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DYNAMIC OPTIMIZATION IN CONTINUOUS TIME

For those of you who are not familiar with optimal control techniques, this handout provides a “cookbook” for deriving optimality conditions in continuous-time dynamic problems and then illustrates this method with the one-sector growth model. This is all that you will need for the purposes of this course. For formal proofs of these techniques, see Kamien and Schwartz (1981) or Chiang (1992). The second part of either book deals with optimal control techniques and contains formal proofs and examples. For extensive use of optimal control techniques in growth models, see Barro and Sala-i-Martin (1995); for applications in open economy macroeconomics, see Chapters 6 and 8 in the book.

1 The general problem

Consider the following standard problem in many areas of economics. Maximize

$$\int_0^{\infty} f(x_t, u_t) e^{-\beta t} dt, \quad (1)$$

subject to

$$\dot{x}_t = g(x_t, u_t), \quad (2)$$

$$x_0 \text{ given.} \quad (3)$$

In optimal control problems, variables are divided into *state* variables and *control* variables. In the problem above, x is the state variable and u is the control variable. (The extension to several control and state variables is straightforward.) The movement of state variables is governed by first-order differential equations such as (2). Note that the state variable may or may

note enter the objective function $f(\cdot)$: in the one-sector growth model below, the state variable does not enter the objective function, in Handout # 2 it does.

Faced with this maximization problem, the first step is to set up the *current value Hamiltonian*, denoted by $H(\cdot)$:

$$H(x_t, u_t, \lambda_t) \equiv f(x_t, u_t) + \lambda_t g(x_t, u_t), \quad (4)$$

where λ_t is an auxiliary variable (analogous to a Lagrange multiplier) which is referred to as the *co-state* variable. This variable λ_t can be interpreted as the marginal valuation at time t of the associated state variable.

It can be shown (see Kamien and Schwartz or Chiang) that, at an optimum, the following necessary conditions must be satisfied:

$$\frac{\partial H}{\partial u_t} = f_u(x_t, u_t) + \lambda_t g_u(x_t, u_t) = 0 \quad (5)$$

$$\dot{\lambda}_t = \beta \lambda_t - \frac{\partial H}{\partial x_t} = \beta \lambda_t - f_x(x, u) - \lambda_t g_x(x_t, u_t) \quad (6)$$

Condition (6) gives the law of motion for the co-state variable. If $f(x_t, u_t)$ and $g(x_t, u_t)$ are concave in both arguments, the necessary conditions are also sufficient for an optimum.

In the typical optimal control problem, one would solve for u as a function of λ and x from (5) (i.e., $u = \tilde{u}(x, \lambda)$) and substitute this into (2) and (6) to obtain:

$$\begin{aligned} \dot{x}_t &= g[x_t, \tilde{u}(x_t, \lambda_t)], \\ \dot{\lambda}_t &= \beta \lambda_t - f_x[x_t, \tilde{u}(x_t, \lambda_t)] - \lambda_t g_x[x_t, \tilde{u}(x_t, \lambda_t)]. \end{aligned}$$

This is a differential equation system in x and λ , which can be solved using standard phase-diagram techniques (an excellent discussion is contained in Kamien and Schwartz in the chapter “Equilibria in infinite horizon autonomous problems.”)

2 An example: the one-sector growth model

Consider the following one-sector growth model. Maximize

$$\int_0^{\infty} \log(c_t) e^{-\beta t} dt, \quad (7)$$

subject to

$$\dot{k}_t = f(k_t) - c_t - \delta k_t, \quad (8)$$

$$k_0 \text{ given}, \quad (9)$$

where c is consumption, k is the capital stock, δ is the rate of depreciation and $f(k)$ is a strictly increasing and strictly concave production function. In this case, c is the control variable and k is the state variable.

The present value Hamiltonian is given by:

$$H(c_t, k_t) \equiv \log(c_t) + \lambda_t [f(k_t) - c_t - \delta k_t]. \quad (10)$$

The variable λ_t can be interpreted as the price or value of an extra unit of capital at time t in terms of utility at time t . The Hamiltonian can be interpreted as follows. At each instant in time, the agent consumes c_t and owns a capital stock k_t . These two variables affect utility through two channels. First, the direct contribution of consumption to utility is captured by the first term, $\log(c_t)$. Second, the choice of consumption affects the change in the capital stock through (8). The *value* of this change (in utility terms) is given by the second term on the RHS of (10). Hence, for a given value of the shadow price λ_t , the Hamiltonian captures the total contribution to utility of the choice of c_t .

Using (5) and (6), the optimality conditions are:

$$\frac{\partial H}{\partial c_t} = \frac{1}{c_t} - \lambda_t = 0 \quad (11)$$

$$\dot{\lambda}_t = \beta \lambda_t - \frac{\partial H}{\partial k_t} = \beta \lambda_t - \lambda_t [f'(k_t) - \delta] = \lambda_t [\beta + \delta - f'(k_t)]. \quad (12)$$

Notice that one can rewrite (12) as

$$\frac{1}{\lambda_t} \frac{\partial H}{\partial k_t} + \frac{\dot{\lambda}_t}{\lambda_t} = \beta,$$

and interpret it as an asset pricing equation. The term $\frac{1}{\lambda_t} \frac{\partial H}{\partial k_t}$ is the “dividend” rate received by the agent (i.e., the marginal contribution of capital to utility divided by the price of the asset), $\frac{\dot{\lambda}_t}{\lambda_t}$ is the rate of capital gain (i.e., the rate of change in the price of the asset) and β is the rate of return of an alternative asset (consumption). Hence, at an optimum, the agent is indifferent between the two types of return because the overall return to investment, $\frac{1}{\lambda_t} \frac{\partial H}{\partial k_t} + \frac{\dot{\lambda}_t}{\lambda_t}$, equals the return to consumption, β .

Solving for c_t from (11) and substituting it into (8) and (12) yields a dynamic system in k and λ :

$$\dot{k}_t = f(k_t) - \frac{1}{\lambda_t} - \delta k_t, \tag{13}$$

$$\dot{\lambda}_t = \lambda_t[\beta + \delta - f'(k_t)]. \tag{14}$$

The steady state is given by:

$$\begin{aligned} f'(\bar{k}) &= \beta + \delta, \\ \frac{1}{\bar{\lambda}} &= f(\bar{k}) - \delta \bar{k}. \end{aligned}$$

By linearizing the system around the steady-state, it is easy to check that this dynamic system is saddle-path stable (you may want to check this as an exercise).¹ (Chiang, Chapter 9, contains a detailed discussion of the one-sector growth model). Hence, for a given value of k_0 , the value of λ_0 will be endogenously determined so as to place the dynamic system on the saddle path. (Alternatively, you could differentiate (11) with respect to time and use (12) to set up a dynamic system in k and c ; this is what Chiang does.)

Finally, note that if the production function were linear (i.e., $f(k) = rk$, $r > 0$) and there were no depreciation (i.e., $\delta = 0$), then it is clear from (14) that we would need to assume that $\beta = r$ for a stationary equilibrium to exist. This is exactly the reason why we assume that $\beta = r$ in small open economy

¹We will see several examples of this type of technique (linearization and qualitative solution of systems of differential equations) in class starting with Handout # 6A.

models. (Intuitively, for a small open economy to have access to perfect international capital markets is like having access to a linear technology with marginal productivity of capital equal to r .) In that case, $\dot{\lambda}_t = 0$ for all t and therefore λ is constant along a perfect foresight equilibrium path.

References

- [1] Barro, Robert J., and Xavier Sala-i-Martin, *Economic Growth* (New York, McGraw-Hill, 1995).
- [2] Chiang, Alpha C., *Elements of Dynamic Optimization* (New York, McGraw-Hill, 1992).
- [3] Kamien, Morton and Nancy Schwartz, *Dynamic Optimization* (Amsterdam and New York: North Holland, 1981).