Thus, we now have a mapping from numbers in the real line into numbers in the real line. By construction, $F(\varepsilon)$ must be maximized when $\varepsilon = 0$. From (88) we know that a necessary condition for local interior maximum is

$$F'(0) = 0.$$

Using Leibniz's rule to differentiate (90),

$$F'(0) = \int_0^T f'(x^*(t))h(t)dt = 0. \tag{91}$$

[35] A functional should not be confused with a function. It involves a different mapping.

For condition (91) to hold for any arbitrary $h(t)$, it must be the case that

$$f'(x^*(t)) = 0 \tag{92}$$

for any value of $t \in [0, T]$.

7.3 Proof that solution is a maximum[36]

We now show that a consumption path characterized by first-order condition (11) and intertemporal constraint (10) characterizes a maximum. Since $u(c)$ is strictly concave, then for any two points, c^1 and c^2, $c^1 \neq c^2$, it must be true that

$$u(c^2) - u(c^1) < u'(c^1)(c^2 - c^1).$$

Let path c_t^1 satisfy first-order condition (11) and intertemporal constraint (10). Let path c_t^2 satisfy intertemporal constraint (10). Then,

$$\int_0^T \left[u(c_t^2) - u(c_t^1)\right] e^{-\beta t} dt \leq \int_0^T u'(c_t^1)(c_t^2 - c_t^1) e^{-\beta t} dt = \lambda \int_0^T (c_t^2 - c_t^1) e^{-\beta t} dt = 0,$$

where we have used first-order condition (11) and the right-most equality follows from the fact that, by assumption, both paths satisfy intertemporal constraint (10). Hence, path c_t^1 is indeed a maximum.

7.4 Derivation of trade balance and current account paths

This appendix computes the path of the current account for the case of an unanticipated and temporary rise in productivity. We follow an iterative procedure which consists in first computing the change in the current account in $t = 0$, then computing the resulting change in net foreign assets, which allows us to compute the change in the current account in $t = 1$, and then repeating the same steps for all subsequent time periods.

Path of the trade balance Since $I_t = 0$ for $t = 1, 2, ..T - 1$, the trade balance for $t = 1, 2, ..T - 1$ is given

$$TB_t = y_t - c_t, \qquad t = 1, 2, ..., T - 1.$$

Using (189) and (79), we obtain:

[36] We follow Calvo (1996).

$$
\begin{aligned}
TB_t|_{t=1,2,\ldots,T-1} &= -rb_0 + \left[\bar{A}^H f(\bar{k}^H) - \bar{A} f(\bar{k})\right] + \left[1 - \left(\frac{1}{1+r}\right)^T\right] f(\bar{k})(\bar{A}^H - \bar{A}) \\
&\quad + \frac{r}{1+r}\left[1 - \left(\frac{1}{1+r}\right)^{T-1}\right]\left[\frac{\bar{A}^H f(\bar{k}^H) - \bar{A}^H f(\bar{k})}{r} - (\bar{k}^H - \bar{k})\right] \\
&> -rb_0.
\end{aligned}
$$

Hence, for $b_0 = 0$, the trade balance goes into surplus in $t = 1$. For $t = T - 1$, we need to add to the above expression the disinvestment effect:

$$
\begin{aligned}
TB_{T-1} &= -rb_0 + \left[\bar{A}^H f(\bar{k}^H) - \bar{A} f(\bar{k})\right] + \left[1 - \left(\frac{1}{1+r}\right)^T\right] f(\bar{k})(\bar{A}^H - \bar{A}) \\
&\quad + \frac{r}{1+r}\left[1 - \left(\frac{1}{1+r}\right)^{T-1}\right]\left[\frac{\bar{A}^H f(\bar{k}^H) - \bar{A}^H f(\bar{k})}{r} - (\bar{k}^H - \bar{k})\right] - \underbrace{(k_T - k_{T-1})}_{-} \\
&> TB_t|_{t=1,2,\ldots,T-1}.
\end{aligned}
$$

Since investment is negative in $T-1$, the trade balance will be larger in $T-1$ than in $T-2$.

What about T? Intuitively, it is clear that there should be a deficit (for $b_0 = 0$). Output is the same as before the shock but we have a higher level of consumption. Indeed, using (189) and (79)

$$
\begin{aligned}
TB_T &= -rb_0 - \left\{\left[1 - \left(\frac{1}{1+r}\right)^T\right] f(\bar{k})(\bar{A}' - \bar{A})\right. \\
&\quad \left. + \frac{r}{1+r}\left[1 - \left(\frac{1}{1+r}\right)^{T-1}\right]\left[\frac{\bar{A}^H f(\bar{k}^H) - \bar{A}^H f(\bar{k})}{r} - (\bar{k}^H - \bar{k})\right]\right\} < -rb_0
\end{aligned}
$$

Path of saving By definition, saving in period 1 is given by

$$
S_1 = rb_1 + \bar{A}^H f(\bar{k}^H) - \bar{c}.
$$

Adding and subtracting $rb_0 + \bar{A}^H f(\bar{k})$ and noting that $S_0 = rb_0 + \bar{A}^H f(\bar{k}) - \bar{c}$ and $b_1 - b_0 = S_0 - I_0$, we can rewrite this expression as

$$
S_1 = (1+r)S_0 + r\left[\bar{A}^H\left(\frac{f(\bar{k}^H) - f(\bar{k})}{r}\right) - I_0\right] > S_0. \tag{93}
$$

Saving in $t = 1$ is thus bigger than in $t = 0$. Proceeding in a similar way, it is easy to check that for periods $t = 2$ through $t = T - 1$ saving can be written recursively as

$$S_t = (1+r)S_{t-1}, \qquad t = 2, ..., T-1. \tag{94}$$

Saving thus keeps increasing over time up to (and including) period $T-1$. By the same token, saving in period T is given by

$$S_T = (1+r)S_{T-1} - rI_{T-1} - \left[\bar{A}^H f(\bar{k}^H) - \bar{A} f(\bar{k})\right].$$

To show that $S_T = 0$, use (94) repeatedly to express S_T as a function of S_0:

$$S_T = (1+r)^T S_0 + (1+r)^{T-1} r \left[\bar{A}^H \left(\frac{f(\bar{k}^H) - f(\bar{k})}{r}\right) - I_0\right] - rI_{T-1} - \left[\bar{A}^H f(\bar{k}^H) - \bar{A} f(\bar{k})\right].$$

Using the expression for $\bar{c}$ given by (79) and the fact that $S_0 = rb_0 + \bar{A}^H f(\bar{k}) - \bar{c}$, it follows that $S_T = 0$.

Path of the current account We have already shown in the text that the current account in period 0 could take any sign. In period 1, investment is zero and therefore $CA_1 = S_1$. Hence, from (93),

$$CA_1 = S_1 = (1+r)S_0 + r\left[\bar{A}^H \left(\frac{f(\bar{k}^H) - f(\bar{k})}{r}\right) - I_0\right] > CA_0.$$

The current account balance in $t = 1$ is larger than in $t = 0$ because $S_1 > S_0$ and there is no investment. Since investment remains zero up to and including period $T-2$, it follows that

$$CA_t = S_t, \qquad t = 2, ..., T-2.$$

Since we have already established that saving increases up to an including $T-2$, the current account balances also increases over time. In period $T-1$, the current account is given by

$$CA_{T-1} = S_{T-1} - I_{T-1} > CA_{T-2},$$

where the inequality follows from the fact that $S_{T-1} > S_{T-2}$ and $I_{T-1} < 0$. In sum, while the level of the current account balance in period 0 is ambiguous, its level keeps increasing over time up to and including period $T-1$. In period T, the current account is zero since, as shown above, saving in T is zero and investment is also zero.

7.5 Recursive formulation

While simple problems – like the one in the text – can be solved with standard Lagrangean-type methods, more elaborate problems are best solved using dynamic programming (i.e., recursive) techniques. To illustrate these methods, this appendix solves the discrete-time version of our basic problem using recursive techniques.[37]

7.5.1 Euler equation derivation

The maximization problem, in sequential form, consists in choosing $\{c_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty} \log(c_t),$$

subject to the sequence of flow constraints

$$c_t + b_{t+1} = y_t + (1+r)\, b_t, \qquad t = 0, 1, ... \tag{95}$$

for a given b_0.

The Bellman equation for this problem is given by

$$V_t(b_t) = \max_{\{c_t, b_{t+1}\}} \{\log(c_t) + \beta V_t(b_{t+1})\} \tag{96}$$

subject to (95). Substituting (95) into (96)

$$V_t(b_t) = \max_{\{b_{t+1}\}} \{\log(y_t + (1+r)\, b_t - b_{t+1}) + \beta V_t(b_{t+1})\}. \tag{97}$$

The first-order condition for the right-hand side problem is given by

$$-\frac{1}{y_t + (1+r)\, b_t - b_{t+1}} + \beta V_t'(b_{t+1}) = 0. \tag{98}$$

The envelope condition follows from differentiating (97) with respect to b_t:

$$V_t'(b_t) = \frac{1}{y_t + (1+r)\, b_t - b_{t+1}} (1+r). \tag{99}$$

Forwarding (99) one period and replacing it in (98),

$$\frac{1}{y_t + (1+r)\, b_t - b_{t+1}} = \beta \frac{1}{y_{t+1} + (1+r)\, b_{t+1} - b_{t+2}} (1+r).$$

Using the budget constraint (95), we can rewrite the last expression as:

$$\frac{c_{t+1}}{c_t} = \beta\,(1+r), \tag{100}$$

which is the Euler equation in discrete time.[38]

[37] We apply recursive techniques to the discrete-time version because they are most often used in that context, as a tool for computational methods. For an application of recursive techniques to continuous-time problems, see Kamien and Schwartz (1991, Section 21). For a detailed treatment of the use of recursive techniques in macroeconomics, see Ljungqvist and Sargent (2000).

[38] Exercise 10 at the end of the chapter asks you to check that we can derive the Euler equation in slightly different ways.

7.5.2 Guess and verify

An alternative way of proceeding is to make a guess for the value function and verify that it satisfies the Bellman equation. This method works, of course, if the value function is time invariant. For this to be the case, both the utility function and the transition equation (i.e., the budget constraint) must be time invariant. To simplify, we will assume that $y = 0$.

Our guess for the value function is

$$V(b_t) = A\log(b_t) + B,$$

where A and B are constants to be determined. Substituting into the Bellman equation (96)

$$V(b_t) = \max_{c_t} \left\{\log(c_t) + \beta \left[A\log((1+r)\, b_t - c_t) + B\right]\right\}. \tag{101}$$

The first-order condition with respect to c_t is given by

$$\frac{1}{c_t} - \frac{\beta A}{(1+r)\, b_t - c_t} = 0.$$

Rearranging this expression:

$$c_t = \left[\frac{(1+r)\, b_t}{1+\beta A}\right]. \tag{102}$$

This is the decision rule (or policy function) for current consumption, but it is expressed in terms of the unknown coefficient A. Substituting (102) into (101), we obtain:

$$V(b_t) = [1+\beta A]\log(b_t) + \log\left(\frac{\beta A}{1+\beta A}\right) + \log(1+r) - \log(\beta A) + \beta A\log(1+r) + \beta A\log\left(1 - \frac{1}{1+\beta A}\right) + \beta B. \tag{103}$$

This equation takes the form:

$$V(b_t) = \Theta\log(b_t) + \Gamma, \tag{104}$$

where

$$\begin{aligned}
\Theta &\equiv 1+\beta A, \\
\Gamma &\equiv \log\left(\frac{\beta A}{1+\beta A}\right) + \log(1+r) - \log(\beta A) + \beta A\log(1+r) + \beta A\log(\left[1 - \frac{1}{1+\beta A}\right]) + \beta B.
\end{aligned}$$

For our guess to be correct, B needs to be equal to Γ and $A = 1+\beta A$. It follows that

$$A = \frac{1}{(1-\beta)}.$$

Substituting A into 102)

$$c_t = (1-\beta)(1+r)\, b_t. \tag{105}$$

If $\beta(1+r) = 1$, then

$$c_t = r b_t. \tag{106}$$

This is the policy rule. This, of course, coincides with the solution that we would obtain solving the problem with pointwise maximization for the case that $y = 0$.

Clearly, for our purposes, the method of guessing and verify is too cumbersome compared with the solution techniques used in the text. But this derivation illustrates the rationale behind the main use of recursive methods in macroeconomics and international finance, which is to solve more complicated models using the computer. As Ljungqvist and Sargent (2000) discuss, the main numerical methods to solve functional equations of the form (96) is value-function iteration or policy-function iteration. In either case, one makes an initial guess for the value function or the policy function and iterates using (96) and until convergence is reached.

7.6 Discrete-time version of endowment model for MATLAB coding

This appendix develops a discrete-time version of the endowment model discussed in the text for the purposes of solving it numerically using MATLAB. In spite of the model's simplicity, such an exercise will prove extremely useful for subsequent chapters where monetary versions of this model will be used to derive various results numerically.

Preferences are given by

$$\sum \beta^t \ln(c_t).$$

The consumer's flow constraint is given by

$$b_{t+1} = (1+r)b_t + y_t - c_t. \tag{107}$$

Setting up the Lagrangean:

$$\underset{\{c_t, b_{t+1}\}}{\text{Max}} \mathcal{L} = \sum \beta^t \ln(c_t) + \sum \beta^t \lambda_t \left[(1+r)b_t + y_t - c_t - b_{t+1}\right].$$

First-order conditions are (assuming $\beta(1+r) = 1$)

$$\frac{1}{c_t} = \lambda_t, \tag{108}$$

$$\lambda_t = \lambda_{t+1}. \tag{109}$$

For further reference, define the trade balance as

$$TB_t \equiv y_t - c_t. \tag{110}$$

7.6.1 Linearization and Matlab code

The system of equations to be linearized and coded into MATLAB consists of three equations (equations (107), (108), and (109)) in three variables (c, b, and λ) for a given path of y_t and an initial value for b, b_0. The three endogenous variables are divided into three categories: controls (c), co-states (λ), and states (b). In addition there is a flow variable (TB) that can be written as function of existing endogenous variables.[39] The exogenous variable in the system is y.

Equations, in turn, are also divided into (i) control equations (equation (108)); (ii) co-state and state equations (equations (109) and (107), respectively; and (iii) flow equations (equation (110)). The number of equations in each category corresponds to the number of variables in each category.

We first present the system of equations, then linearize it around the steady-state, and then coded it for the purposes of using the King-Plosser-Rebelo (1988) MATLAB routine.[40]

System of equations The system of equations to be linearized consists of one control equation (C1), one co-state equation (S1), one state equation (S2), and one flow equation (FV1):

$$\begin{aligned} \frac{1}{c_t} &= \lambda_t, & (\text{ C1}) \\ \lambda_t &= \lambda_{t+1}, & (111) \\ b_{t+1} &= (1+r)b_t + y_t - c_t, & (\text{ S2}) \\ TB_t &\equiv y_t - c_t. & (\text{FV1}) \end{aligned}$$

Steady state The steady-state follows from equations (C1), (S2), and (FV1): C1

$$\begin{aligned} c_{ss} &= y + rb_0, & (112) \\ c_{ss} &= \frac{1}{\lambda_{ss}}, \\ TB_{ss} &= y - c_{ss}. \end{aligned}$$

Notice that (112) is an approximation because – as the text has made clear – the economy may not return to its initial stationary equilibrium. It is to avoid this problem that, from a numerical point of view, it may make sense to "close" the model by inducing stationarity (see Schmitt-Grohe and Uribe (2003)).

[39] For a detailed presentation of this linearization method, see King, Plosser, and Rebelo (2001).

[40] The accuracy of the linear approximation, however, may suffer because the system has a unit root and thus the system does not necessarily return to its initial stationary state. Alternatively, one could induce stationary in the model for numerical purposes along the lines suggested by Schmitt-Grohe and Uribe (2003). It is important to notice, however, that Schmitt-Grohe and Uribe (2003) find that the impulse responses are virtually identical when comparing the stationary model to the non-stationary one.

Linearization Linearizing equations (C1), (S1), (S2) and (FV1):

$$-\widehat{c_t} = \widehat{\lambda}_t, \tag{C1L}$$

$$\widehat{\lambda}_t - \widehat{\lambda_{t+1}} = 0, \tag{S1L}$$

$$\widehat{b_{t+1}} - (1+r)\widehat{b}_t = \frac{y}{b_0}\widehat{y_t} - \frac{c_{ss}}{b_0}\widehat{c}_t, \tag{S2L}$$

$$\widehat{TB_t} = \frac{y}{TB_{ss}}\widehat{y_t} - \frac{c_{ss}}{TB_{ss}}\widehat{c}_t. \tag{FV1L}$$

We should note that since the parametrization that we will choose implies a negative value of TB_{ss}, we will multiply the coefficient on the RHS of (FV1L) by -1 so that a positive (negative) percentage in the trade balance reflects an improvement (worsening) of the trade balance.

We will now write the linearized system in matrix form for the purposes of inputting it into MATLAB. The first sub-system consists of equation C1L and links control variables to states, co-states, and exogenous variables:[41]

$$\underbrace{M_{cc}}_{1x1}\ \widehat{c_t} = \underbrace{M_{cs}}_{1x2}\begin{bmatrix} \widehat{b_{t-1}} \\ \widehat{\lambda}_t \end{bmatrix},$$

where the elements of M_{cc} and M_{cs} follow from equations (C1L)-(C4L):[42]

$$M_{cc} = \begin{bmatrix} -1 \end{bmatrix},$$
$$M_{cs} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The second sub-system comprises (S1L) and (S2L) and links changes in states and co-states variables to changes in control and exogenous variables.

$$\underbrace{M_{ss1}}_{2x2}\begin{bmatrix} \widehat{b_{t-1}} \\ \widehat{\lambda}_t \end{bmatrix} + \underbrace{M_{ss0}}_{2x2}\begin{bmatrix} \widehat{b}_t \\ \widehat{\lambda_{t+1}} \end{bmatrix} = \underbrace{M_{sc1}}_{2x1}\begin{bmatrix} \widehat{c_t} \end{bmatrix} + \underbrace{M_{se1}}_{2x1}\widehat{y_t},$$

$$M_{ss1} = \begin{bmatrix} 0 & 1 \\ -(1+r) & 0 \end{bmatrix},$$

$$M_{ss0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$M_{sc1} = \begin{bmatrix} 0 \\ -\frac{c_{ss}}{b_0} \end{bmatrix},$$

[41] In this particular model, M_{ce} is a zero vector because the exogenous variable, y, does not enter the control equations. In the standard growth model, some elements of M_{ce} may differ from zero (see King, Plosser, and Rebelo (1988)).

[42] The mnemonic of this matrix notation is that M_{cc} relates control variables to control variables and M_{cs} relates control variables to states and co-states variables.

$$M_{se1} = \begin{bmatrix} 0 \\ \frac{y}{b_0} \end{bmatrix}.$$

The last and third subsystem comprises equation (FV1L) and links changes in flow variables to changes in controls and co-states:

$$\widehat{TB}_t = \underbrace{FV_c}_{1x1}\widehat{c}_t + \underbrace{FV_e}_{1x1}\widehat{y}_t,$$

where

$$FV_c = \left[\frac{c_{ss}}{TB_{ss}}\right],$$

$$FV_e = \left[-\frac{y}{TB_{ss}}\right].$$

MATLAB program The above system has been inputted into MATLAB and the corresponding program files can be found on the book website at http://www.econ.umd.edu/~vegh/book/book.htm. There are five files: (1) *dynendowment.m*; (2) *mdrendowment.m*; (3) *implendowment.m*; (4) *pathendowment.m* and (5) *plotimpendowment.m*.

To replicate numerically the experiments conducted in the text, we can proceed as follows:

- To solve for an unanticipated and permanent change in the endowment, you first need to run *dynendowment.m*. You then need to run *mdrendowment.m*, which will ask you for the serial correlation of the shock, which should be one (because the shock is permanent). You then need to run the program *implendowment.m*, which will ask you to choose the y shock and specify the size of the shock (say, -10 for a reduction of 10 percent in the endowment). This program will then automatically call on the program *plotimpmbs.m* to generate the plot. 9 illustrates the output of the program for a reduction of 10 percent in the endowment.[43] All plots represent percentage deviations relative to the initial stationary equilibrium. As expected, consumption falls by the permanent income component and there is no change in the trade balance or the current account.[44] These results are, of course identical to the ones derived analytically (recall Figure 2).

- To solve for an unanticipated and temporary reduction in the endowment, you first need to run *dynendowment.m*. You then need to specify a path for the endowment in *pathendowment.m*. Finally, you need to run *plotimpmbs.m* to generate the plot. Figure 10 illustrates the output of

[43] The parametriztion for Figures 10 and 11 is as follows: $r = 0.015$; $b_0 = 5$; and $y = 1$.
[44] The response in Panel C and D is essentially zero (note the scale).

the program for a reduction of 10 percent in the endowment for ten periods. The results are exactly in line with those derived theoretically (recall Figure 3).

If you want to change the parameterization of the model, you need to edit the *dynendowment.m* program.

We conclude that, in spite of the non-stationarity of the model, the linearization method pioneered by King, Plosser, and Rebelo (1998, 2001) offers a simple method to derive a qualitatively correct solution. We will frequently resort to this method in subsequent chapters.

Exercises[45]

1. Time inconsistency: the role of discounting

This exercise takes as given the time separability of preferences and illustrates the role of hyperbolic discounting in generating time inconsistency. To this end, the exercise first asks you to verify that time-separable preferences with exponential discounting (i.e., the preferences used in the text) are time consistent. It then asks you to work out an example in which the introduction of hyperbolic discounting renders preferences time-inconsistent.[46]

(a) *Exponential consumer*

Suppose that the preferences of the "exponential" consumer are given by:

$$U_0(c_0, c_1, ...) = \sum_{t=0}^{\infty} \delta^t \log(c_t), \tag{113}$$

where c_t is consumption in period t, $\delta \in (0,1)$, and the subscript 0 on the lifetime utility function indicates that the consumption path is being evaluated as of time $t = 0$. Assume $\delta(1+r) = 1$.

The flow constraint for period t is given by

$$b_{t+1} = (1+r)b_t + y - c_t,$$

where y is the constant endowment, r is the exogenously-given world real interest rate, and b_t are net foreign assets held between period t and period $t+1$. Iterating forward the flow constraint and imposing the condition that

$$\lim_{t\to\infty} \frac{b_{t+1}}{(1+r)^t} = 0$$

yields the following intertemporal constraint:

$$\sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t c_t = (1+r)\left(b_0 + \frac{y}{r}\right). \tag{114}$$

In this context:

[45] An answer key is available from the author upon request.

[46] See Backus, Routledge, and Zin (2004) for a detailed discussion on the role preferences in generating time-inconsistency problems. See also the discussion in Calvo (1996).

i. Define the discount factor at time t (a measure of the degree of the consumer's impatience) as the marginal rate of substitution between consumption at two consecutive dates for a constant consumption path, $\bar{c}$:

$$\text{Discount factor}_t \equiv MRS^t_{t,t+1} = \left. \frac{\partial U(c_0, c_1, ...)/\partial c_{t+1}}{\partial U(c_0, c_1, ...)/\partial c_t} \right|_{c_0=c_1=...=\bar{c}}.$$

Suppose that the consumer is standing at time t. Compute the discount factor for consumption between $t+1$ and $t+2$ (i.e., compute $MRS^t_{t+1,t+2}$). Next suppose that the consumer is standing at $t+1$. Recompute the discount factor for consumption between $t+1$ and $t+2$ (i.e., compute $MRS^{t+1}_{t+1,t+2}$). Verify $MRS^t_{t+1,t+2} = MRS^{t+1}_{t+1,t+2}$.

ii. Denote by c^0_t, $t = 0, 1, ...$ the optimal consumption path chosen at $t = 0$. Compute reduced-forms for c^0_t, $t = 0, 1, ...$

iii. Denote by c^1_t, $t = 1, 2, ...$ the optimal consumption path chosen at $t = 1$. Verify that the optimal consumption path chosen at $t = 1$ is time consistent (i.e., coincides with the path chosen at $t = 0$). (Hint: Compute reduced-forms for c^1_t, $t = 1, 2, ...$ and check that $c^0_1 = c^1_1$, $c^0_2 = c^1_2$, and so forth).

(b) *Hyperbolic consumer*

Suppose now that preferences are of the "quasi-hyperbolic" type:

$$U_0(c_0, c_1, ...) = \log(c_0) + \rho \sum_{t=1}^{\infty} \delta^t \log(c_t), \tag{115}$$

where $\rho \in (0, 1)$. Assume $\delta(1 + r) = 1$.

In this context:

i. Suppose that the consumer is standing at time t. Compute the discount factor for consumption between $t+1$ and $t+2$ (i.e., compute $MRS^t_{t+1,t+2}$). Next suppose that the consumer is standing at $t + 1$. Recompute the discount factor for consumption between $t + 1$ and $t + 2$ (i.e., compute $MRS^{t+1}_{t+1,t+2}$). Verify that $MRS^t_{t+1,t+2} > MRS^{t+1}_{t+1,t+2}$. Notice that this implies that the consumer is more patient about consuming between tomorrow and the day after tomorrow from the standpoint of today than from the standpoint of tomorrow. In this sense, the consumer is more impatient in the "short-run" than in the "long-run", which is the defining characteristic of hyperbolic discounting.

ii. Compute reduced forms for c^0_t, $t = 0, 1, ...$

iii. Compute reduced forms for c^1_t, $t = 1, 2, ...$ Assume, for simplicity, that $b_0 = 0$. Show that the optimal consumption path chosen at $t = 1$ is time inconsistent (i.e., does not coincide with the path chosen at $t = 0$). (Hint: Compute reduced-forms for c_1^j, $j = 0, 1, ...$ and verify that $c_1^1 > c_1^0$.) Explain intuitively the source of the time inconsistency.

2. Time inconsistency: the role of non time-separability

This exercise – which complements the previous one – takes as given the presence of exponential discounting and illustrates the role of non time-separability in generating time inconsistency.

(a) *Past consumption affects today's utility*

Let preferences be given by:

$$U_0(c_0, c_1, ...) = \log(c_0) + \sum_{t=1}^{\infty} \delta^t [\log(c_t) + \alpha \log(c_{t-1})].$$

The sign of α defines the type of preferences. Naturally, if $\alpha = 0$, these are the standard time-separable preferences. If $\alpha < 0$, there is habit persistence (or habit formation) in the sense that higher consumption in period $t - 1$ decreases utility in t (capturing the idea that consumers get used or "addicted" to that level of consumption). If $\alpha > 0$, there is durability of consumption goods in the sense that higher consumption at $t-1$ increases utility at t.[47] The intertemporal constraint remains given by (114). (Assume $\delta(1 + r) = 1$.)

In this context:

i. Compute reduced forms for c_t^0, $t = 0, 1, ...$

ii. Compute reduced forms for c_t^1, $t = 0, 1, ...$Verify that the optimal consumption path chosen at $t = 1$ is time consistent.

(b) *Future consumption yields utility*

Let preferences be given by:

$$U_0(c_0, c_1, ...) = \sum_{t=0}^{\infty} \delta^t [\log(c_t) + \log(c_{t+1})].$$

In this case the consumer derives utility not only from today's consumption but also from next period's consumption. Assume $\delta(1 + r) = 1$.

i. Compute reduced forms for c_t^0, $t = 0, 1, ...$

[47] Habit persistence has been used extensively in the finance literature as a possible explanation for the equity-premium puzzle (see Constantinides, 1990). In the area of development macroeconomics, habit persistence has been used to explain some of the stylized facts associated with exhange rate-based stabilization (see Uribe (2002)). We will examine these issues in detail in Chapter 13.

ii. Compute reduced forms for c_t^1, $t = 0, 1, ...$ Assume, for simplicity, that $b_0 = 0$. Show that the optimal consumption path chosen at $t = 1$ is time inconsistent. Explain intuitively the source of time inconsistency.

3. Roots of the system

Consider the infinite horizon model of Section 2.2 with logarithmic preferences. In this context:

(a) Show that the system has roots $r - \beta$ and r. (If $r = \beta$, then the roots are zero and r.)

(b) Consider a discrete time version of the model. Show that for the $(1 + r)\beta = 1$ case, the roots are 1 and $1 + r$.

4. Consumption tilting

This exercise analyzes consumption tilting (i.e., optimal consumption plans when β is not necessarily equal to r) in both the finite and infinite horizon settings.

Let the instantaneous utility function be given by:

$$u(c_t) = \frac{c_t^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}}, \tag{116}$$

where $\sigma > 0$ denotes the intertemporal rate of substitution in consumption.

(a) *Finite horizon*

Consider the finite horizon problem analyzed in the text, in which the consumer maximizes (1) (with the instantaneous utility function given by (116)) subject to (10). In this context:

i. Derive the first-order conditions for the consumer's problem and show how the rate of growth of consumption depends on the relation between β and r. Explain the intuition behind the results.

ii. Derive a closed-form solution for c_t.

(b) *Infinite horizon (based on Calvo (1996))*

In an infinite horizon setting, the existence of a well-defined optimal consumption path when r is different from β cannot be taken for granted. (By well-defined optimal consumption path we mean a consumption path whose present discounted value is finite.)

Consider the basic infinite horizon model described in the text, in which the consumer maximizes (13) (with the instantaneous utility function given by (116)) subject to (18). For simplicity, let the endowment stream be constant over time and equal to y and $b_0 = 0$. In this context:

i. Derive a condition involving r, β, and σ that guarantees the existence of a well-defined optimal consumption path. In particular, show that $\sigma \leq 1$ is a *sufficient* condition for existence. [Hint: Solve for the optimal consumption path, write the intertemporal budget constraint in terms of c_0, and establish the condition for the integral to converge.]

ii. To illustrate the fact that $\sigma \leq 1$ is not a *necessary* condition to guarantee the existence of a well-defined optimal consumption path, consider the case in which $\sigma = 1.5$. What is the condition involving r and β for which existence is guaranteed?

iii. Restrict your attention to cases in which a well-defined optimal consumption path exists. Show that:

   A. If $r = \beta$, $c_t = y$ for all $t \geq 0$.

   B. If $r > \beta$, $c_0 < y$ and consumption increases over time.

   C. If $r < \beta$, $c_0 > y$ and consumption falls over time.

iv. Check that the same condition that you derived in (i) above guarantees that the utility functional (13), with $u(c)$ given by (116), converges.

5. Fluctuating real interest rate

Consider the infinite horizon model analyzed in Subsection 2.2 but suppose that the world real interest rate fluctuates over time. In particular – and to fix ideas – assume that the time path of the real interest rate is given by

$$r = \begin{cases} r^H & \text{for} \quad 0 \leqslant t \leqslant T, \\ r^L & \text{for} \quad t > T, \end{cases}$$

where $r^H > \beta$ and $r^L < \beta$. Assume logarithmic preferences.

In this context, derive a reduced-form solution for the path of consumption.

6. Adding labor supply to the basic model

This exercise adds labor supply to the basic infinite horizon model of Section 2.2. Production is thus endogenous. Specifically, consider the economy of Section 2.2 with the following modifications (same notation is used).

Households. Let preferences be given by

$$\int_0^{\infty} \log[c_t - \phi(\ell_t^s)^{\upsilon}]e^{-\beta t}dt,$$

where ℓ^s is labor supply and $\phi(>0)$ and $\upsilon(>1)$ are positive parameters. The household's flow constraint is given by

$$\dot{b}_t = rb_t + w_t\ell_t^s - c_t + \Omega_t,$$

where w is the real wage and Ω_t are the profits from firms (i.e., households own the firms). The corresponding intertemporal constraint is given by

$$b_0 + \int_0^{\infty} \left(w_t\ell_t^s + \Omega_t\right) e^{-rt}dt = \int_0^{\infty} c_t e^{-rt}dt.$$

Firms Firms face a static problem. Production is given by

$$y_t^T = \Psi_t \left(\ell_t^d\right)^{\alpha}, \qquad \alpha < 1,$$

where ℓ^d is labor demand and Ψ is a productivity parameter.

In the context of this model:

(a) Solve the household's maximization problem. Using the first-order conditions, derive a labor supply function (i.e., express ℓ_t^s as a function of w_t). Explain the intuition behind your derivations.

(b) Solve for the firms' problem. Explain the intuition behind the results.

(c) After imposing labor market equilibrium (i.e., $\ell_t^s = \ell_t^d$), solve for a perfect foresight path along which Ψ_t is at some high value between time 0 and time T and low afterwards. (For expositional clarity, make sure that you plot the paths of all endogenous variables against time, including the trade balance and current account.) Explain the intuition behind all of your results.

(d) What key difference do you notice in the behavior of consumption relative to the model of Section 2.2? Show that if the labor supply elasticity is small (as is the case in practice) then, from a quantitative point, the behavior of consumption in response to a fluctuating path of Ψ_t will not differ significantly from the behavior of consumption in response to a fluctuating endowment path in the model of Section 2.2.[48]

[48] In practice, however, total labor hours do fluctuate considerably over the business cycle due to entry and exit from the labor force (the extensive margin), as opposed to changes in hours worked by existing agents (the intensive margin). To generate the observed comovement between total labor hours and the cycle at an aggregate level, one needs to incorporate this "extensive margin" into the model (see King and Rebelo (1999) for a detailed discussion.)

7. Decentralized economy

This exercise asks you to check that the centralized production economy analyzed in the text can be decentralized. Suppose that there are two agents in the economy: consumers and firms. Consumers own the capital stock and own the firms. There is a market for physical capital in which consumers rent the capital stock to firms at a rate r^k. Firms produce the good using the capital stock and give back profits to consumers. Preferences and technology are the same as in the text.

   (a) Write down the consumer's flow constraint and then derive the consumer's intertemporal constraint. Derive the consumer's first-order conditions.
   (b) Write down the firm's flow constraint and derive the first-order condition.
   (c) Show that the optimality conditions characterizing consumption and the capital stock are the same as in the centralized economy.
   (d) Derive aggregate constraints (both the flow constraint, or current account, and the intertemporal constraint) and show that they correspond exactly to those for the centralized economy (equations (41) and (42)).

8. Small changes in productivity

This exercise asks you to revisit some of the experiments performed in the text for the model with investment for the case in which the change in productivity is small (i.e., the change is given by dA). This exercise will shed light on the reaction of saving to changes in productivity.

Consider the model with investment described in Section (5). In this context:

   (a) Analyze the effects of a (small) unanticipated and permanent increase in A. [In other words, assume that A changes by dA.] In particular, show that there will be no change in saving. Explain the intuition behind the results.
   (b) Analyze the effects of a (small) unanticipated and temporary increase in A that lasts for $T(\geq 2)$ periods. Derive a reduced form for the change in the current account in period 0 and show how it depends on T. Explain the intuition behind the results.

9. Numerical example

Consider the model with investment of Chapter 1 and focus on an temporary increase in productivity that lasts for 4 periods (i.e., $T = 4$). Using any computer program that you feel comfortable with (Excel, Matlab, Mathematica), solve the model numerically and generate the following pictures:

(a) Plot as a function of *time* output, investment, consumption, saving, trade balance, and current account for a case in which the current account initially worsens.

(b) Plot as a function of *time* output, investment, consumption, saving, trade balance, and current account for a case in which the current account initially improves.

Provide some intuition regarding the paths of the different variables in each of the two cases

[Note that these are NOT the plots presented in Figure 8 in Chapter 1. Those plots in the book depict the response in $t = 0$ of saving, investment, and the current account as a function of the duration of the shock (T). Here you are asked to take T as given and plot the path of the variables over time]

10. Alternative ways of deriving the Euler equation in the recursive formulation

Show that one can also derive the Euler equation (100) by proceeding in either of the two following ways:

(a) Substitute for b_{t+1} in the Bellman equation (96) and differentiate with respect to c_t

(b) Incorporate constraint (95) into the Bellman equation (96) using a Lagrange multiplier and differentiate with respect to c_t and b_{t+1}.

11. Numerical solutions using MATLAB

To get acquainted with the MATLAB programs detailed in the Appendix to this Chapter – and using the same parameterization as in the text – compute the time paths for consumption, trade balance, and net foreign assets corresponding to a temporary fall in output for four different values of T: 5, 10, 20, and 100. Compare the different time paths for consumption and the trade balance and discuss the intuition behind the results.

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Box 1. Is saving procyclical?

Our model predicts that, in response to an unanticipated and temporary increase (fall) in output, there should be a positive co-movement between output and saving as households save (dissave) in order to smooth out consumption over time. In a stochastic version of our model, this comovement would translate into a positive correlation between saving and output.[49] In other words, saving should be procyclical.

What do the data say? Table 1 reports correlations between the cyclical component of saving (as a proportion of GDP) and GDP. Series were detrended using the HP filter. On average – and as predicted by the model – the correlation is positive for all three groups of countries: industrial, Latin America, and other developing countries. The averages are 0.27, 0.13, and 0.13, respectively. In only 9 cases (out of 57) is the correlation negative, with most negative correlations in Latin America (5 out of 19). The evidence thus suggests that the saving/GDP correlation is lower in developing countries than in industrial countries. In fact, Lane and Tornell (1998) have argued that saving rates in Latin America tend to be lower than in industrial countries – and even countercyclical in some instances – and have proposed a political-economy explanation whereby, in response to a positive shock, pressure groups fight for the newly available resources, leading to higher spending than is socially optimal and resulting in a lower, and possibly negative, saving rate.

While Lane and Tornell's story may well be valid, our simple model can in fact rationalize a negative comovement between saving and output. Think of an unanticipated output shock that raises today's endowment but increases it even more in the future. Households will dissave to smooth out consumption because today's endowment is lower than in the future. Hence, today's comovement between saving and output will be negative.[50] Our model can also rationalize a smaller comovement in developing countries than industrial countries. To see this, think of a temporary and positive shock. Then equation (32) would become $S_0 = (y^H - y^L)e^{-rT} > 0$. It follows that the larger is T, the smaller is the response of saving. Hence, to the extent that shocks in developing countries have a higher permanent component – as argued by Aguiar and Gopinath (2004) – our model could explain a lower, and possibly negative, correlation in developing countries.

In sum, by and large, the evidence clearly indicates that saving is procyclical, as predicted by our basic model. Other features of the data – such as a lower correlation in developing countries and occasionally negative correlations – are also consistent with our model.

[49] As will become clear in Chapter 2, a stochastic version of our basic model (with an uncertain endowment path) with incomplete markets will generate correlations that match exactly the comovements predicted by unanticipated temporary shocks in this model's chapter. The reason is that in both cases the shocks generate wealth effects.

[50] This is in fact the case analyzed in Figure 4 below (once we introduce investment into our model). Output increases in period 0 while saving falls.

Box 2. The Feldstein-Horioka puzzle

In a widely-cited 1980 paper in the *Economic Journal*, Martin Feldstein and Charles Horioka documented a puzzle regarding saving and investment in open economies.[51] They argued that, if perfect capital mobility prevailed, investors should invest in those countries with the highest marginal productivity of capital until marginal productivities of capital are equalized across countries. Since there is no need to finance domestic investment with domestic saving, we should expect a very low or no correlation between saving and investment in any particular country. Feldstein and Horioka presented data for OECD countries that showed that, contrary to our theoretical expectations, saving and investment are highly correlated. Table 2 supports this stylized fact by presenting the saving-investment correlation for 15 developing and 15 industrial countries. The average correlation is positive for both groups of countries (0.47 for developing countries and 0.61 for industrial countries) and, even more remarkably, is positive for every single country. Feldstein and Horioka argued that the explanation for such a puzzle was that the assumption of very high capital mobility was, in fact, unrealistic and that, by and large, capital mobility – particularly of the kind that would support long-term investment – was limited at best.

While the facts themselves are uncontroversial, Feldstein and Horioka's take on this puzzle as reflecting limited capital mobility is much more debatable. In fact, our simple model with investment is able to solve this puzzle, at least qualitatively speaking. Indeed, as analyzed in Section 5.6, our model predicts that, in response to an unanticipated and positive increase in productivity, saving may actually increase. This is the case illustrated in Figure 7, where we can see that both investment and saving increase at time 0. Intuitively, by increasing output, a positive and temporary productivity shock induces household to save in order to spread the benefits of the higher level of output over time. Quantitatively, calibrated versions of our basic model such as Mendoza's (1991) are also consistent with a positive saving-investment correlation even in the presence of perfect capital mobility. In fact, for a version of the model with adjustment costs calibrated for Canada, Mendoza (1991) reports a saving-investment correlation of around 0.5-0.6, which is fully consistent with Table 1. We thus conclude that, in terms of our model, the Feldstein-Horioka puzzle is actually no puzzle at all!

[51] The Felstein-Horioka paper spanned a large literature; see Coakleya, Kulasib, and Smith (1998) for a survey.

Box 3. Confronting the model with the data

How does our basic model fare in practice? A highly popular method of confronting the model with the data – which is the hallmark of the real business cycle approach – is to compare the correlations generated by a stochastic version of our basic model with investment to the correlations actually observed in the data.[52] In particular, think of the impact effect of a temporary increase in productivity (Figure 7) as capturing the relevant comovements. Assuming that the investment effect dominates, our model would predict the following comovements (translated into correlations):

- A positive correlation between consumption and output
- A positive correlation between investment and output
- A negative correlation between the trade balance and output

Table 3 shows the contemporaneous correlations with output of consumption, investment, and next exports (computed using the cyclical components obtained with the Hodrick-Prescott filter) for 13 emerging markets and 13 developed countries that can be regarded as "small open economies.". The data indicate that the correlation between consumption and output is, on average, 0.72 for emerging markets and 0.66 for industrial countries (and in fact positive for every single country). The average correlation between investment and output is 0.77 for emerging countries and 0.67 for industrial countries (and, again, positive for every single country). The average correlation between next exports (a proxy for the trade balance) and output is -0.51 for emerging markets (and negative for 11 of the 13 countries in the sample) and -0.17 for industrial countries (and negative for 10 out of the 13 countries in the sample). We thus conclude that, at least based on this metric, the now standard intertemporal model of the current account works remarkably well in terms of describing the cyclical behavior of the main macroeconomic aggregates – consumption, investment, and trade balance (net exports).

As analyzed in the text, whether the trade balance turns negative in response to an increase in productivity will critically depend on the length of the shock: the longer is the duration of the shock, the more likely that the investment effect will dominate and that the trade balance will become negative. Aguiar and Gopinath (2004) show empirically that the "permanent" component of shocks in emerging markets is larger than in industrial countries. In terms of our model, if the temporary increase in productivity lasts longer, the negative response of

[52] The real business cycle approach was pioneered by Finn Kydland (a Norwegian economist born in 1943) and Edward Prescott (an American economist born in 1940) in a 1982 *Econometrica* paper. They were awarded the 2004 Nobel Prize in Economics partly because of this contribution. While originally conceived as a theory of the business cycle – arguing that productivity shocks could be the source of business cycles – it has become much more influential as a methodological tool. For an excellent survey of the real business cycle literature, see King and Rebelo (2000). Open economy versions of the standard real business cycle model may be found in, among others, Aguiar and Gopinath (2004), Correia, Neves, and Rebelo (1995), and Mendoza (1991).

the trade balance will be larger (in absolute value) because the saving effect is smaller. This would be consistent with the larger correlation (in absolute value) for emerging markets than in industrial countries indicated in Table 3.[53]

[53] In this spirit, Aguiar and Gopinath (2004) show, in the context of a stochastic model, that more persistent shocks are critical in generating a larger correlation (in absolute value) for emerging markets.

Box 4. Testing the intertemporal approach to the current account

When contrasting a model with the data, it is useful to start by deriving the model's testable implications (that is, those related to observable and quantifiable data). In the case of the model derived in Section 5, there are two testable implications that have been explored in the empirical literature:[54]

- Foreign indebtedness implies future trade surpluses and vice versa (i.e., the transversality condition holds).
- The current account does not respond to permanent shocks to output, but it reacts to transitory ones.

These implications can be tested using different time series tools. The first testable implication, for example, has been the focus of the Deficit Sustainability literature. The basic idea consists in exploiting the fact that if the (unobservable) transversality condition holds (as when deriving equation 42), then the current account must follow a stationary stochastic process. One way to test the stationarity of the current account is using the information contained in equation 52; that is, $CA = IB + TB$. If the income and trade balances follow non-stationary processes, then current account stationarity requires the existence of a cointegration relationship between IB and TB. Ahmed and Rogers (1995) test this cointegration relationship using a set of very long annual time series for the US and the UK and find that the intertemporal constraint holds over the whole sample period and even in the presence of unusual events that may change the short-run dynamic behavior of the variables.

The second implication is a result of the shock-absorber role of the current account that allows the economy to finance temporary shocks. In this regard, Gosh and Ostry (1995) develop a model similar to the one presented in Section 5 and show that, under certain functional forms, the current account in a given period can be expressed as the present discounted value of expected changes (first differences) in output net of investment and government expenditure. The current account, then, reacts only to transitory shocks to net output (after a permanent shock, the expected change in net output is zero). In this case, any news regarding future changes in net output would translate into a change in the current account. Then, the model implies that the current account should help forecast subsequent movements in net output. Gosh and Ostry study this hypothesis applying Granger-causality tests on current account and net output time series for a very large sample of developing countries in Africa, Asia, the

[54] A third implication usually studied in the literature corresponds to the fact that the current account in a small open economy model is independent of global shocks, a result not emphasized in this chapter since it requires working with a two-country model different to the ones here developed. Glick and Rogoff (1995) construct a small open economy model with country-specific and global productivity shocks, and estimate the model's structural equations for investment and the current account using annual data for the G-7 countries in the 1961-1990 period. They find that the current account responds negatively and usually significantly to country-specific shocks and shows little or no response to global shocks.

Middle East and Latin America and the Caribbean. They run regressions of the change in net output against its lagged value and the lagged current account balance; if the current account helps to forecast (that is, Granger-cause) net output, the coefficient on the lagged current account should be significant. The authors find that this is the case for around 60 percent of the countries in their sample.

Time series techniques also help in disentangling permanent and transitory shocks to macroeconomic variables, providing further room for studying the second implication. This is usually done through a structural vector autoregression (SVAR), which consists of a system of equations in which each variable is regressed against lagged values of all the variables in the system. By imposing certain restrictions on the coefficients of the system, it is possible to give economic meaning to the estimated innovations. Kano (2008), for example, estimates two SVAR systems, one for the United Kingdom and the other for Canada, identifying three types of shocks to output: global, country-specific transitory, and country-specific permanent. He then constructs estimated responses of the current account to these shocks finding that, in the two economies, there is no response to a permanent shock and there is a positive response to a transitory one. These results, then, seem to provide further empirical support for the intertemporal approach to the current account. The work of Kano, however, also raises some concerns on the performance of the basic model, since he finds that the current account response to transitory shocks is too large, and that fluctuations in the current account are dominated by shocks that explain almost none of the fluctuations in net output. Therefore, he concludes highlighting the importance of consumption-tilting factors in explaining current account dynamics.

More generally, concerns like the ones just mentioned correspond not necessarily to a failure of the intertemporal approach to the current account, but rather point to the need of enriching the theoretical structure developed so far. In that spirit, Chapters 2, 3, and 4 will relax several critical assumption behind our basic model and study how these changes affect the economy's ability and/or desire to smooth consumption.

Figure 1. Perfect foresight paths

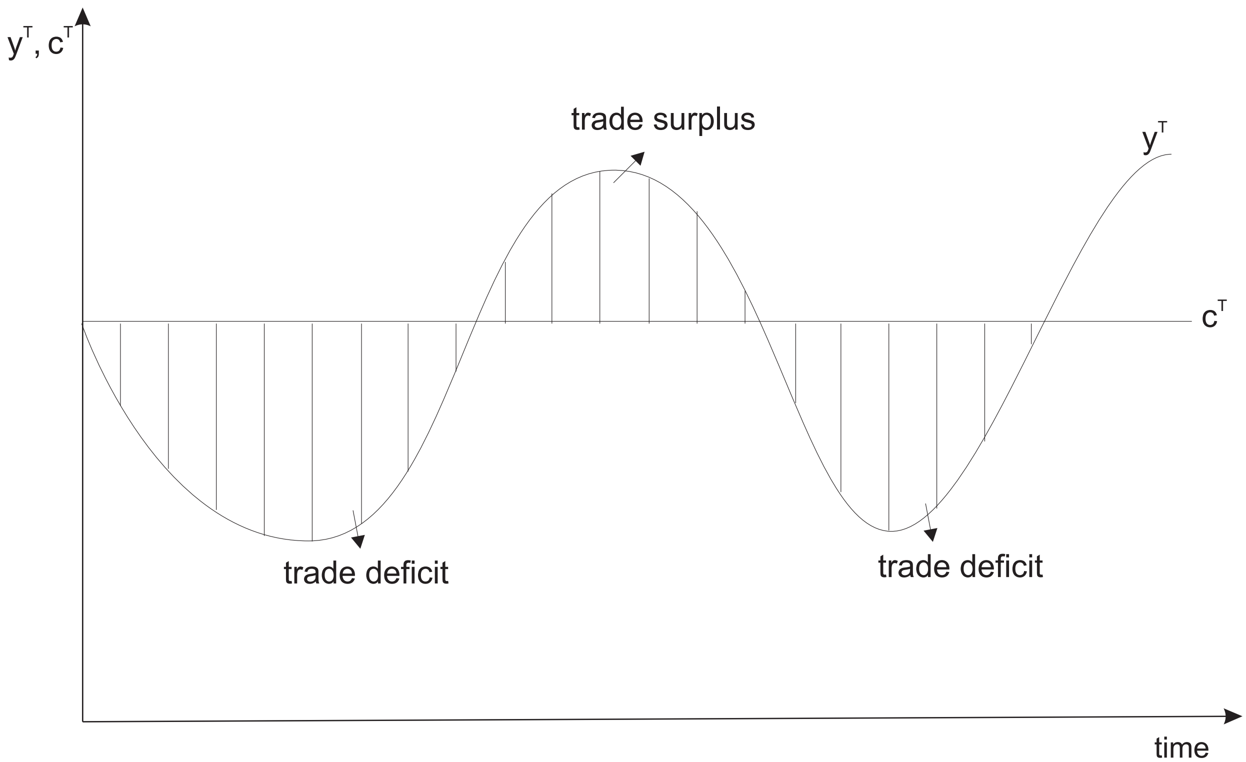


Figure 2. Permanent fall in output

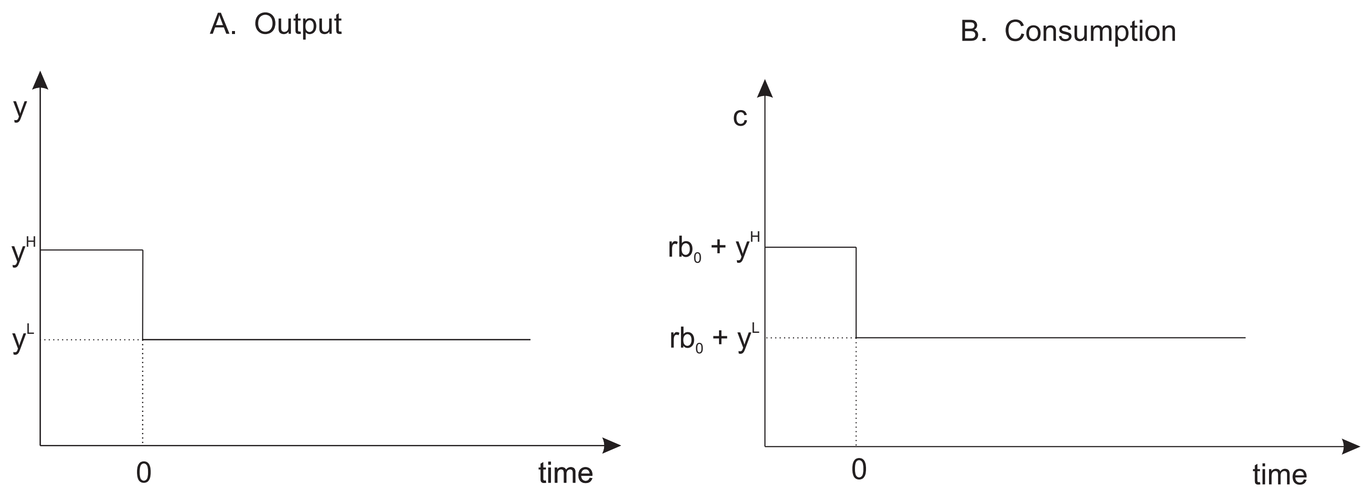


Figure 3. Temporary fall in output

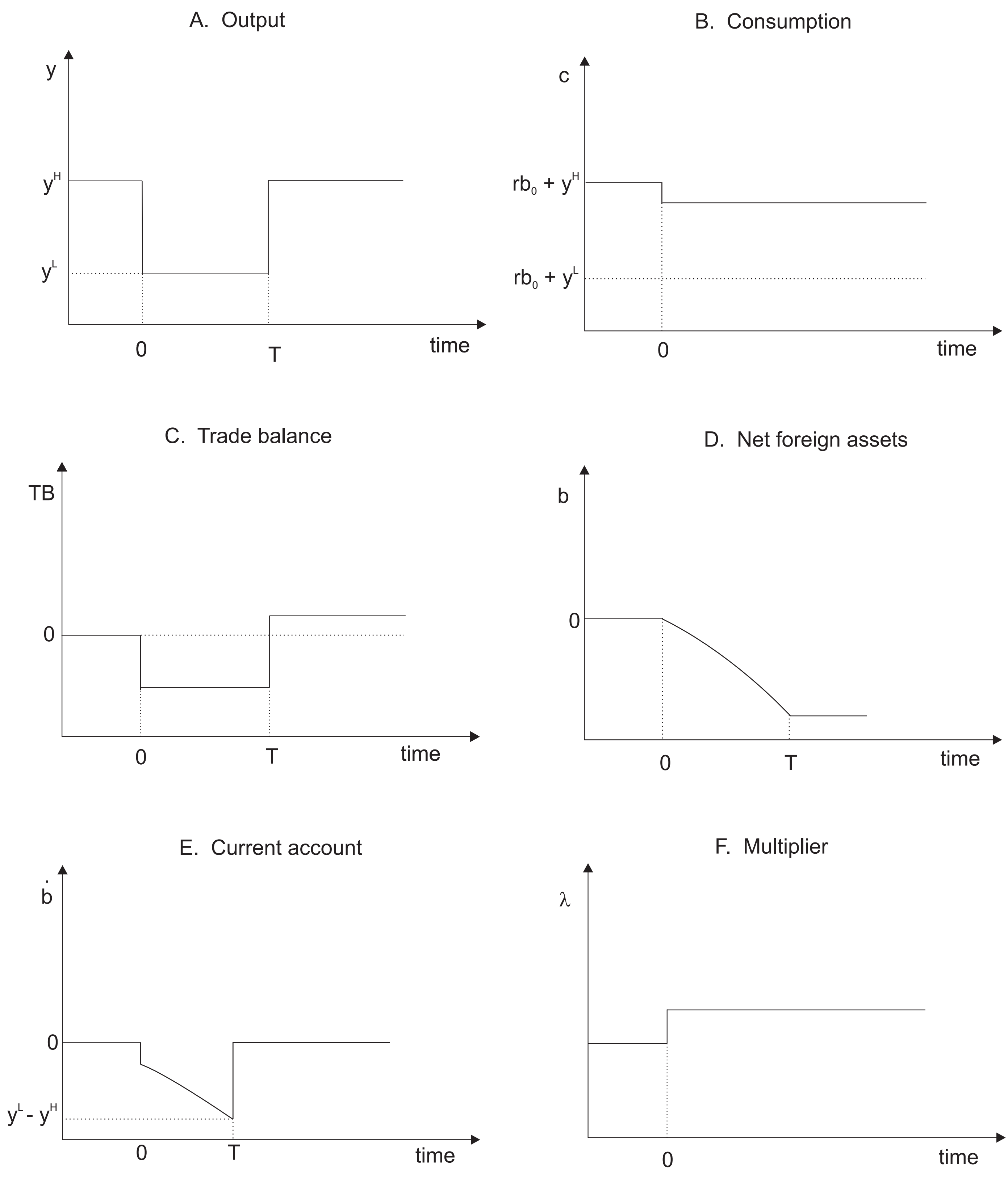


Figure 4. Permanent increase in productivity

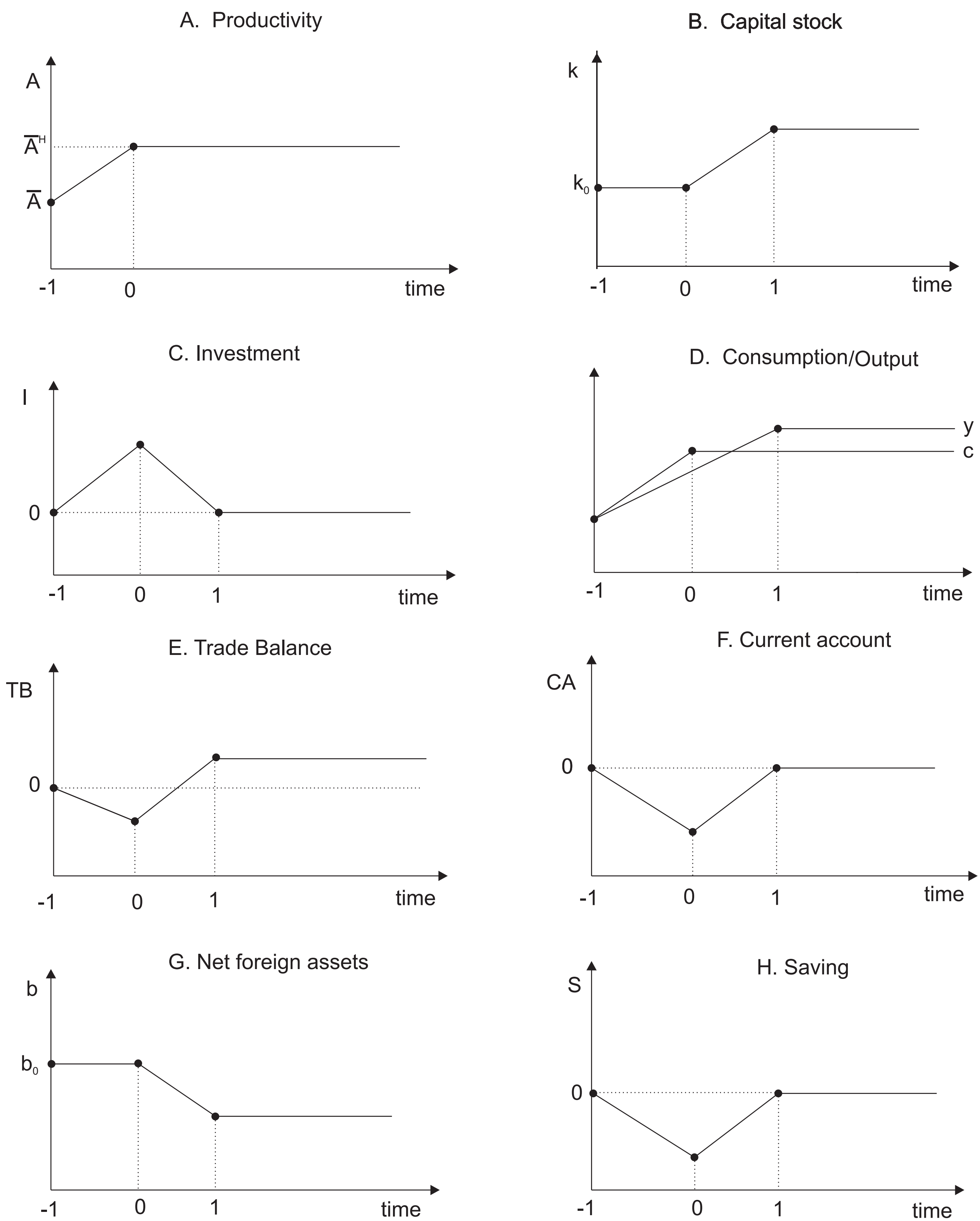


Figure 5. Production and Investment

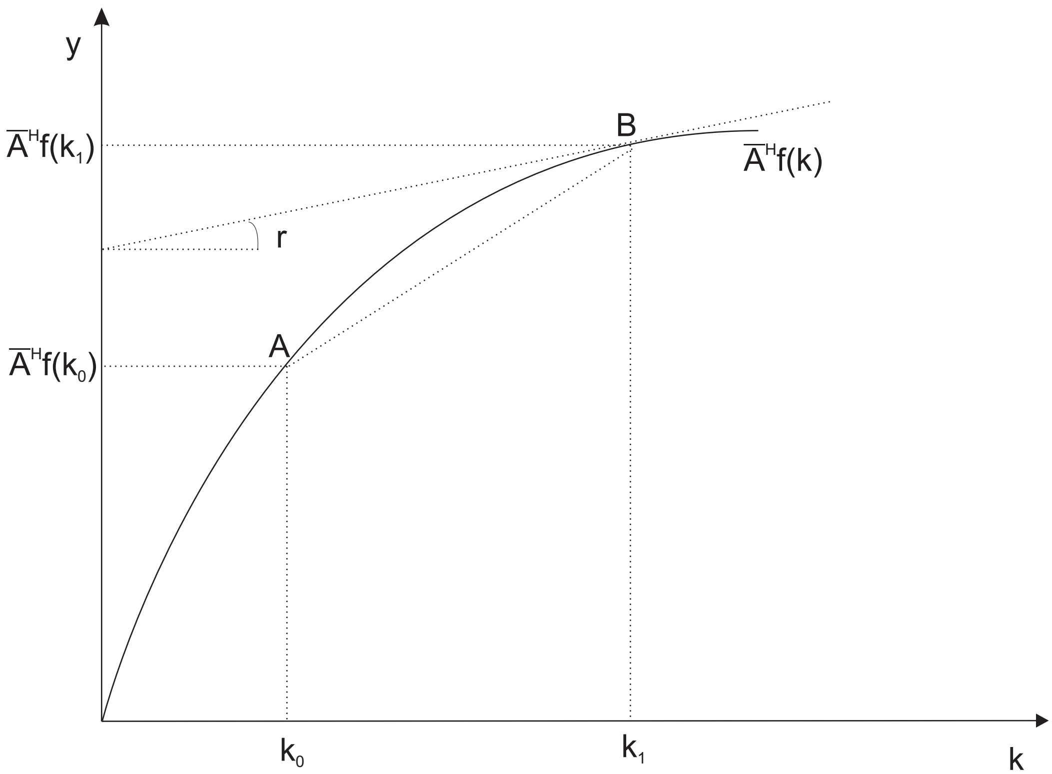


Figure 6. One-period increase in productivity

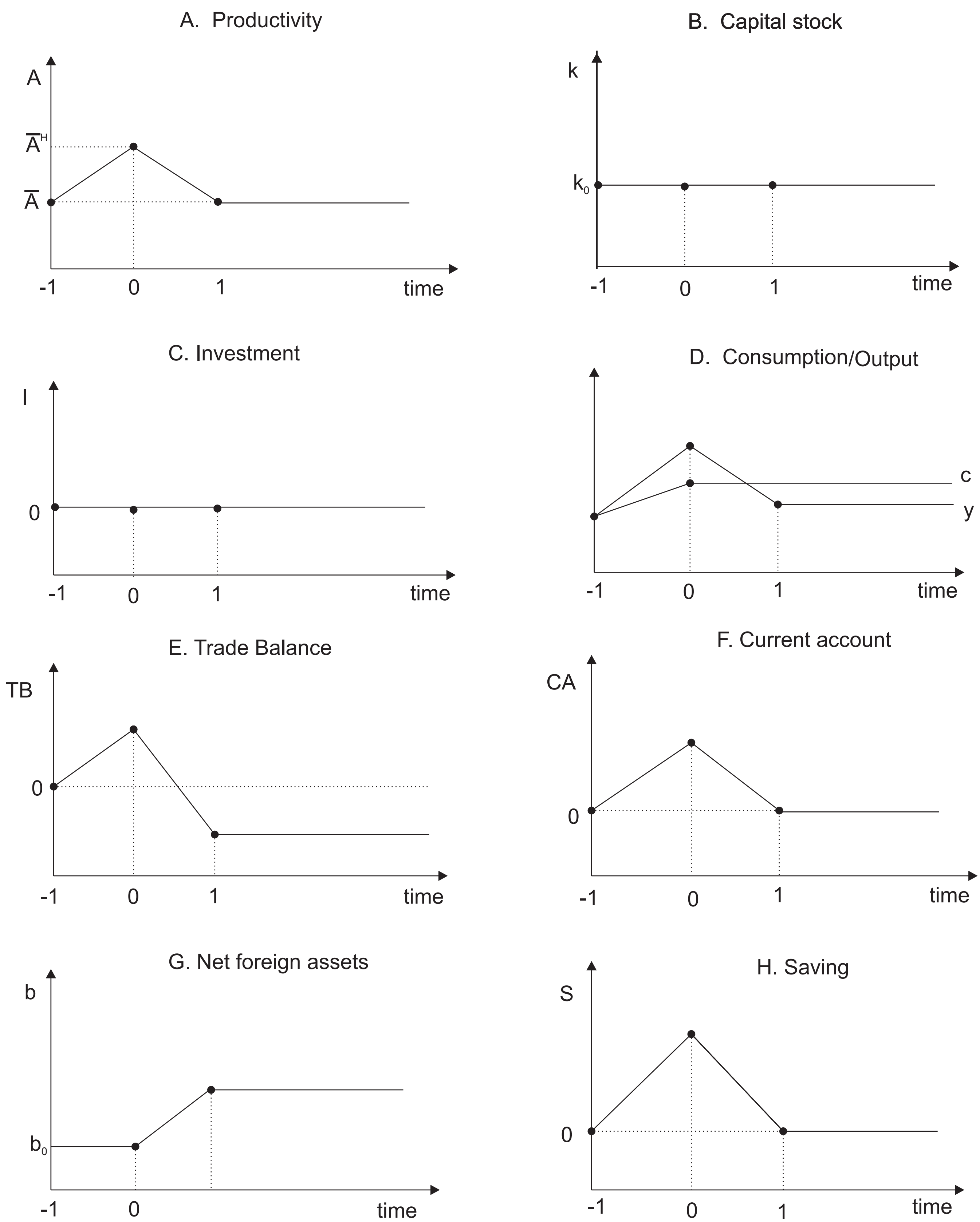


Figure 7. Temporary increase in productivity

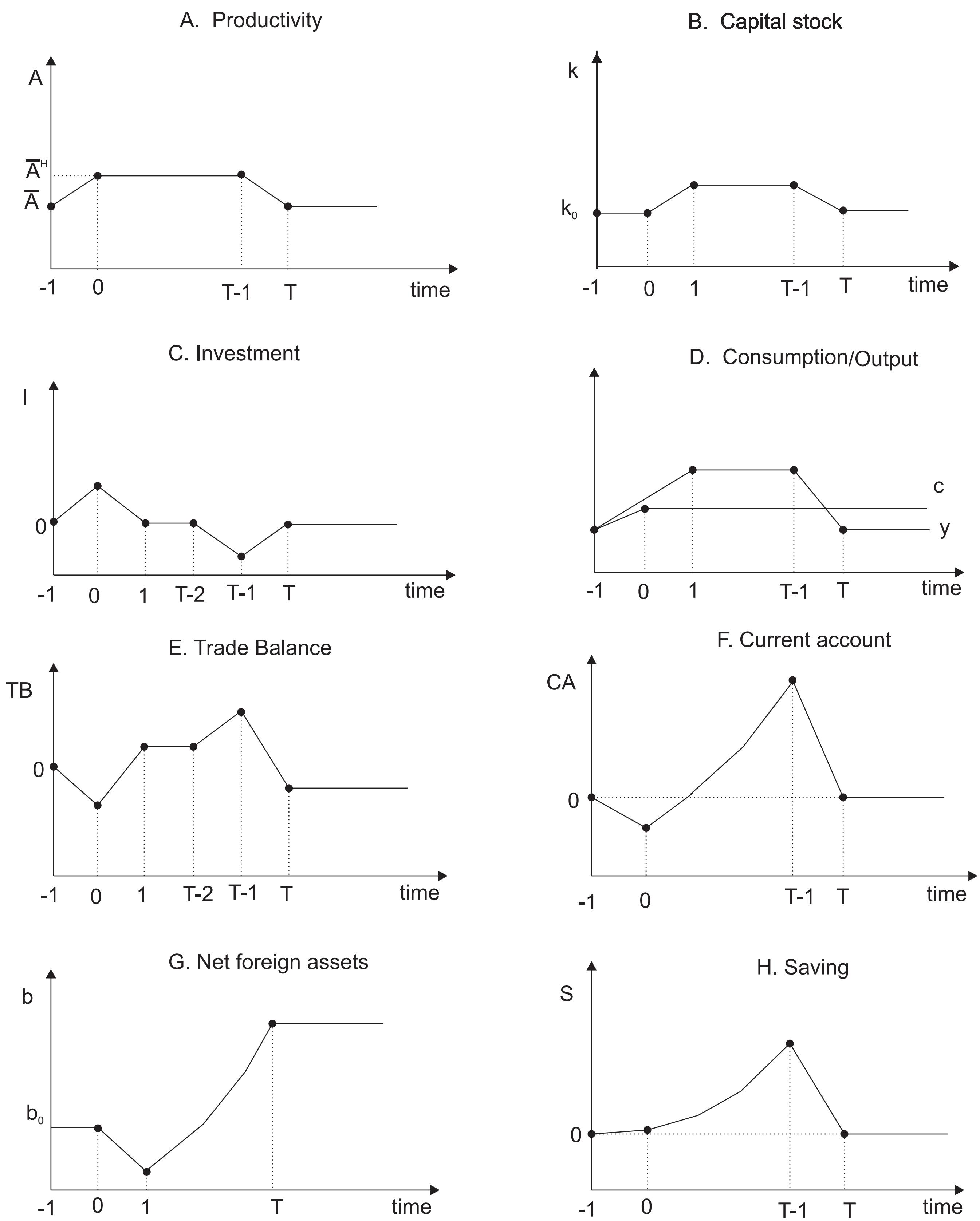
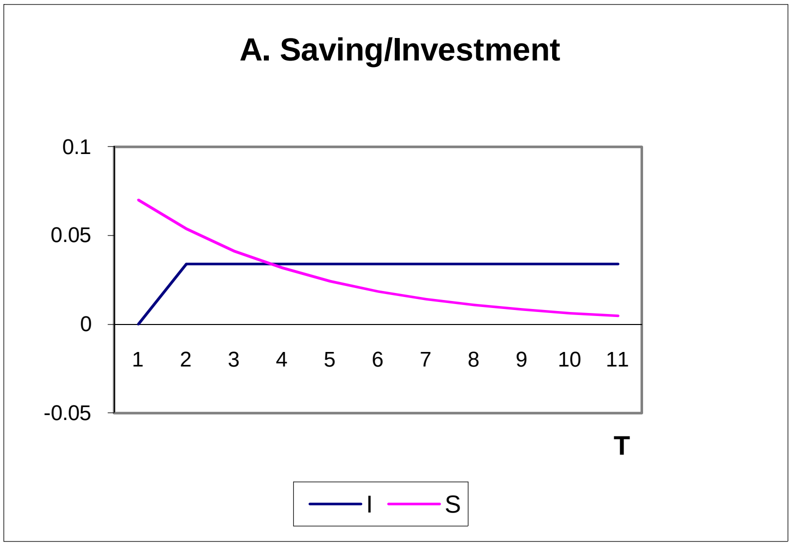


Figure 8. Saving, investment, and current account



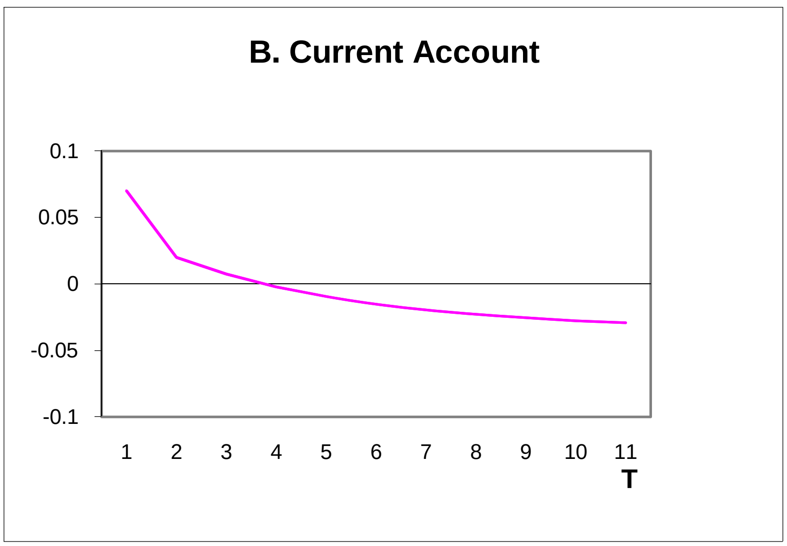


Figure 9. Permanent fall in endowment (10 percent)

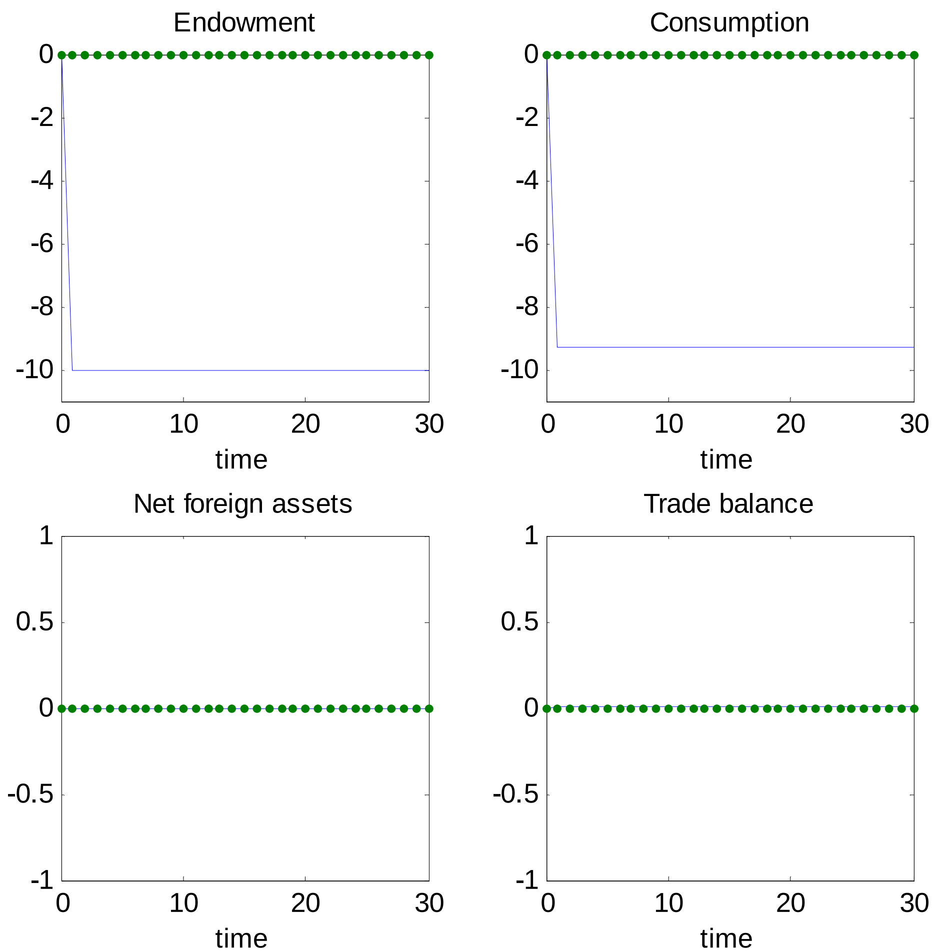


Figure 10. Temporary fall in endowment (10 percent)

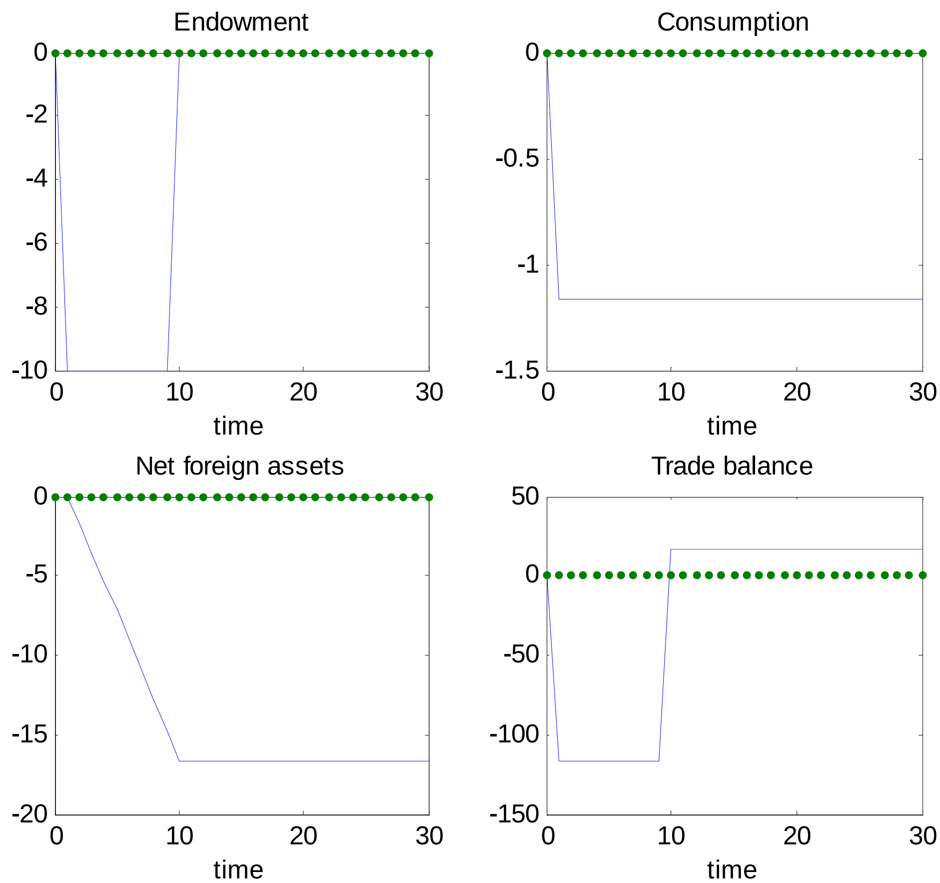


Table 1. Correlation between GDP and saving/GDP

Industrial		Latin America		Other developing	
Australia	0.31	Argentina	-0.17	Algeria	0.13
Austria	0.21	Bolivia	0.32	Gambia	0.00
Belgium	0.25	Brazil	0.20	Hong Kong	0.19
Canada	0.51	Chile	0.02	Hungary	0.45
Denmark	0.22	Colombia	0.31	Iceland	-0.05
Finland	0.58	Costa Rica	0.14	India	0.28
France	0.28	Dominican Republic	-0.02	Indonesia	0.10
Germany	0.24	Ecuador	0.19	Iran	0.44
Greece	-0.19	El Salvador	0.37	Israel	0.11
Ireland	0.05	Guatemala	0.16	Kenya	-0.05
Italy	0.11	Guyana	0.16	Malaysia	0.03
Japan	0.28	Honduras	0.25	Pakistan	0.00
Netherlands	0.09	Mexico	0.03	Saudi Arabia	0.43
New Zealand	0.46	Nicaragua	0.42	Singapore	0.23
Norway	0.18	Trinidad and Tobago	-0.10	South Africa	0.11
Portugal	0.23	Paraguay	-0.05	Syria	0.03
Sweden	0.55	Peru	0.36	Thailand	0.16
Switzerland	0.55	Uruguay	-0.16	Tunisia	-0.25
United States	0.20	Venezuela	0.01	Turkey	0.10
Average	**0.27**	**Average**	**0.13**	**Average**	**0.13**

Annual data from the World Bank. World Development Indicators (2010)
Sample period: 1970-2007

Table 2. Correlation between saving and investment for selected countries

Developing countries		Industrial Countries	
Argentina	0.70	Australia	0.89
Bolivia	0.11	Austria	0.62
Brazil	0.64	Belgium	0.86
Chile	0.71	Canada	0.63
Colombia	0.16	Denmark	0.07
Costa Rica	0.41	Finland	0.62
Ecuador	0.21	France	0.86
El Salvador	0.40	Germany	0.52
Honduras	0.44	Italy	0.68
Mexico	0.20	Japan	0.97
Paraguay	0.43	Netherlands	0.31
Peru	0.60	New Zealand	0.33
Turkey	0.79	Sweden	0.39
Uruguay	0.77	Switzerland	0.85
Venezuela	0.48	United States	0.58
Average	**0.47**	**Average**	**0.61**

Note: Based on annual data 1970-2007
Source: World Development Indicators 2010 (World Bank)

Table 3. Business cycle correlations

Emerging markets:

	ρ(c,y)	ρ(y,I)	ρ(NX/y,y)	σ(c)/σ(y)
Argentina	0.9	0.96	-0.7	1.38
Brazil	0.41	0.62	0.01	2.01
Ecuador	0.73	0.89	-0.79	2.39
Israel	0.45	0.49	0.12	1.60
Korea	0.85	0.78	-0.61	1.23
Malaysia	0.76	0.86	-0.74	1.70
Mexico	0.92	0.91	-0.74	1.24
Peru	0.78	0.85	-0.24	0.92
Philippines	0.59	0.76	-0.41	0.62
Slovak Republic	0.42	0.46	-0.44	2.04
South Africa	0.72	0.75	-0.54	1.61
Thailand	0.92	0.91	-0.83	1.09
Turkey	0.89	0.83	-0.69	1.09
Mean	**0.72**	**0.77**	**-0.51**	**1.45**

Small industrial countries

Australia	0.48	0.80	-0.43	0.69
Austria	0.74	0.75	0.10	0.87
Belgium	0.67	0.62	-0.04	0.81
Canada	0.88	0.77	-0.20	0.77
Denmark	0.36	0.51	-0.08	1.19
Finland	0.84	0.88	-0.45	0.94
Netherlands	0.72	0.70	-0.19	1.07
New Zealand	0.76	0.82	-0.26	0.90
Norway	0.63	0.00	0.11	1.32
Portugal	0.75	0.70	-0.11	1.02
Spain	0.83	0.83	-0.60	1.11
Sweden	0.35	0.68	0.01	0.97
Switzerland	0.58	0.69	-0.03	0.51
Mean	**0.66**	**0.67**	**-0.17**	**0.94**

Source: Aguiar and Gopinath (2004)