

# Risk Aversion and the Labor Margin in Dynamic Equilibrium Models

Eric T. Swanson

Federal Reserve Bank of San Francisco

[eric.swanson@sf.frb.org](mailto:eric.swanson@sf.frb.org)

<http://www.ericswanson.us>

## Abstract

The household's labor margin has a substantial effect on risk aversion, and hence asset prices, in dynamic equilibrium models even when utility is additively separable between consumption and labor. This paper derives simple, closed-form expressions for risk aversion that take into account the household's labor margin. Ignoring this margin can wildly overstate the household's true aversion to risk. Risk premia on assets priced with the stochastic discount factor increase essentially linearly with risk aversion, so measuring risk aversion correctly is crucial for asset pricing in the model. Closed-form expressions for risk aversion in models with generalized recursive preferences and internal and external habits are also derived.

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## 1. Introduction

In a static, one-period model with household utility  $u(\cdot)$  defined over a single consumption good, Arrow (1964) and Pratt (1965) defined the coefficients of absolute and relative risk aversion,  $-u''(c)/u'(c)$  and  $-cu''(c)/u'(c)$ . Difficulties immediately arise, however, when one attempts to generalize these concepts to the case of many periods or many goods (e.g., Kihlstrom and Mirman, 1974). These difficulties are particularly pronounced in a dynamic equilibrium model with labor, in which there is a double infinity of goods to consider—consumption and labor in every future period and state of nature—all of which may vary in response to a typical shock to household income or wealth.

The present paper shows how to compute risk aversion in dynamic equilibrium models in general. First, we verify that risk aversion depends on the partial derivatives of the household's value function  $V$  with respect to wealth  $a$ —that is, the coefficients of absolute and relative risk aversion are essentially  $-V_{aa}/V_a$  and  $-aV_{aa}/V_a$ , respectively. Even though closed-form solutions for the value function do not exist in general, we nevertheless can derive simple, closed-form expressions for risk aversion because derivatives of the value function are much easier to compute than the value function itself. For example, in many dynamic models the derivative of the value function with respect to wealth equals the current-period marginal utility of consumption (Benveniste and Scheinkman, 1979).

The main result of the paper is that the household's labor margin has substantial effects on risk aversion, and hence asset prices. Even when labor and consumption are additively separable in utility, they remain connected by the household's budget constraint: in particular, the household can absorb income shocks either through changes in consumption, changes in hours worked, or some combination of the two. This ability to absorb shocks along either or both margins greatly alters the household's attitudes toward risk. For example, if the household's utility kernel is given by  $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t$ , the quantity  $-cu_{11}/u_1 = \gamma$  is often referred to as the household's coefficient of relative risk aversion, but in fact the household is *risk neutral* with respect to gambles over income or wealth—the proper measure of risk aversion for asset pricing, as we show in Section 2. Intuitively, the household is indifferent at the margin between using labor or consumption to absorb a shock to income or assets, and the household in this example is clearly risk neutral with respect to gambles over hours. More generally, when  $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$ , risk aver-

sion equals  $(\gamma^{-1} + \chi^{-1})^{-1}$ , a combination of the parameters on the household's consumption and labor margins, reflecting that the household absorbs shocks using both margins.<sup>1</sup>

While modeling risk neutrality is not a main goal of the present paper, risk neutrality nevertheless can be a desirable feature for some applications, such as labor market search or financial frictions, since it allows for closed-form solutions to key features of the model.<sup>2</sup> Thus, an additional contribution of the present paper is to show ways of modeling risk neutrality that do not require utility to be linear in consumption, which has undesirable implications for interest rates and consumption growth. Instead, any utility kernel with a singular Hessian matrix can be used.

A final result of the paper is that risk premia computed using the Lucas-Breeden stochastic discounting framework are essentially linear in risk aversion. That is, measuring risk aversion correctly—taking into account the household's labor margin—is necessary for understanding asset prices in the model. Since much recent research has focused on bringing dynamic stochastic general equilibrium (DSGE) models into closer agreement with asset prices,<sup>3</sup> it is surprising that so little attention has been paid to measuring risk aversion correctly in these models. The present paper aims to fill that void.

There are a few previous studies that extend the Arrow-Pratt definition beyond the one-good, one-period case. In a static, multiple-good setting, Stiglitz (1969) measures risk aversion using the household's indirect utility function rather than utility itself, essentially a special case of Proposition 1 of the present paper. Constantinides (1990) measures risk aversion in a dynamic, consumption-only (endowment) economy using the household's value function, another special case of Proposition 1. Boldrin, Christiano, and Fisher (1997) apply Constantinides' definition to some very simple endowment economy models for which they can compute closed-form expressions for the value function, and hence risk aversion. The present paper builds on these studies by deriving closed-form solutions for risk aversion in dynamic equilibrium models in general, demonstrating the importance of the labor margin,

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<sup>1</sup>Note that the intertemporal elasticity of substitution in this example is still  $1/\gamma$ , so a corollary of this result is that risk aversion and the intertemporal elasticity of substitution are nonreciprocal when labor supply can vary.

<sup>2</sup>See, e.g., Mortensen and Pissarides (1994), and Bernanke, Gertler, and Gilchrist (1999).

<sup>3</sup>See, e.g., Boldrin, Christiano, and Fisher (2001), Tallarini (2000), Rudebusch and Swanson (2008, 2009), Van Binsbergen, Fernandez-Villaverde, Kojien, and Rubio-Ramirez (2010), and Backus, Routledge, and Zin (2009).

and showing the tight link between risk aversion and asset prices in these models.

The remainder of the paper proceeds as follows. Section 2 presents the main ideas of the paper, deriving Arrow-Pratt risk aversion in dynamic equilibrium models for the time-separable expected utility case and demonstrating the importance of risk aversion for asset pricing. Section 3 extends the analysis to the case of generalized recursive preferences (Epstein and Zin, 1989), which have been the focus of much recent research at the boundary between macroeconomics and finance. Section 4 extends the analysis to the case of internal and external habits, two of the most common intertemporal nonseparabilities in preferences in both the macroeconomics and finance literatures. Section 5 discusses some general implications and concludes. An Appendix provides details of derivations and proofs that are outlined in the main text.

## 2. Time-Separable Expected Utility Preferences

To highlight the intuition and methods of the paper, we consider first the case where the household has additively time-separable expected utility preferences.

### 2.1 The Household's Optimization Problem and Value Function

Time is discrete and continues forever. At each time  $t$ , the household seeks to maximize the expected present discounted value of utility flows:

$$E_t \sum_{\tau=t}^{\infty} \beta^{\tau-t} u(c_{\tau}, l_{\tau}), \quad (1)$$

subject to the sequence of asset accumulation equations:

$$a_{\tau+1} = (1 + r_{\tau})a_{\tau} + w_{\tau}l_{\tau} + d_{\tau} - c_{\tau}, \quad \tau = t, t + 1, \dots \quad (2)$$

and the no-Ponzi-scheme condition:

$$\lim_{T \rightarrow \infty} \prod_{\tau=t}^T (1 + r_{\tau+1})^{-1} a_{T+1} \geq 0, \quad (3)$$

where  $E_t$  denotes the mathematical expectation conditional on the household's information set at time  $t$ ,  $\beta \in (0, 1)$  is the household's discount factor,  $(c_t, l_t) \in \Omega \subseteq \mathbb{R}^2$  denotes the

household's choice of consumption and labor in period  $t$ ,  $a_t$  is the household's beginning-of-period assets, and  $w_t$ ,  $r_t$ , and  $d_t$  denote the real wage, interest rate, and net transfer payments at time  $t$ . There is a finite-dimensional Markovian state vector  $\theta_t$  that is exogenous to the household and governs the processes for  $w_t$ ,  $r_t$ , and  $d_t$ . Conditional on  $\theta_t$ , the household knows the time- $t$  values for  $w_t$ ,  $r_t$ , and  $d_t$ . The state vector and information set of the household's optimization problem at each date  $t$  is thus  $(a_t; \theta_t)$ , and we denote the domain of  $(a_t; \theta_t)$  by  $X$ . Let  $\Gamma : X \rightarrow \Omega$  describe the set-valued correspondence of feasible choices for  $(c_t, l_t)$  for each given  $(a_t; \theta_t)$ .

We make the following regularity assumptions regarding the utility kernel  $u$ :

**Assumption 1.** *The function  $u : \Omega \rightarrow \mathbb{R}$  is increasing in its first argument, decreasing in its second, twice-differentiable, and concave.*

In addition to Assumption 1, a few more technical conditions are required to ensure the value function for the household's optimization problem exists and satisfies the Bellman equation (Stokey and Lucas (1990), Rincón-Zapatera and Rodríguez-Palmero (2003), and Marinacci and Montrucchio (2010) give different sets of such sufficient conditions). The details of these conditions are tangential to the present paper, so we simply assume:

**Assumption 2.** *The value function  $V : X \rightarrow \mathbb{R}$  for the household's optimization problem exists and satisfies the Bellman equation:*

$$V(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} u(c_t, l_t) + \beta E_t V(a_{t+1}; \theta_{t+1}), \quad (4)$$

where  $a_{t+1}$  is given by (2).

Together, Assumptions 1–2 guarantee the existence of a unique optimal choice for  $(c_t, l_t)$  at each point in time, given  $(a_t; \theta_t)$ . Let  $c_t^* \equiv c^*(a_t; \theta_t)$  and  $l_t^* \equiv l^*(a_t; \theta_t)$  denote the household's optimal choices of  $c_t$  and  $l_t$  as functions of the state  $(a_t; \theta_t)$ . Then  $V$  can be written as:

$$V(a_t; \theta_t) = u(c_t^*, l_t^*) + \beta E_t V(a_{t+1}^*; \theta_{t+1}), \quad (5)$$

where  $a_{t+1}^* \equiv (1 + r_t)a_t + w_t l_t^* + d_t - c_t^*$ .

To avoid having to consider boundary solutions, we make the following standard assumption:

**Assumption 3.** *For any  $(a_t; \theta_t) \in X$ , the household's optimal choice  $(c_t^*, l_t^*)$  lies in the interior of  $\Gamma(a_t; \theta_t)$ .*

Intuitively, Assumption 3 requires the partial derivatives of  $u$  to grow sufficiently large toward the boundary that only interior solutions for  $c_t^*$  and  $l_t^*$  are optimal for all  $(a_t; \theta_t) \in X$ .

Assumptions 1–3 guarantee that  $V$  is continuously differentiable and satisfies the Benveniste-Scheinkman equation, but we will require slightly more than this below:

**Assumption 4.** *The value function  $V(\cdot; \cdot)$  is twice-differentiable.*

Assumption 4 also implies differentiability of the optimal policy functions,  $c^*$  and  $l^*$ . Santos (1991) provides relatively mild sufficient conditions for Assumption 4 to be satisfied; intuitively,  $u$  must be strongly concave.

## 2.2 Representative Household and Steady State Assumptions

Up to this point, we have considered the case of a single household in isolation, leaving the other households of the model and the production side of the economy unspecified. Implicitly, the other households and production sector jointly determine the process for  $\theta_t$  (and hence  $w_t$ ,  $r_t$ , and  $d_t$ ), and much of the analysis below does not need to be any more specific about these processes than this. However, to move from general expressions for risk aversion to more concrete, closed-form expressions, we adopt two standard assumptions from the DSGE literature.<sup>4</sup>

**Assumption 5.** *The household is atomistic and representative.*

Assumption 5 implies that the individual household's choices for  $c_t$  and  $l_t$  have no effect on the aggregate quantities  $w_t$ ,  $r_t$ ,  $d_t$ , and  $\theta_t$ . It also implies that, when the economy is at the nonstochastic steady state (described shortly), any individual household finds it optimal to choose the steady-state values of  $c$  and  $l$  given  $a$  and  $\theta$ .

**Assumption 6.** *The model has a nonstochastic steady state, or a balanced growth path that can be renormalized to a nonstochastic steady state after a suitable change of variables. At*

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<sup>4</sup>Alternative assumptions about the nature of the other households in the model or the production sector may also allow for closed-form expressions for risk aversion. However, the assumptions used here are standard and thus the most natural to pursue.

the nonstochastic steady state,  $x_t = x_{t+1} = x_{t+k}$  for  $k = 1, 2, \dots$ , and  $x \in \{c, l, a, w, r, d, \theta\}$ , and we drop the subscript  $t$  to denote the steady-state value.

It is important to note that Assumptions 5–6 do not prohibit us from offering an individual household a hypothetical gamble of the type described below. The steady state of the model serves only as a reference point around which the *aggregate* variables  $w$ ,  $r$ ,  $d$ , and  $\theta$  and the *other households'* choices of  $c$ ,  $l$ , and  $a$  can be predicted with certainty. This reference point is important because it makes it much easier to compute closed-form expressions for many features of the model.

### 2.3 The Coefficient of Absolute Risk Aversion

The household's risk aversion at time  $t$  generally depends on the household's state vector at time  $t$ ,  $(a_t; \theta_t)$ . Given this state, we consider the household's aversion to a hypothetical one-shot gamble in period  $t$  of the form:

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t + \sigma \varepsilon_{t+1}, \quad (6)$$

where  $\varepsilon_{t+1} \in [\underline{\varepsilon}, \bar{\varepsilon}]$  is a random variable, representing the gamble, that has mean zero, unit variance, is independent of  $\theta_\tau$  for all  $\tau$ , and is independent of  $a_\tau$ ,  $c_\tau$ , and  $l_\tau$  for all  $\tau \leq t$ . A few words about (6) are in order: First, the gamble is dated  $t + 1$  to clarify that its outcome is not in the household's information set at time  $t$ . Second,  $c_t$  cannot be made the subject of the gamble without substantial modifications to the household's optimization problem, because  $c_t$  is a choice variable under control of the household at time  $t$ . However, (6) is clearly equivalent to a one-shot gamble over net transfers  $d_t$  or asset returns  $r_t$ , both of which are exogenous to the household. Indeed, thinking of the gamble as being over  $r_t$  helps to illuminate the connection between (6) and the price of risky assets, to which we will return in Section 2.6. As shown there, the household's aversion to the gamble in (6) is directly linked to the premium households require to hold risky assets.

Following Arrow (1964) and Pratt (1965), we can ask what one-time fee  $\mu$  the household would be willing to pay in period  $t$  to avoid the gamble in (6):

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t - \mu. \quad (7)$$

The quantity  $\mu$  that makes the household just indifferent between (6) and (7), for infinitesimal  $\mu$  and  $\sigma$ , is the household's coefficient of absolute risk aversion:<sup>5</sup>

**Definition 1.** *Let  $(a_t; \theta_t)$  be an interior point of  $X$ . The household's coefficient of absolute risk aversion is given by  $\lim_{\sigma \rightarrow 0} 2\mu(\sigma)/\sigma^2$ , where  $\mu(\sigma)$  denotes the value of  $\mu$  that satisfies  $V(a_t - \frac{\mu}{1+r_t}; \theta_t) = \tilde{V}(a_t; \theta_t; \sigma)$ , and where  $\tilde{V}(a_t; \theta_t; \sigma)$  denotes the value function of the household's optimization problem inclusive of the one-shot gamble (6).*

The following proposition verifies that the coefficient of absolute risk aversion is well-defined and equals the “folk wisdom” value of  $-V_{11}/V_1$ :<sup>6</sup>

**Proposition 1.** *The household's coefficient of absolute risk aversion in Definition 1 exists and is given by:*

$$\frac{-E_t V_{11}(a_{t+1}^*; \theta_{t+1})}{E_t V_1(a_{t+1}^*; \theta_{t+1})}, \quad (8)$$

where  $V_1$  and  $V_{11}$  denote the first and second partial derivatives of  $V$  with respect to its first argument. Evaluated at the steady state, (8) simplifies to:

$$\frac{-V_{11}(a; \theta)}{V_1(a; \theta)}. \quad (9)$$

PROOF: See Appendix.

Equations (8)–(9) are essentially Constantinides' (1990) definition of risk aversion, and have obvious similarities to Arrow (1964) and Pratt (1965). Here, of course, it is the curvature of the value function  $V$  with respect to assets that matters, rather than the curvature of the utility kernel  $u$  with respect to consumption.<sup>7</sup>

Deriving the coefficient of absolute risk aversion in Proposition 1 is simple enough, but the problem with (8)–(9) is that closed-form expressions for  $V$  (and hence  $V_1$  and  $V_{11}$ ) do not exist in general, even for the simplest dynamic models with labor. This difficulty may help to explain the popularity of “shortcut” approaches to measuring risk aversion, notably  $-u_{11}(c_t^*, l_t^*)/u_1(c_t^*, l_t^*)$ , which has no clear relationship to (8)–(9) except in the one-good one-period case. Boldrin, Christiano, and Fisher (1997) derive closed-form solutions for  $V$ —and

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<sup>5</sup>We defer discussion of relative risk aversion until the next subsection because defining total household wealth is complicated by the presence of human capital—that is, the household's labor income.

<sup>6</sup>See, e.g., Constantinides (1990), Farmer (1990), Cochrane (2001), and Flavin and Nakagawa (2008).

<sup>7</sup>Arrow (1964) and Pratt (1965) occasionally refer to utility as being defined over “money”, so one could argue that they always intended for risk aversion to be measured using indirect utility or the value function.



hence risk aversion—for some very simple, consumption-only endowment economy models. This approach is a nonstarter for even the simplest dynamic models with labor.

We solve this problem by observing that  $V_1$  and  $V_{11}$  often can be computed even when closed-form solutions for  $V$  cannot be. For example, the Benveniste-Scheinkman equation:

$$V_1(a_t; \theta_t) = (1 + r_t) u_1(c_t^*, l_t^*), \quad (10)$$

states that the marginal value of a dollar of assets equals the marginal utility of consumption times  $1 + r_t$  (the interest rate appears here because beginning-of-period assets in the model generate income in period  $t$ ). In (10),  $u_1$  is a known function. Although closed-form solutions for the functions  $c^*$  and  $l^*$  are not known in general, the points  $c_t^*$  and  $l_t^*$  often are known—for example, when they are evaluated at the nonstochastic steady state,  $c$  and  $l$ . Thus, we can compute  $V_1$  at the nonstochastic steady state by evaluating (10) at that point.

We compute  $V_{11}$  by noting that (10) holds for general  $a_t$ ; hence we can differentiate (10) to yield:

$$V_{11}(a_t; \theta_t) = (1 + r_t) \left[ u_{11}(c_t^*, l_t^*) \frac{\partial c_t^*}{\partial a_t} + u_{12}(c_t^*, l_t^*) \frac{\partial l_t^*}{\partial a_t} \right]. \quad (11)$$

All that remains is to find the derivatives  $\partial c_t^*/\partial a_t$  and  $\partial l_t^*/\partial a_t$ .

We solve for  $\partial l_t^*/\partial a_t$  by differentiating the household's intratemporal optimality condition:

$$-u_2(c_t^*, l_t^*) = w_t u_1(c_t^*, l_t^*), \quad (12)$$

with respect to  $a_t$ , and rearranging terms to yield:

$$\frac{\partial l_t^*}{\partial a_t} = -\lambda_t \frac{\partial c_t^*}{\partial a_t}, \quad (13)$$

where

$$\lambda_t \equiv \frac{w_t u_{11}(c_t^*, l_t^*) + u_{12}(c_t^*, l_t^*)}{u_{22}(c_t^*, l_t^*) + w_t u_{12}(c_t^*, l_t^*)} = \frac{u_1(c_t^*, l_t^*) u_{12}(c_t^*, l_t^*) - u_2(c_t^*, l_t^*) u_{11}(c_t^*, l_t^*)}{u_1(c_t^*, l_t^*) u_{22}(c_t^*, l_t^*) - u_2(c_t^*, l_t^*) u_{12}(c_t^*, l_t^*)}. \quad (14)$$

Note that, if consumption and leisure in period  $t$  are normal goods, then  $\lambda_t > 0$ , although we do not require this restriction below. It now only remains to solve for the derivative  $\partial c_t^*/\partial a_t$ .

Intuitively,  $\partial c_t^*/\partial a_t$  should not be too difficult to compute: it is just the household's marginal propensity to consume today out of a change in assets, which we can deduce from the household's Euler equation and budget constraint. Differentiating the Euler equation:

$$u_1(c_t^*, l_t^*) = \beta E_t(1 + r_{t+1}) u_1(c_{t+1}^*, l_{t+1}^*), \quad (15)$$

with respect to  $a_t$  yields:<sup>8</sup>

$$u_{11}(c_t^*, l_t^*) \frac{\partial c_t^*}{\partial a_t} + u_{12}(c_t^*, l_t^*) \frac{\partial l_t^*}{\partial a_t} = \beta E_t(1 + r_{t+1}) \left[ u_{11}(c_{t+1}^*, l_{t+1}^*) \frac{\partial c_{t+1}^*}{\partial a_t} + u_{12}(c_{t+1}^*, l_{t+1}^*) \frac{\partial l_{t+1}^*}{\partial a_t} \right] \quad (16)$$

Substituting in for  $\partial l_t^*/\partial a_t$  gives:

$$(u_{11}(c_t^*, l_t^*) - \lambda_t u_{12}(c_t^*, l_t^*)) \frac{\partial c_t^*}{\partial a_t} = \beta E_t(1 + r_{t+1}) (u_{11}(c_{t+1}^*, l_{t+1}^*) - \lambda_{t+1} u_{12}(c_{t+1}^*, l_{t+1}^*)) \frac{\partial c_{t+1}^*}{\partial a_t}. \quad (17)$$

Evaluating (17) at steady state,  $\beta = (1 + r)^{-1}$ ,  $\lambda_t = \lambda_{t+1} = \lambda$ , and the  $u_{ij}$  cancel, giving:

$$\frac{\partial c_t^*}{\partial a_t} = E_t \frac{\partial c_{t+1}^*}{\partial a_t} = E_t \frac{\partial c_{t+k}^*}{\partial a_t}, \quad k = 1, 2, \dots \quad (18)$$

$$\frac{\partial l_t^*}{\partial a_t} = E_t \frac{\partial l_{t+1}^*}{\partial a_t} = E_t \frac{\partial l_{t+k}^*}{\partial a_t}, \quad k = 1, 2, \dots \quad (19)$$

In other words, whatever the change in the household's consumption today, it must be the same as the expected change in consumption tomorrow, and the expected change in consumption at each future date  $t + k$ .<sup>9</sup>

The household's budget constraint is implied by asset accumulation equation (2) and transversality condition (3). Differentiating (2) with respect to  $a_t$ , evaluating at steady state, and applying (3), (18), and (19) gives:

$$\frac{1 + r}{r} \frac{\partial c_t^*}{\partial a_t} = (1 + r) + \frac{1 + r}{r} w \frac{\partial l_t^*}{\partial a_t}. \quad (20)$$

That is, the expected present value of changes in household consumption must equal the change in assets (times  $1 + r$ ) plus the expected present value of changes in labor income.

Combining (20) with (13), we can solve for  $\partial c_t^*/\partial a_t$  evaluated at the steady state:

$$\frac{\partial c_t^*}{\partial a_t} = \frac{r}{1 + w\lambda}. \quad (21)$$

In response to a unit increase in assets, the household raises consumption in every period by the extra asset income,  $r$ , adjusted downward by the amount  $1 + w\lambda$  that takes into account the household's decrease in hours worked.

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<sup>8</sup>By  $\partial c_{t+1}^*/\partial a_t$  we mean:

$$\frac{\partial c_{t+1}^*}{\partial a_t} = \frac{\partial c_{t+1}^*}{\partial a_{t+1}} \frac{da_{t+1}^*}{da_t} = \frac{\partial c_{t+1}^*}{\partial a_{t+1}} \left[ 1 + r_{t+1} + w_t \frac{\partial l_t^*}{\partial a_t} - \frac{\partial c_t^*}{\partial a_t} \right],$$

and analogously for  $\partial l_{t+1}^*/\partial a_t$ ,  $\partial c_{t+2}^*/\partial a_t$ ,  $\partial l_{t+2}^*/\partial a_t$ , etc.

<sup>9</sup>Note that this equality does not follow from the steady state assumption. For example, in a model with internal habits, which we will consider in Section 4, the individual household's optimal consumption response to a change in assets increases with time, even starting from steady state.

We can now compute the household's coefficient of absolute risk aversion. Substituting (10), (11), (13)–(14), and (21) into (9), we have proved:

**Proposition 2.** *The household's coefficient of absolute risk aversion in Proposition 1, evaluated at steady state, satisfies:*

$$\frac{-V_{11}(a; \theta)}{V_1(a; \theta)} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{r}{1 + w\lambda}, \quad (22)$$

where  $u_1$ ,  $u_{11}$ , and  $u_{12}$  denote the corresponding partial derivatives of  $u$  evaluated at the steady state  $(c, l)$ , and  $\lambda$  is given by (14) evaluated at steady state.

When there is no labor margin in the model, Proposition 2 has the following corollary:

**Corollary 3.** *Suppose that  $l_t$  is fixed exogenously at  $l \geq 0$  for all  $t$  and the household chooses  $c_t$  optimally at each  $t$  given this constraint. Then the household's coefficient of absolute risk aversion (22), evaluated at steady state, is given by:*

$$\frac{-V_{11}(a; \theta)}{V_1(a; \theta)} = \frac{-u_{11}}{u_1} r. \quad (23)$$

PROOF: The assumptions and steps leading up to Proposition 2, adjusted to the one-dimensional case, are essentially the same as the above with  $\lambda_t = 0$ .

Proposition 2 and Corollary 3 are remarkable. First, the household's coefficient of absolute risk aversion in (23) is just the traditional measure,  $-u_{11}/u_1$ , times  $r$ , which translates assets into current-period consumption. In other words, for *any* utility kernel  $u$ , the traditional, static measure of risk aversion is also the correct measure in the dynamic context, regardless of whether  $u$  or the rest of the model is homothetic, whether or not we can solve for  $V$ , and no matter what the functional forms of  $u$  and  $V$ .

More generally, when households have a labor margin, Proposition 2 shows that risk aversion is less than the traditional measure by the factor  $1 + w\lambda$ , *even when consumption and labor are additively separable in  $u$*  (i.e.  $u_{12} = 0$ ). Even in the additively separable case, households can partially absorb shocks to income through changes in hours worked. As a result,  $c_t^*$  depends on household labor supply, so labor and consumption are indirectly connected through the budget constraint. When  $u_{12} \neq 0$ , risk aversion in Proposition 2 is further attenuated or amplified by the direct interaction between consumption and labor in utility,  $u_{12}$ .

The household's labor margin can have dramatic effects on risk aversion. For example, no matter how large the traditional measure  $-u_{11}/u_1$ , the household can still be risk neutral:

**Corollary 4.** *Let  $\Delta$  denote the discriminant of  $u$ ,  $\Delta = u_{11}u_{22} - u_{12}^2$ . The household's coefficient of absolute risk aversion (22) vanishes as  $\Delta \rightarrow 0$ , so long as either  $u_1 \neq -u_2$  or  $u_{12} < \max\{|u_{11}|, |u_{22}|\}$  in the limit.*

PROOF: The corollary is stated as a limiting result to respect concavity in Assumption 1. Substituting out  $\lambda$  and  $w$ , (22) vanishes as  $\Delta \rightarrow 0$  except for the special case  $u_1 = -u_2$  and  $u_{11} = -u_{12} = u_{22}$ —that is, the special case  $u(c, l) = \tilde{u}(c - l)$  to second order for some function  $\tilde{u}$ . The corollary rules out that case by assumption.

In other words, risk aversion depends on the concavity of  $u$  in all dimensions rather than just in one dimension. Even when  $u_{11}$  is very large, the household still can be risk neutral if  $u_{22}$  is small or the cross-effect  $u_{12}$  is sufficiently large. Geometrically, if there exists any direction in  $(c, l)$ -space along which  $u$  is flat, the household will optimally choose to absorb shocks to income along that line, resulting in risk-neutral behavior.

We provide some more concrete examples of risk aversion calculations in Section 2.5, below, after first defining relative risk aversion.

## 2.4 The Coefficient of Relative Risk Aversion

The difference between absolute and relative risk aversion is the size of the hypothetical gamble faced by the household. If the household faces a one-shot gamble of size  $A_t$  in period  $t$ , that is:

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t + A_t \sigma \varepsilon_{t+1}, \quad (24)$$

or the household can pay a one-time fee  $A_t \mu$  in period  $t$  to avoid this gamble, then it follows from Proposition 1 that the household's coefficient of risk aversion,  $\lim_{\sigma \rightarrow 0} 2\mu(\sigma)/\sigma^2$ , for this gamble is given by:

$$\frac{-A_t E_t V_{11}(a_{t+1}^*; \theta_{t+1})}{E_t V_1(a_{t+1}^*; \theta_{t+1})}. \quad (25)$$

The natural definition of  $A_t$ , considered by Arrow (1964) and Pratt (1965), is the household's wealth at time  $t$ . The gamble in (24) is then over a fraction of the household's wealth and (25) is referred to as the household's coefficient of relative risk aversion.

In models with labor, however, household wealth can be more difficult to define because of the presence of human capital. In these models, there are two natural definitions of human

capital, so we consequently define two measures of household wealth  $A_t$  and two coefficients of relative risk aversion (25).

First, when the household's time endowment is not well-defined—as when  $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}$  and no upper bound  $\bar{l}$  on  $l_t$  is specified, or  $\bar{l}$  is specified but is completely arbitrary—it is most natural to define human capital as the present discounted value of labor income,  $w_t l_t^*$ . Equivalently, total household wealth  $A_t$  equals the present discounted value of consumption, which follows from the budget constraint (2)–(3). We state this formally as:

**Definition 2.** *The household's consumption-based coefficient of relative risk aversion is given by (25), with  $A_t \equiv (1+r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau} c_{\tau}^*$ , the present discounted value of household consumption, and where  $m_{t,\tau}$  denotes the stochastic discount factor  $\beta^{\tau-t} u_1(c_{\tau}^*, l_{\tau}^*)/u_1(c_t^*, l_t^*)$ .*

The factor  $(1+r_t)^{-1}$  in the definition expresses wealth  $A_t$  in beginning- rather than end-of-period- $t$  units, so that in steady state  $A = c/r$  and the consumption-based coefficient of relative risk aversion is given by:

$$\frac{-A V_{11}(a; \theta)}{V_1(a; \theta)} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1 + w\lambda}. \quad (26)$$

Alternatively, when the household's time endowment  $\bar{l}$  is well specified, we can define human capital to be the present discounted value of the household's time endowment,  $w_t \bar{l}$ . In this case, total household wealth  $\tilde{A}_t$  equals the present discounted value of leisure  $w_t(\bar{l} - l_t^*)$  plus consumption  $c_t^*$ , from (2)–(3). We thus have:

**Definition 3.** *The household's leisure-and-consumption-based coefficient of relative risk aversion is given by (25), with  $A_t = \tilde{A}_t \equiv (1+r_t)^{-1} E_t \sum_{\tau=t}^{\infty} m_{t,\tau} (c_{\tau}^* + w_{\tau}(\bar{l} - l_{\tau}^*))$ .*

In steady state,  $\tilde{A} = (c + w(\bar{l} - l))/r$ , and the leisure-and-consumption-based coefficient of relative risk aversion is given by:

$$\frac{-\tilde{A} V_{11}(a; \theta)}{V_1(a; \theta)} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c + w(\bar{l} - l)}{1 + w\lambda}. \quad (27)$$

Of course, (26) and (27) are related by the ratio of the two gambles,  $(c + w(\bar{l} - l))/c$ .

Other definitions of relative risk aversion, corresponding to alternative definitions of wealth and the size of the gamble  $A_t$ , are also possible, but Definitions 2–3 are the most natural for several reasons. First, both definitions reduce to the usual present discounted value of income or consumption when there is no human capital in the model. Second, both

measures of risk aversion reduce to the traditional  $-cu_{11}/u_1$  when there is no labor margin in the model—that is, when  $\lambda = 0$ . Third, in steady state the household consumes exactly the flow of income from its wealth,  $rA$ , consistent with standard permanent income theory (where one must include the value of leisure  $w(\bar{l} - l)$  as part of consumption when the value of leisure is included in wealth).

We close this section by noting that neither measure of relative risk aversion is reciprocal to the intertemporal elasticity of substitution:

**Corollary 5.** *Evaluated at steady state: i) the consumption-based coefficient of relative risk aversion and intertemporal elasticity of substitution are reciprocal if and only if  $\lambda = 0$ ; ii) the leisure-and-consumption-based coefficient of relative risk aversion and intertemporal elasticity of substitution are reciprocal if and only if  $\lambda = (\bar{l} - l)/c$ .*

PROOF: Note that the case  $w = 0$  is ruled out by Assumptions 1 and 3. The household's intertemporal elasticity of substitution, evaluated at steady state, is given by  $(dc_{t+1}^* - dc_t^*)/d\log(1+r_{t+1})$ , which equals  $-u_1/(c(u_{11} - \lambda u_{12}))$  by a calculation along the lines of (17), holding  $w_t$  fixed but allowing  $l_t^*$  and  $l_{t+1}^*$  to vary endogenously. The corollary follows from comparing this expression to (26) and (27).

## 2.5 Examples

Some simple examples illustrate how ignoring the household's labor margin can lead to wildly inaccurate measures of the household's true attitudes toward risk.

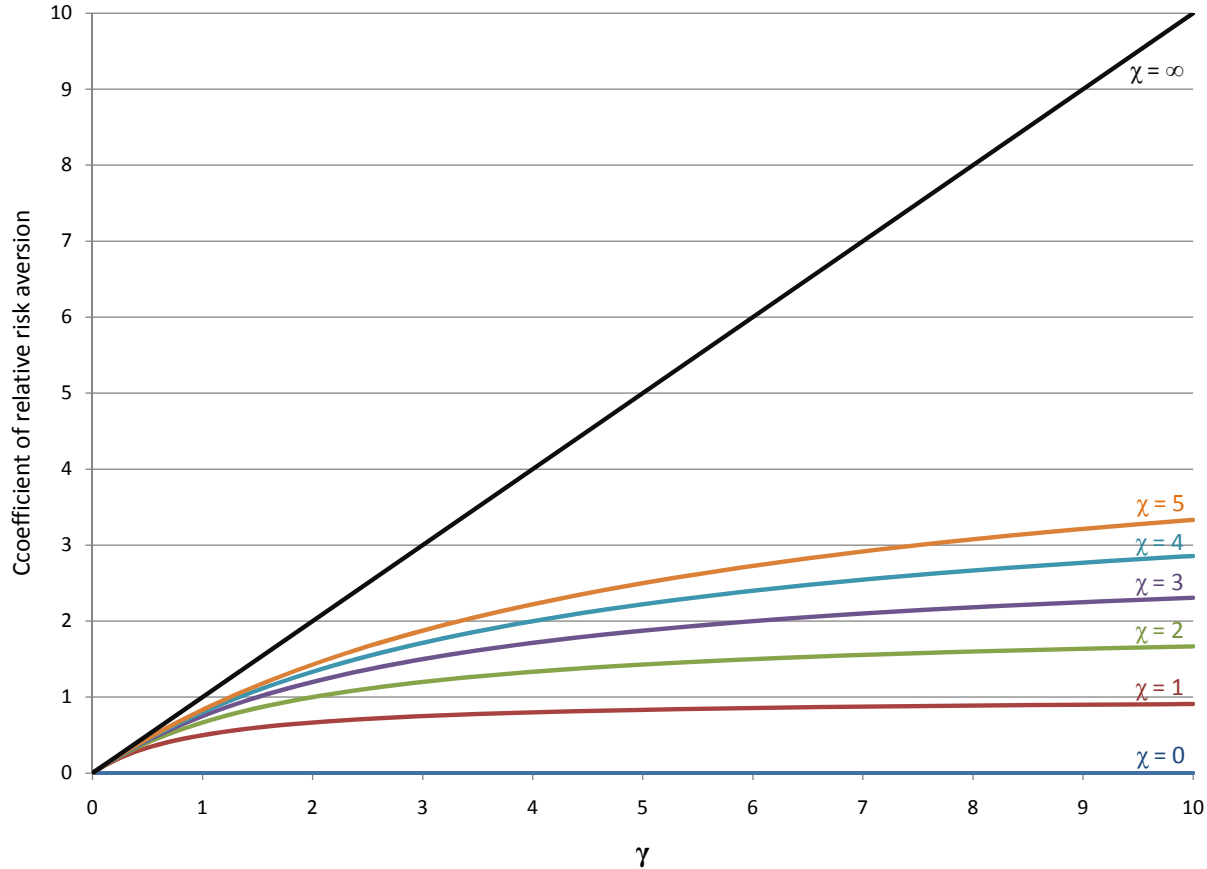
**Example 2.1.** Consider the additively separable utility kernel:

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi}, \quad (28)$$

where  $\gamma, \chi, \eta > 0$ . The traditional measure of risk aversion for this utility kernel is  $-cu_{11}/u_1 = \gamma$ , but the household's consumption-based coefficient of relative risk aversion is given by (26):

$$\frac{-AV_{11}}{V_1} = \frac{-cu_{11}}{u_1} \frac{1}{1 + w \frac{wu_{11}}{u_{22}}} = \frac{\gamma}{1 + \frac{\gamma}{\chi} \frac{wl}{c}}. \quad (29)$$

The household's leisure-and-consumption-based coefficient of relative risk aversion (27) is not well defined in this example (the household's risk aversion can be made arbitrarily large or small just by varying the household's time endowment  $\bar{l}$ ), so we focus only on the consumption-based measure (29).



**Figure 1.** Consumption-based coefficient of relative risk aversion for the utility kernel  $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma) - \eta l_t^{1+\chi}/(1+\chi)$  in Example 2.1, as a function of the traditional measure  $\gamma$ , for different values of  $\chi$ . See text for details.

In steady state,  $c \approx wl$ ,<sup>10</sup> so (29) can be written as:

$$\frac{-AV_{11}}{V_1} \approx \frac{1}{\frac{1}{\gamma} + \frac{1}{\chi}}. \quad (30)$$

Note that (30) is less than the traditional measure of risk aversion by a factor of  $1 + \gamma/\chi$ . Thus, if  $\gamma = 2$  and  $\chi = 1$ —parameter values that are well within the range of estimates in the literature—then the household’s true risk aversion is less than the traditional measure by a factor of about three. This point is illustrated in Figure 1, which graphs the coefficient of relative risk aversion for this example as a function of the traditional measure,  $\gamma$ , for several different values of  $\chi$ . If  $\chi$  is very large, then the bias from using the traditional measure is small because household labor supply is essentially fixed.<sup>11</sup> However, as  $\chi$  approaches 0, a

<sup>10</sup>In steady state,  $c = ra + wl + d$ , so  $c = wl$  holds exactly if there is neither capital nor transfers in the model. In any case,  $ra + d$  is typically small for standard calibrations in the literature.

<sup>11</sup>Similarly, if  $\gamma$  is very small, the bias from using the traditional measure is small because the household chooses to absorb income shocks almost entirely along its consumption margin. As a result, the labor margin is again almost inoperative.

common benchmark in the literature, the bias explodes and true risk aversion approaches zero—the household becomes risk neutral. Intuitively, households with linear disutility of work are risk neutral with respect to gambles over wealth because they can completely offset those gambles at the margin by working more or fewer hours, and households with linear disutility of work are clearly risk neutral with respect to gambles over hours.

Expression (30) also helps to clarify several points. First, risk aversion in the model is a combination of both parameters  $\gamma$  and  $\chi$ , reflecting that the household absorbs income gambles along both of its margins, consumption and labor. Second, for any given  $\gamma$ , actual risk aversion in the model can lie anywhere between 0 and  $\gamma$ , depending on  $\chi$ . That is, having an additional margin with which to absorb income gambles reduces the household's aversion to risk. Third, (30) is symmetric in  $\gamma$  and  $\chi$ , reflecting that labor and consumption enter symmetrically into  $u$  in this example and play an essentially equal role in absorbing income shocks. Put another way, ignoring the labor margin in this example would be just as erroneous as ignoring the consumption margin.

**Example 2.2.** Consider the King-Plosser-Rebelo-type (1988) utility kernel:

$$u(c_t, l_t) = \frac{c_t^{1-\gamma} (1-l_t)^{\chi(1-\gamma)}}{1-\gamma}, \quad (31)$$

where  $\gamma > 0$ ,  $\gamma \neq 1$ ,  $\chi > 0$ ,  $\bar{l} = 1$ , and  $\chi(1-\gamma) < \gamma$  for concavity. The traditional measure of risk aversion for (31) is  $\gamma$ , but the household's actual leisure-and-consumption-based coefficient of relative risk aversion is given by:

$$\frac{-\tilde{A} V_{11}}{V_1} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c + w(1-l)}{1+w\lambda} = \gamma - \chi(1-\gamma). \quad (32)$$

Note that concavity of (31) implies that (32) is positive. As in the previous example, (32) depends on both  $\gamma$  and  $\chi$ , and can lie anywhere between 0 and the traditional measure  $\gamma$ , depending on  $\chi$ . In this example, risk aversion is less than the traditional measure by the amount  $\chi(1-\gamma)$ . As  $\chi$  approaches  $\gamma/(1-\gamma)$ —that is, as utility approaches Cobb-Douglas—the household becomes risk neutral; in this case, household utility along the line  $c_t = w_t(1-l_t)$  is linear, so the household finds it optimal to absorb shocks to wealth along that line.

The household's consumption-based coefficient of relative risk aversion is a bit more complicated than (32):

$$\frac{-A V_{11}}{V_1} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1+w\lambda} = \frac{\gamma - \chi(1-\gamma)}{1+\chi}. \quad (33)$$



Again, (33) is a combination of the parameters  $\gamma$  and  $\chi$ , and can lie anywhere between 0 and  $\gamma$ , depending on  $\chi$ . Neither (32) nor (33) equals the traditional measure  $\gamma$ , except for the special case  $\chi = 0$ .

## 2.6 Risk Aversion and Asset Pricing

In the preceding sections, we showed that the labor margin has important implications for Arrow-Pratt risk aversion with respect to gambles over income or wealth. We now show that risk aversion with respect to these gambles is also the right concept for asset pricing.

### 2.6.1 Measuring Risk Aversion with $V$ As Opposed to $u$

Some comparison of the expressions  $-V_{11}/V_1$  and  $-u_{11}/u_1$  helps to clarify why the former measure is the relevant one for pricing assets, such as stocks or bonds, in the model. From Proposition 1,  $-V_{11}/V_1$  is the Arrow-Pratt coefficient of absolute risk aversion for gambles over income or wealth in period  $t$ . In contrast, the expression  $-u_{11}/u_1$  is the risk aversion coefficient for a hypothetical gamble in which the household is *forced to consume immediately* the outcome of the gamble. Clearly, it is the former concept that corresponds to the stochastic payoffs of a standard asset, such as a stock or bond, in a DSGE model. In order for  $-u_{11}/u_1$  to be the relevant measure for pricing a security, it is not enough that the security pay off in units of consumption in period  $t+1$ . The household would additionally have to be prevented from adjusting its consumption and labor choices in period  $t+1$  in response to the security's payoffs, so that the household is forced to absorb those payoffs into period  $t+1$  consumption. It is difficult to imagine such a security in the real world—all standard securities in financial markets correspond to gambles over income or wealth, for which the  $-V_{11}/V_1$  measure of risk aversion is the appropriate one.

### 2.6.2 Risk Aversion, the Stochastic Discount Factor, and Risk Premia

Arrow-Pratt risk aversion, and hence the labor margin, is also closely tied to asset prices in the standard Lucas-Breeden stochastic discounting framework.

Let  $m_{t+1} = \beta u_1(c_{t+1}^*, l_{t+1}^*)/u_1(c_t^*, l_t^*)$  denote the household's stochastic discount factor and let  $p_t$  denote the cum-dividend price of a risky asset at time  $t$ , with  $E_t p_{t+1}$  normalized

to unity. The percentage difference between the risk-neutral price of the asset and its actual price—the risk premium on the asset—is given by:

$$(E_t m_{t+1} E_t p_{t+1} - E_t m_{t+1} p_{t+1}) / E_t m_{t+1} = -\text{Cov}_t(dm_{t+1}, dp_{t+1}) / E_t m_{t+1} \quad (34)$$

where  $\text{Cov}_t$  denotes the covariance conditional on information at time  $t$ , and  $dx \equiv x_{t+1} - E_t x_{t+1}$ ,  $x \in \{m, p\}$ . For small changes  $dc_{t+1}^*$  and  $dl_{t+1}^*$ , we have, to first order:

$$dm_{t+1} = \frac{\beta}{u_1(c_t^*, l_t^*)} [u_{11}(c_{t+1}^*, l_{t+1}^*) dc_{t+1}^* + u_{12}(c_{t+1}^*, l_{t+1}^*) dl_{t+1}^*], \quad (35)$$

conditional on information at time  $t$ . In (35), the household's labor margin affects  $m_{t+1}$  and hence asset prices for two reasons: First, if  $u_{12} \neq 0$ , changes in  $l_{t+1}$  directly affect the household's marginal utility of consumption. Second, even if  $u_{12} = 0$ , the presence of the labor margin affects how the household responds to shocks and hence affects  $dc_{t+1}^*$ .

Intuitively, one can already see the relationship between risk aversion and  $dm_{t+1}$  in (35): if  $dl_{t+1}^* = -\lambda dc_{t+1}^*$  and  $dc_{t+1}^* = rda_{t+1}/(1+w\lambda)$ , as in Section 2.3, then  $dm_{t+1}$  equals the coefficient of absolute risk aversion times  $da_{t+1}$ . In actuality, the relationship is slightly more complicated than this because  $\theta$  (and hence  $w$ ,  $r$ , and  $d$ ) may change as well as  $a$ . For example, differentiating (12) and evaluating at steady state, we have, to first order:

$$dl_{t+1}^* = -\lambda dc_{t+1}^* - \frac{u_1}{u_{22} + wu_{12}} dw_{t+1}, \quad (36)$$

Similarly, combining (2), (3), and (15), differentiating, and evaluating at steady state, we show in the Appendix that:

$$\begin{aligned} dc_{t+1}^* &= \frac{r}{1+w\lambda} \left[ da_{t+1} + E_{t+1} \sum_{k=1}^{\infty} \frac{1}{(1+r)^k} (l dw_{t+k} + dd_{t+k} + adr_{t+k}) \right] \\ &+ \frac{u_1}{u_{11}u_{22} - u_{12}^2} dw_{t+1} + \frac{-ru_1}{u_{11} - \lambda u_{12}} E_{t+1} \sum_{k=1}^{\infty} \frac{1}{(1+r)^k} \left( \frac{\lambda}{1+w\lambda} dw_{t+k} - d \log R_{t+1, t+k} \right), \end{aligned} \quad (37)$$

where  $R_{t+1, t+k} \equiv \prod_{i=2}^k (1+r_{t+i})$ . Note that for the Arrow-Pratt one-shot gamble considered in Section 2.3, the aggregate variables  $w$ ,  $r$ , and  $d$  were all held constant, so (36)–(37) reduce to (13) and (21) in that case. The term in square brackets in (37) describes the change in the present value of household income, and thus the first line of (37) describes the income effect on consumption. The last line of (37) describes the substitution effect: changes in consumption due to changes in current and future interest rates and wages. (Recall that  $-u_1/(c(u_{11} - \lambda u_{12}))$  is the intertemporal elasticity of substitution.)

Substituting (36)–(37) into (35) yields:

$$dm_{t+1} = \beta \frac{u_{11} - \lambda u_{12}}{u_1} \frac{r}{1 + w\lambda} \left[ da_{t+1} + E_{t+1} \sum_{k=1}^{\infty} \frac{1}{(1+r)^k} (l dw_{t+k} + dd_{t+k} + adr_{t+k}) \right] - \beta r E_{t+1} \sum_{k=1}^{\infty} \frac{1}{(1+r)^k} \left( \frac{\lambda}{1+w\lambda} dw_{t+k} - d \log R_{t+1,t+k} \right). \quad (38)$$

The risk premium (34), evaluated at steady state, is then given by:

$$\frac{-u_{11} + \lambda u_{12}}{u_1} \frac{r}{1 + w\lambda} \text{Cov}_t(dp_{t+1}, dA_{t+1}) + r \text{Cov}_t(dp_{t+1}, d\Psi_{t+1}), \quad (39)$$

where  $dA_{t+1}$  denotes the quantity in square brackets in (38)—the change in household wealth—and  $d\Psi_{t+1}$  denotes the summation on the second line of (38)—the change in current and future wages and interest rates. Equations (38)–(39) are essentially Merton’s (1973) ICAPM, generalized to include labor. In (39), the first term is the covariance of the asset price with household wealth, multiplied by the coefficient of absolute risk aversion, while the second term captures the asset’s ability to hedge what Merton calls “changes in investment opportunities.” Intuitively, even for households that are Arrow-Pratt risk neutral, an asset that pays off well when future interest rates are high or wages are low—and hence future consumption is low—is preferable to an asset that has no correlation with future  $r$  or  $w$ .

Equation (39) shows the importance of risk aversion—and hence the labor margin—for asset pricing in the model. Risk premia are essentially linear in the coefficient of absolute risk aversion, a relationship which also holds for the more general cases of Epstein-Zin preferences and habits, considered below.<sup>12</sup> This link between risk aversion and risk premia should not be too surprising: Arrow-Pratt risk aversion describes the risk premium for the most basic gambles over household income or wealth. Here we have shown that the same coefficient also appears for completely general gambles that may be correlated with aggregate variables such as interest rates, wages, and net transfers.<sup>13</sup> The risk premia on these gambles are determined by the household’s stochastic discount factor, but the stochastic discount factor is itself directly linked to risk aversion and the household’s labor margin.

<sup>12</sup>See the Appendix. For an example of this linearity, see Figure 1 of Rudebusch and Swanson (2009).

<sup>13</sup>Boldrin, Christiano, and Fisher (1997) argue that it is  $u_{11}/u_1$  rather than  $V_{11}/V_1$  that matters for the equity premium in their Figure 2. As shown above, it is in fact  $V_{11}/V_1$  that is crucial. What explains Boldrin et al.’s Figure 2 is that the covariance of equity prices with the short-term interest rate is not being held constant in their model—in particular, the variance of the risk-free rate in their model changes tremendously over the points in their Figure 2.

### 3. Generalized Recursive Preferences

We now turn to the case of generalized recursive preferences, as in Epstein and Zin (1989) and Weil (1989). The household's asset accumulation equation (2) and transversality condition (3) are the same as in Section 2, but now instead of maximizing (1), the household chooses  $c_t$  and  $l_t$  to maximize the recursive expression:<sup>14</sup>

$$V(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} u(c_t, l_t) + \beta \left( E_t V(a_{t+1}; \theta_{t+1})^{1-\alpha} \right)^{1/(1-\alpha)}, \quad (40)$$

where  $\alpha \in \mathbb{R}$ ,  $\alpha \neq 1$ .<sup>15</sup> Note that (40) is the same as (4), but with the value function “twisted” and “untwisted” by the coefficient  $1 - \alpha$ . When  $\alpha = 0$ , the preferences given by (40) reduce to the special case of expected utility.

If  $u \geq 0$  everywhere, then the proof of Theorem 3.1 in Epstein and Zin (1989) shows that there exists a solution  $V$  to (40) with  $V \geq 0$ . If  $u \leq 0$  everywhere, then it is natural to let  $V \leq 0$  and reformulate the recursion as:

$$V(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} u(c_t, l_t) - \beta \left( E_t (-V(a_{t+1}; \theta_{t+1}))^{1-\alpha} \right)^{1/(1-\alpha)}. \quad (41)$$

The proof in Epstein and Zin (1989) also demonstrates the existence of a solution  $V$  to (41) with  $V \leq 0$  in this case.

To avoid the possibility of complex numbers arising in the maximand of (40) or (41), we restrict the range of  $u$  to be either  $\mathbb{R}_+$  or  $\mathbb{R}_-$ :<sup>16</sup>

**Assumption 7.** *Either  $u : \Omega \rightarrow \mathbb{R}_+$ , or  $u : \Omega \rightarrow \mathbb{R}_-$ .*

The main advantage of generalized recursive preferences (40) is that they allow for greater flexibility in modeling risk aversion and the intertemporal elasticity of substitution. In (40), the intertemporal elasticity of substitution over deterministic consumption paths is exactly the same as in (4), but the household's risk aversion to gambles can be amplified (or attenuated) by the additional parameter  $\alpha$ .

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<sup>14</sup>Note that, traditionally, Epstein-Zin preferences over consumption streams have been written as:

$$\tilde{V}(a_t; \theta_t) = \max_{c_t} \left[ c_t^\rho + \beta \left( E_t \tilde{V}(a_{t+1}; \theta_{t+1})^{\tilde{\alpha}} \right)^{\rho/\tilde{\alpha}} \right]^{1/\rho},$$

but by setting  $V = \tilde{V}^\rho$  and  $\alpha = 1 - \tilde{\alpha}/\rho$ , this can be seen to correspond to (40).

<sup>15</sup>We exclude the case  $\alpha = 1$  here for simplicity.

<sup>16</sup>Alternatively, one can restrict the domain of  $u$  to ensure  $u \geq 0$  or  $u \leq 0$ ; e.g., by requiring  $c \geq 1$  for  $u(c, l) = \log c + \chi(\bar{l} - l)$ .

### 3.1 Coefficients of Absolute and Relative Risk Aversion

We consider the household's aversion to the same hypothetical gamble as in (6):

**Proposition 6.** *With generalized recursive preferences (40) or (41), the household's coefficient of absolute risk aversion with respect to the gamble described by (6) is given by:*

$$\frac{-E_t V(a_{t+1}^*; \theta_{t+1})^{-\alpha} \left[ V_{11}(a_{t+1}^*; \theta_{t+1}) - \alpha \frac{V_1(a_{t+1}^*; \theta_{t+1})^2}{V(a_{t+1}^*; \theta_{t+1})} \right]}{E_t V(a_{t+1}^*; \theta_{t+1})^{-\alpha} V_1(a_{t+1}^*; \theta_{t+1})}. \quad (42)$$

*Evaluated at steady state, (42) simplifies to:*

$$\frac{-V_{11}(a; \theta)}{V_1(a; \theta)} + \alpha \frac{V_1(a; \theta)}{V(a; \theta)}. \quad (43)$$

PROOF: See Appendix.

The first term in (43) is the same as the expected utility case (9), while the second term in (43) reflects the amplification or attenuation of risk aversion from the additional curvature parameter  $\alpha$ . When  $\alpha = 0$ , (42)–(43) reduce to (8)–(9). When  $u \geq 0$  and hence  $V \geq 0$ , higher values of  $\alpha$  correspond to greater degrees of risk aversion; when  $u$  and  $V \leq 0$ , the opposite is true: higher values of  $\alpha$  correspond to lesser degrees of risk aversion.

The household's coefficient of relative risk aversion is given by  $A_t$  times (42), which, evaluated at steady state, simplifies to:

$$\frac{-AV_{11}(a; \theta)}{V_1(a; \theta)} + \alpha \frac{AV_1(a; \theta)}{V(a; \theta)}. \quad (44)$$

We define the household's total wealth  $A_t$  based on the present discounted value of its lifetime consumption or lifetime leisure and consumption, as in Section 2.4, and we refer to (44) as the consumption-based or leisure-and-consumption-based coefficient of relative risk aversion, depending on the definition of  $A$ .<sup>17</sup>

Expressions (43) and (44) highlight an important feature of risk aversion with generalized recursive preferences: it is not invariant with respect to level shifts of the utility kernel,

<sup>17</sup>Note that, with generalized recursive preferences, the household's stochastic discount factor is given by:

$$\frac{\beta u_1(c_{t+1}^*, l_{t+1}^*)}{u_1(c_t^*, l_t^*)} \left( \frac{V(a_{t+1}^*; \theta_{t+1})}{(E_t V(a_{t+1}^*; \theta_{t+1})^{1-\alpha})^{1/(1-\alpha)}} \right)^{-\alpha},$$

which must be used to compute household wealth. At steady state, however, this simplifies to the usual  $\beta$ .

except for the special case of expected utility ( $\alpha = 0$ ). That is, the utility kernels  $u(\cdot, \cdot)$  and  $u(\cdot, \cdot) + k$ , where  $k$  is a constant, lead to different household attitudes toward risk. The household's preferences are invariant, however, with respect to multiplicative transformations of the utility kernel.

When it comes to computing the risk aversion coefficients (43)–(44), expressions (10)–(21) for  $V_1$ ,  $V_{11}$ ,  $\partial l_t^*/\partial a_t$ , and  $\partial c_t^*/\partial a_t$  continue to apply in the current context. Moreover,  $V = u(c, l)/(1 - \beta)$  at the steady state. Substituting these into (43)–(44) gives:

**Proposition 7.** *The household's coefficient of absolute risk aversion in Proposition 6, evaluated at steady state, is given by:*

$$\frac{-V_{11}}{V_1} + \alpha \frac{V_1}{V} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{r}{1 + w\lambda} + \alpha \frac{r u_1}{u}. \quad (45)$$

*The household's consumption-based coefficient of relative risk aversion, evaluated at steady state, is given by:*

$$\frac{-AV_{11}}{V_1} + \alpha \frac{AV_1}{V} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1 + w\lambda} + \alpha \frac{c u_1}{u}. \quad (46)$$

*The household's leisure-and-consumption-based coefficient of relative risk aversion, evaluated at steady state, is given by  $(c + w(\bar{l} - l))/c$  times (46).*

Proposition 7 is important because risk aversion for Epstein-Zin preferences has only been computed previously in homothetic, isoelastic, consumption-only models, where the value function can be computed in closed form. Proposition 7 does not require homotheticity, is valid for general functional forms  $u$ , unknown functional forms  $V$ , and allows for the presence of labor.

## 3.2 Examples

**Example 3.1.** Consider the additively separable utility kernel:

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi}, \quad (47)$$

with generalized recursive preferences (41) and  $\chi > 0$ ,  $\eta > 0$ , and  $\gamma > 1$ , which was used by Rudebusch and Swanson (2009).<sup>18</sup> In this case, where  $u(\cdot, \cdot) < 0$ , risk aversion is decreasing in  $\alpha$ , and  $\alpha < 0$  corresponds to preferences that are more risk averse than expected utility.

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<sup>18</sup>We restrict attention here to the case  $\gamma > 1$ , consistent with Assumption 7. The case  $\gamma \leq 1$  can be considered if we place restrictions on the domain of  $c_t$  and  $l_t$  such that  $u(\cdot, \cdot) < 0$ ; one can always choose units for  $c_t$  and  $l_t$  such that this doesn't represent much of a constraint in practice. Of course, one can also consider alternative utility kernels with  $\gamma \leq 1$  for which  $u(\cdot, \cdot) > 0$ .

In models without labor, period utility  $u(c_t, l_t) = c_t^{1-\gamma}/(1-\gamma)$  implies a coefficient of relative risk aversion of  $\gamma + \alpha(1-\gamma)$ , which we will refer to as the traditional measure.<sup>19</sup> Taking into account both the consumption and labor margins of (47), the household's consumption-based coefficient of relative risk aversion (46) is given by:

$$\begin{aligned} \frac{-AV_{11}}{V_1} + \alpha \frac{AV_1}{V} &= \frac{\gamma}{1 + \frac{\gamma}{\chi} \frac{wl}{c}} + \frac{\alpha(1-\gamma)}{1 + \frac{\gamma-1}{1+\chi} \frac{wl}{c}}, \\ &\approx \frac{\gamma}{1 + \frac{\gamma}{\chi}} + \frac{\alpha(1-\gamma)}{1 + \frac{\gamma-1}{1+\chi}}, \end{aligned} \quad (48)$$

using  $c \approx wl$ . As in Example 2.1, the household's leisure-and-consumption-based coefficient of relative risk aversion is not well defined in this example, so we restrict attention to the consumption-based measure (48).

As  $\chi$  becomes large, household labor becomes less flexible and the bias from ignoring the labor margin shrinks to zero ((48) approaches  $\gamma + \alpha(1-\gamma)$ ). As  $\chi$  approaches zero, (48) decreases to  $\alpha(1-\gamma)/\gamma$ , which is close to zero if we think of  $\gamma$  as being small (close to unity). Thus, for given values of  $\gamma$  and  $\alpha$ , actual household risk aversion can lie anywhere between about zero and  $\gamma + \alpha(1-\gamma)$ , depending on the value of  $\chi$ .

**Example 3.2.** Van Binsbergen et al. (2010) and Backus, Routledge, and Zin (2008) consider generalized recursive preferences with:

$$u(c_t, l_t) = \frac{(c_t^\nu (1-l_t)^{1-\nu})^{1-\gamma}}{1-\gamma}, \quad (49)$$

where  $\gamma > 0$ ,  $\gamma \neq 1$ , and  $\nu \in (0, 1)$ . Van Binsbergen et al. call  $\gamma + \alpha(1-\gamma)$  the coefficient of relative risk aversion, while Backus et al. use  $\gamma\nu + \alpha(1-\gamma)\nu + (1-\nu)$ , after mapping each study's notation over to the present paper's. The former measure effectively treats consumption and leisure as a single composite commodity, while the latter measure allows  $\nu$ —the importance of the labor margin—to affect the household's attitudes toward risk.

Substituting (49) into (46), the household's consumption-based coefficient of relative risk aversion is:

$$\frac{-AV_{11}}{V_1} + \alpha \frac{AV_1}{V} = \gamma\nu + \alpha(1-\gamma)\nu, \quad (50)$$

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<sup>19</sup>Set  $\eta = 0$  and  $\lambda = 0$  and substitute (47) into (46). This is the case, for example, in Epstein and Zin (1989) and Boldrin, Christiano, and Fisher (1997), which do not have labor. In models with variable labor, Rudebusch and Swanson (2009) refer to  $\gamma + \alpha(1-\gamma)$  as the *quasi* coefficient of relative risk aversion.

while the leisure-and-consumption-based coefficient of relative risk aversion is:

$$\frac{-\tilde{A}V_{11}}{V_1} + \alpha \frac{\tilde{A}V_1}{V} = \gamma + \alpha(1 - \gamma). \quad (51)$$

The latter agrees with the Van Binsbergen et al. (2010) measure of risk aversion, while the former is similar to (though not quite the same as) the Backus et al. (2008) measure. In this paper, we have provided the formal justification for both measures, (50) and (51).<sup>20</sup> Again, the leisure-and-consumption-based measure of risk aversion corresponds to treating the Cobb-Douglas aggregate of consumption and leisure as a single, composite good.

**Example 3.3.** Tallarini (2000) considers an alternative Epstein-Zin specification:

$$\tilde{V}(a_t; \theta_t) \equiv u(c_t^*, l_t^*) + \frac{\beta(1 + \theta)}{(1 - \beta)(1 - \chi)} \log E_t \exp \left[ \frac{(1 - \beta)(1 - \chi)}{1 + \theta} \tilde{V}(a_{t+1}^*; \theta_{t+1}) \right], \quad (52)$$

with utility kernel:

$$u(c_t, l_t) = \log c_t + \theta \log(\bar{l} - l_t). \quad (53)$$

We can compute the coefficient of absolute risk aversion for (52) by following along the steps in the proof of Proposition 6, which yields:

$$\frac{-\tilde{V}_{11}(a; \theta)}{\tilde{V}_1(a; \theta)} = \frac{(1 - \beta)(1 - \chi)}{1 + \theta} \tilde{V}_1(a; \theta). \quad (54)$$

The other steps leading up to Proposition 7 are all the same, so substituting in for  $\tilde{V}_1$  and  $\tilde{V}_{11}$  in (54) yields a consumption-based coefficient of relative risk aversion of:

$$\frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1 + w\lambda} - \frac{1 - \chi}{1 + \theta} cu_1 = \frac{\chi}{1 + \theta}. \quad (55)$$

The leisure-and-consumption-based coefficient of relative risk aversion is  $\chi$ , which again corresponds to treating consumption and leisure as a single, composite good.

Both coefficients of relative risk aversion differ from the value  $(\chi + \theta)/(1 + \theta)$  emphasized by Tallarini (2000). Tallarini applies the traditional, one-good measure of risk aversion for Epstein-Zin preferences,  $\frac{-cu_{11}}{u_1} - \frac{1 - \chi}{1 + \theta} cu_1$ , to the case where  $\theta > 0$  but labor is held fixed. This ignores the fact that, when  $\theta > 0$ , households will vary their labor endogenously in response to shocks. As a result, Tallarini's measure lies somewhere in between the consumption-based and the leisure-and-consumption-based relative risk aversion coefficients.

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<sup>20</sup>As  $\nu \rightarrow 0$ ,  $w/c \rightarrow \infty$ , so consumption becomes trivial to insure with variations in labor supply. This explains why the consumption-based coefficient of relative risk aversion in (50) vanishes as  $\nu \rightarrow 0$ .



## 4. Internal and External Habits

Many studies in macroeconomics and finance assume that households derive utility not from consumption itself, but from consumption relative to some reference level, or habit stock. Habits, in turn, can have substantial effects on the household's attitudes toward risk (e.g., Campbell and Cochrane, 1999, Boldrin, Christiano, and Fisher, 1997). In this section, we investigate how habits affect risk aversion in the DSGE framework.

We generalize the household's utility kernel in this section to  $u(c_t - h_t, l_t)$ , where  $h_t$  denotes the household's reference level of consumption, or habits. We focus on an additive rather than multiplicative specification for habits because the implications for risk aversion are typically more interesting in the additive case.

If the habit stock  $h_t$  is external to the household ("keeping up with the Joneses" utility), then the parameters that govern the process for  $h_t$  can be incorporated into the exogenous state vector  $\theta_t$ , and the analysis proceeds much as in the previous sections. However, if the habit stock  $h_t$  is a function of the household's own past levels of consumption, then the state variables of the household's optimization problem must be augmented to include the state variables that govern  $h_t$ . We consider each of these cases in turn.

### 4.1 External Habits

When the reference consumption level  $h_t$  in the utility kernel  $u(c_t - h_t, l_t)$  is external to the household, then the parameters that govern  $h_t$  can be incorporated into the exogenous state vector  $\theta_t$  and the analysis of the previous sections carries over essentially as before. In particular, the coefficient of absolute risk aversion continues to be given by (9) in the case of expected utility and (43) in the case of generalized recursive preferences. The household's intratemporal optimality condition (12) still implies:

$$\frac{\partial l_t^*}{\partial a_t} = -\lambda_t \frac{\partial c_t^*}{\partial a_t}, \quad (56)$$

where  $\lambda_t$  is given by (14), and the household's Euler equation (15) still implies:

$$\frac{\partial c_t^*}{\partial a_t} = E_t \frac{\partial c_{t+1}^*}{\partial a_t} = E_t \frac{\partial c_{t+k}^*}{\partial a_t}, \quad k = 1, 2, \dots \quad (57)$$

$$\frac{\partial l_t^*}{\partial a_t} = E_t \frac{\partial l_{t+1}^*}{\partial a_t} = E_t \frac{\partial l_{t+k}^*}{\partial a_t}, \quad k = 1, 2, \dots \quad (58)$$

evaluated at steady state. Together with the budget constraint (2)–(3), (56)–(58) imply:

$$\frac{\partial c_t^*}{\partial a_t} = \frac{r}{1 + w\lambda}. \quad (59)$$

The only real differences that arise relative to the case without habits is, first, that the steady-state point at which the derivatives of  $u(\cdot, \cdot)$  are evaluated is  $(c - h, l)$  rather than  $(c, l)$ , and second, that relative risk aversion confronts the household with a hypothetical gamble over  $c$  rather than  $c - h$ , which has a tendency to make the household more risk averse for a given functional form  $u(\cdot, \cdot)$ , because the stakes are effectively larger.

**Example 4.1.** Consider the case of expected utility with additively separable utility kernel:

$$u(c_t - h_t, l_t) = \frac{(c_t - h_t)^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi}, \quad (60)$$

where  $\gamma, \chi, \eta > 0$ . The traditional measure of risk aversion for this example is  $-cu_{11}/u_1 = \gamma c/(c - h)$ , which exceeds  $\gamma$  by a factor that depends on the importance of habits relative to consumption. The consumption-based coefficient of relative risk aversion is:

$$\begin{aligned} \frac{-AV_{11}}{V_1} &= \frac{-cu_{11}}{u_1} \frac{1}{1 + w \frac{wu_{11}}{u_{22}}}, \\ &= \frac{\gamma c}{(c - h)} \frac{1}{1 + \frac{\gamma c}{\chi(c-h)} \frac{wl}{c}}. \end{aligned} \quad (61)$$

When there is no labor margin in the model ( $\lambda = 0$ ), the consumption-based measure agrees with the traditional measure. When there is a labor margin, the household's consumption-based coefficient of relative risk aversion (61) is less than the traditional measure by the factor  $1 + \frac{\gamma c}{\chi(c-h)}$ , using  $wl \approx c$ . Ignoring the labor margin in (61) thus leads to an even greater bias in the model with habits ( $h > 0$ ) than in the model without habits ( $h = 0$ ). If  $\gamma = 2$ ,  $\chi = 1$ , and  $h = .8c$ , then the household's true risk aversion is smaller than the traditional measure by a factor of more than ten.

When the household has generalized recursive preferences rather than expected utility preferences, the consumption-based coefficient of relative risk aversion for (60) is:

$$\frac{\gamma c}{(c - h)} \frac{1}{1 + \frac{\gamma c}{\chi(c-h)} \frac{wl}{c}} + \frac{\alpha(1 - \gamma)c}{(c - h)} \frac{1}{1 + \frac{c}{(c-h)} \frac{\gamma-1}{1+\chi} \frac{wl}{c}}. \quad (62)$$

Again, the bias from ignoring the labor margin in (62) is even greater in the model with habits ( $h > 0$ ) than without habits ( $h = 0$ ).

## 4.2 Internal Habits

When habits are internal to the household, we must specify how the household's actions affect its future habits. In order to minimize notation and emphasize intuition, in the present section we focus on the case where habits are proportional to last period's consumption:

$$h_t = bc_{t-1}, \quad (63)$$

$b \in (0, 1)$ , and we assume the household has expected utility preferences. In the Appendix, we derive the corresponding closed-form expressions for the more complicated case where the habit stock evolves according to the longer-memory process:

$$h_t = \rho h_{t-1} + bc_{t-1}, \quad (64)$$

with  $\rho \in (-1, 1)$ .

With internal habits, the value of  $h_{t+1}$  depends on the household's choices in period  $t$ , so we write out the dependence of the household's value function on  $h_t$  explicitly:

$$V(a_t, h_t; \theta_t) = u(c_t^* - h_t, l_t^*) + \beta (E_t V(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})^{1-\alpha})^{1/(1-\alpha)}, \quad (65)$$

where  $c_t^* \equiv c^*(a_t, h_t; \theta_t)$  and  $l_t^* \equiv l^*(a_t, h_t; \theta_t)$  denote the household's optimal choices for consumption and labor in period  $t$  as functions of the household's state vector, and  $a_{t+1}^*$  and  $h_{t+1}^*$  denote the optimal stocks of assets and habits in period  $t+1$  that are implied by  $c_t^*$  and  $l_t^*$ ; that is,  $a_{t+1}^* \equiv (1+r_t)a_t + w_t l_t^* + d_t - c_t^*$  and  $h_{t+1}^* \equiv bc_t^*$ .

The household's coefficient of absolute risk aversion can be derived in the same manner as in Propositions 1 and 6:

**Proposition 8.** *With generalized recursive preferences (40) or (41), utility kernel  $u(c_t - h_t, l_t)$ , and internal habits  $h_t$  given by (63), the household's coefficient of absolute risk aversion with respect to the gamble (6) is given by:*

$$\frac{-E_t V(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})^{-\alpha} \left[ V_{11}(a_{t+1}^*, h_{t+1}^*; \theta_{t+1}) - \alpha \frac{V_1(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})^2}{V(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})} \right]}{E_t V(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})^{-\alpha} V_1(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})}. \quad (66)$$

*Evaluated at steady state, (66) simplifies to:*

$$\frac{-V_{11}(a, h; \theta)}{V_1(a, h; \theta)} + \alpha \frac{V_1(a, h; \theta)}{V(a, h; \theta)}. \quad (67)$$

PROOF: Essentially identical to the proof of Proposition 6.

Computing closed-form expressions for  $V_1$  and  $V_{11}$  in (67) is substantially more complicated for the case of internal habits, however, because of the dynamic relationship between the household's current consumption and its future habits. In order to minimize notation and simplify this derivation as much as possible, we restrict attention in the main text to the case of expected utility preferences ( $\alpha = 0$ ). In the Appendix, we derive the corresponding closed-form expressions for the more complicated case of generalized recursive preferences.

The household's first-order conditions for (65) with respect to consumption and labor (and imposing  $\alpha = 0$ ) are given by:

$$u_1(c_t^* - h_t, l_t^*) = \beta E_t V_1(a_{t+1}^*, h_{t+1}^*; \theta_{t+1}) - \beta b E_t V_2(a_{t+1}^*, h_{t+1}^*; \theta_{t+1}), \quad (68)$$

$$u_2(c_t^* - h_t, l_t^*) = -\beta w_t E_t V_1(a_{t+1}^*, h_{t+1}^*; \theta_{t+1}). \quad (69)$$

Equation (69) is essentially the same as in the case without habits. The first-order condition (68), however, includes the future effect of consumption on habits in the second term on the right-hand side.

Differentiating (65) with respect to its first two arguments and applying the envelope theorem yields:

$$V_1(a_t, h_t; \theta_t) = \beta(1 + r_t) E_t V_1(a_{t+1}^*, h_{t+1}^*; \theta_{t+1}), \quad (70)$$

$$V_2(a_t, h_t; \theta_t) = -u_1(c_t^* - h_t, l_t^*). \quad (71)$$

Equations (69) and (70) can be used to solve for  $V_1$  in terms of current-period utility:

$$V_1(a_t, h_t; \theta_t) = -\frac{(1 + r_t)}{w_t} u_2(c_t^* - h_t, l_t^*), \quad (72)$$

which states that the marginal value of wealth equals the marginal utility of working fewer hours.<sup>21</sup> This solves for  $V_1$ .

To solve for  $V_{11}$ , differentiate (72) with respect to  $a_t$  to yield:

$$V_{11}(a_t, h_t; \theta_t) = -\frac{(1 + r_t)}{w_t} \left( u_{12} \frac{\partial c_t^*}{\partial a_t} + u_{22} \frac{\partial l_t^*}{\partial a_t} \right), \quad (73)$$

---

<sup>21</sup> Using the marginal utility of labor is simpler than the marginal utility of consumption in (72) because it avoids having to keep track of future habits and the value function next period. However, in steady state it is also true that  $V_1 = u_1(1 - \beta b)/\beta$ , which we will use to express risk aversion in terms of  $u_1$  and  $u_{11}$  below.

where we drop the arguments of the  $u_{ij}$  to reduce notation. It now remains to solve for  $\partial c_t^*/\partial a_t$  and  $\partial l_t^*/\partial a_t$ , which we do in the same manner as before, except that the dynamics of internal habits require us to solve for  $\partial c_\tau^*/\partial a_t$  and  $\partial l_\tau^*/\partial a_t$  for all dates  $\tau \geq t$  at the same time. To better keep track of these dynamics, we henceforth let a time subscript  $\tau \geq t$  denote a generic future date and reserve the subscript  $t$  to denote the date of the current period—the period in which the household faces the hypothetical one-shot gamble.

We solve for  $\partial l_\tau^*/\partial a_t$  in terms of  $\partial c_\tau^*/\partial a_t$  in much the same way as without habits. The household's intratemporal optimality condition ((68) combined with (69)) implies:

$$-u_2(c_\tau^* - h_\tau^*, l_\tau^*) = w_\tau [u_1(c_\tau^* - h_\tau^*, l_\tau^*) + b\beta E_\tau V_2(a_{\tau+1}^*, h_{\tau+1}^*; \theta_{\tau+1})], \quad (74)$$

$$= w_\tau (1 - \beta b F) u_1(c_\tau^* - h_\tau^*, l_\tau^*), \quad (75)$$

where  $F$  denotes the forward operator, that is  $Fx_\tau \equiv E_\tau x_{\tau+1}$  for any expression  $x$  dated  $\tau$ .

Differentiating (75) with respect to  $a_t$  yields:

$$-u_{12} \left( \frac{\partial c_\tau^*}{\partial a_t} - \frac{\partial h_\tau^*}{\partial a_t} \right) - u_{22} \frac{\partial l_\tau^*}{\partial a_t} = w_\tau (1 - \beta b F) \left[ u_{11} \left( \frac{\partial c_\tau^*}{\partial a_t} - \frac{\partial h_\tau^*}{\partial a_t} \right) + u_{12} \frac{\partial l_\tau^*}{\partial a_t} \right], \quad (76)$$

where  $Fu_{11} \partial c_\tau^*/\partial a_t$  denotes  $E_\tau u_{11}(c_{\tau+1}^* - h_{\tau+1}^*, l_{\tau+1}^*) \partial c_{\tau+1}^*/\partial a_t$ , and  $\partial h_\tau^*/\partial a_t = 0$  for  $\tau = t$  since  $h_t$  is given. Evaluating (76) at steady state and solving for  $\partial l_\tau^*/\partial a_t$  yields:

$$\frac{\partial l_\tau^*}{\partial a_t} = -\frac{u_{12} + wu_{11} - \beta b w u_{11} F}{u_{22} + wu_{12}} \left[ 1 - \frac{\beta b w u_{12}}{u_{22} + wu_{12}} F \right]^{-1} (1 - bL) \frac{\partial c_\tau^*}{\partial a_t}. \quad (77)$$

where the  $u_{ij}$  are evaluated at steady state,  $L$  denotes the lag operator—that is,  $Lx_\tau \equiv x_{\tau-1}$  for any expression  $x$  dated  $\tau$ —and we assume  $|\beta b w u_{12}/(u_{22} + wu_{12})| < 1$  in order to ensure convergence. Note that when  $b = 0$ , (77) reduces to  $-\frac{wu_{11} + u_{12}}{u_{22} + wu_{12}} \frac{\partial c_\tau^*}{\partial a_t}$ , as in Section 2. This solves for  $\partial l_\tau^*/\partial a_t$  in terms of (current and future)  $\partial c_\tau^*/\partial a_t$ .

As before, we solve for  $\partial c_\tau^*/\partial a_t$  using the household's Euler equation and budget constraint. Differentiating the household's Euler equation:

$$\frac{1}{w_\tau} u_2(c_\tau^* - h_\tau^*, l_\tau^*) = \beta E_\tau \frac{1 + r_{\tau+1}}{w_{\tau+1}} u_2(c_{\tau+1}^* - h_{\tau+1}^*, l_{\tau+1}^*), \quad (78)$$

with respect to  $a_t$  and evaluating at steady state yields:

$$u_{12} [(1 + b) - F - bL] \frac{\partial c_\tau^*}{\partial a_t} = -u_{22} (1 - F) \frac{\partial l_\tau^*}{\partial a_t}. \quad (79)$$

Substituting (77) into (79) yields the following difference equation for  $c_\tau$ :

$$\left[ u_{12}(u_{22} + wu_{12} - \beta bwu_{12}F) [(1+b) - F - bL] - u_{22}(1-F)(u_{12} + wu_{11} - \beta bwu_{11}F)(1-bL) \right] \frac{\partial c_\tau^*}{\partial a_t} = 0. \quad (80)$$

Since  $FL = 1$ ,<sup>22</sup> equation (80) simplifies to:

$$(1 - \beta bF)(1 - F)(1 - bL) \frac{\partial c_\tau^*}{\partial a_t} = 0, \quad (81)$$

which, from (79), also implies:

$$(1 - \beta bF)(1 - F) \frac{\partial l_\tau^*}{\partial a_t} = 0. \quad (82)$$

Equations (81) and (82) hold for all  $\tau \geq t$ , hence we can invert the  $(1 - \beta bF)$  operator forward to get:

$$(1 - F)(1 - bL) \frac{\partial c_\tau^*}{\partial a_t} = 0, \quad (83)$$

$$(1 - F) \frac{\partial l_\tau^*}{\partial a_t} = 0. \quad (84)$$

In other words, whatever the initial responses  $\partial c_t^*/\partial a_t$  and  $\partial l_t^*/\partial a_t$  are, we must have:

$$E_t \frac{\partial c_{t+1}^*}{\partial a_t} = (1+b) \frac{\partial c_t^*}{\partial a_t},$$

$$E_t \frac{\partial c_{t+k}^*}{\partial a_t} = (1+b+\dots+b^k) \frac{\partial c_t^*}{\partial a_t}, \quad (85)$$

$$\text{and } E_t \frac{\partial l_{t+k}^*}{\partial a_t} = \frac{\partial l_t^*}{\partial a_t}, \quad k = 1, 2, \dots \quad (86)$$

evaluated at steady state. Because of habits, consumption responds only gradually to a surprise change in wealth, asymptoting over time to its new steady-state level, but labor moves immediately to its new steady-state level in response to surprises in wealth.

From (85), we can now solve (79) to get:

$$\frac{\partial l_t^*}{\partial a_t} = -\lambda \frac{\partial c_t^*}{\partial a_t}, \quad (87)$$

where

$$\lambda \equiv \frac{w(1-\beta b)u_{11} + u_{12}}{u_{22} + w(1-\beta b)u_{12}} = \frac{u_1 u_{12} - u_2 u_{11}}{u_1 u_{22} - u_2 u_{12}}, \quad (88)$$

---

<sup>22</sup>To be precise,  $FLx_\tau = E_{\tau-1}x_\tau$ , but since the household evaluates these expressions from the perspective of the initial period  $t$ ,  $E_t FLx_\tau = E_t x_\tau$ . Formally, take the expectation of (80) at time  $t$  and then apply  $E_t FL = E_t$  to get (81).

and where the latter equality follows because  $w = -(1 - \beta b)^{-1}u_2/u_1$  in steady state. Thus, that (87)–(88) are essentially identical to (13)–(14).<sup>23</sup> Again,  $\lambda$  must be positive if leisure and consumption are normal goods.

It now remains to solve for  $\partial c_t^*/\partial a_t$ . From the household's budget constraint and condition (86), we have:

$$E_t \sum_{\tau=t}^{\infty} (1+r)^{-(\tau-t)} \frac{\partial c_{\tau}^*}{\partial a_t} = (1+r) + w \frac{1+r}{r} \frac{\partial l_t^*}{\partial a_t}. \quad (89)$$

Substituting (85)–(87) into (89) and solving for  $\partial c_t^*/\partial a_t$  yields:

$$\frac{\partial c_t^*}{\partial a_t} = \frac{(1 - \beta b)r}{1 + (1 - \beta b)w\lambda}. \quad (90)$$

Without habits or labor, an increase in assets would cause consumption to rise by the amount of the income flow from the change in assets,  $r$ . The presence of habits attenuates this change by the amount  $\beta b$  in the numerator of (90), and the consumption response is further attenuated by the household's change in hours worked, which is accounted for by the denominator.

Substituting (72), (73), (87), (88), and (90) into (67), we have established:<sup>24</sup>

**Proposition 9.** *The household's coefficient of absolute risk aversion in Proposition 8, evaluated at steady state, is given by:*

$$\frac{-V_{11}}{V_1} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{(1 - \beta b)r}{1 + (1 - \beta b)w\lambda}. \quad (91)$$

*The household's consumption-based coefficient of relative risk aversion, evaluated at steady state, is given by:*

$$\frac{-AV_{11}}{V_1} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{(1 - \beta b)c}{1 + (1 - \beta b)w\lambda}. \quad (92)$$

*The household's leisure-and-consumption-based coefficient of relative risk aversion, evaluated at steady state, is given by  $(c + w(\bar{l} - l))/c$  times (92).*

Equations (91)–(92) have essentially the same form as the corresponding expressions in the model without habits.

<sup>23</sup> However, unlike the model without habits, (87)–(88) only hold here in steady state.

<sup>24</sup> In order to express (91) in terms of  $u_1$  and  $u_{11}$  instead of  $u_2$  and  $u_{22}$ , we use  $V_1 = (1 - \beta b)u_1/\beta$  and differentiate the first-order condition:

$$u_1(c_t^* - h_t, l_t^*) = \frac{1}{1+r_t} V_1(a_t, h_t; \theta_t) + \beta b E_t u_1(c_{t+1}^* - h_{t+1}^*, l_{t+1}^*),$$

with respect to  $a_t$  to solve for  $V_{11}$  using (85)–(88) and (90).

**Example 4.2.** Consider the utility kernel of example 4.1:

$$u(c_t - h_t, l_t) = \frac{(c_t - h_t)^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi}, \quad (93)$$

where  $\gamma, \chi, \eta > 0$ , but now with habits  $h_t = bc_{t-1}$  internal to the household rather than external. In this case, the household's consumption-based coefficient of relative risk aversion is given by:

$$\begin{aligned} \frac{-AV_{11}}{V_1} &= \frac{-cu_{11}}{u_1} \frac{1-\beta b}{1+(1-\beta b)w\lambda}, \\ &= \gamma \frac{1-\beta b}{1-b} \frac{1}{1+\frac{\gamma}{\chi} \frac{1-\beta b}{1-b} \frac{wl}{c}}, \\ &\approx \frac{\gamma}{1+\frac{\gamma}{\chi}}, \end{aligned} \quad (94)$$

where the last line uses  $\beta \approx 1$  and  $wl \approx c$ .

The most striking feature of equation (94) is that it is independent of  $b$ , the importance of habits. This is in sharp contrast to the case of external habits, where risk aversion is strongly increasing in  $b$  (cf. equation (61)).

## 5. Discussion and Conclusions

The ability to vary labor supply has dramatic effects on household risk aversion and asset prices in dynamic equilibrium models. The traditional measure of risk aversion,  $-cu_{11}/u_1$ , ignores the household's ability to partially offset shocks to income with changes in hours worked. For reasonable parameterizations, the traditional measure can easily overstate risk aversion by a factor of three or more. Indeed, households can even be risk neutral when the traditional measure of risk aversion is far from zero. Many studies in the macroeconomics, macro-finance, and international literatures thus may be overstating the actual degree of risk aversion in their models by a substantial degree.

Risk aversion matters for asset pricing. Risk premia on assets computed using the stochastic discount factor are essentially linear in the degree of risk aversion. As a result, asset prices in DSGE models can be very different and can behave very differently depending on how the household's labor margin is specified. Understanding how labor supply affects asset prices is thus important for bringing DSGE-type models closer to financial market data.



If risk aversion is measured incorrectly because the labor margin is ignored, then risk premia in the model are also more likely to be surprising or puzzling. An extreme example of this is when household utility has a zero discriminant—implying risk neutrality—even when the traditional measure of risk aversion is large.

Risk neutrality itself can be a desirable feature for some applications, such as labor market search or financial frictions. In these applications, risk neutrality allows for much simpler or even closed-form solutions to key aspects of the model. The present paper suggests new ways of modeling risk neutrality in a dynamic framework. The traditional approach—linearity of utility in consumption—has undesirable implications for interest rates and consumption growth, but the present paper shows that any utility kernel with a singular Hessian can be used instead.

A related observation is that risk aversion and the intertemporal elasticity of substitution are nonreciprocal, even for expected utility preferences. There is a wedge between the two concepts that depends on the household's labor margin.

The simple, closed-form expressions for risk aversion that this paper derives, and the methods of the paper more generally, should be useful to researchers interested in pricing any asset—stocks, bonds, or futures, in foreign or domestic currency—within the framework of dynamic equilibrium models. Since these models are a mainstay of research in academia, at central banks, and international financial institutions, the applicability of the results should be widespread.

## Appendix: Mathematical Derivations

### Proof of Proposition 1

Since  $(a_t; \theta_t)$  is an interior point of  $X$ ,  $V(a_t + \frac{\sigma \underline{\varepsilon}}{1+r_t}; \theta_t)$  and  $V(a_t + \frac{\sigma \bar{\varepsilon}}{1+r_t}; \theta_t)$  exist for sufficiently small  $\sigma$ , and  $V(a_t + \frac{\sigma \underline{\varepsilon}}{1+r_t}; \theta_t) \leq \tilde{V}(a_t; \theta_t; \sigma) \leq V(a_t + \frac{\sigma \bar{\varepsilon}}{1+r_t}; \theta_t)$ , hence  $\tilde{V}(a_t; \theta_t; \sigma)$  exists. Moreover, since  $V(\cdot; \cdot)$  is continuous and increasing in its first argument, the intermediate value theorem implies there exists a unique  $-\mu(\sigma) \in [\sigma \underline{\varepsilon}, \sigma \bar{\varepsilon}]$  with  $V(a_t - \frac{\mu(\sigma)}{1+r_t}; \theta_t) = \tilde{V}(a_t; \theta_t; \sigma)$ .

For a sufficiently small fee  $\mu$  in (7), the change in household welfare (5) is given to first order by:

$$\frac{-V_1(a_t; \theta_t)}{1+r_t} d\mu. \quad (\text{A1})$$

Using the envelope theorem, we can rewrite (A1) as:

$$-\beta E_t V_1(a_{t+1}^*; \theta_{t+1}) d\mu. \quad (\text{A2})$$

Turning now to the gamble in (6), note that the household's optimal choices for consumption and labor in period  $t$ ,  $c_t^*$  and  $l_t^*$ , will generally depend on the size of the gamble  $\sigma$ —for example, the household may undertake precautionary saving when faced with this gamble. Thus, in this section we write  $c_t^* \equiv c^*(a_t; \theta_t; \sigma)$  and  $l_t^* \equiv l^*(a_t; \theta_t; \sigma)$  to emphasize this dependence on  $\sigma$ . The household's value function, inclusive of the one-shot gamble in (6), satisfies:

$$\tilde{V}(a_t; \theta_t; \sigma) = u(c_t^*, l_t^*) + \beta E_t V(a_{t+1}^*; \theta_{t+1}), \quad (\text{A3})$$

where  $a_{t+1}^* \equiv (1+r_t)a_t + w_t l_t^* + d_t - c_t^*$ . Because (6) describes a one-shot gamble in period  $t$ , it affects assets  $a_{t+1}^*$  in period  $t+1$  but otherwise does not affect the household's optimization problem from period  $t+1$  onward; as a result, the household's value-to-go at time  $t+1$  is just  $V(a_{t+1}^*; \theta_{t+1})$ , which does not depend on  $\sigma$  except through  $a_{t+1}^*$ .

Differentiating (A3) with respect to  $\sigma$ , the first-order effect of the gamble on household welfare is:

$$\left[ u_1 \frac{\partial c^*}{\partial \sigma} + u_2 \frac{\partial l^*}{\partial \sigma} + \beta E_t V_1 \cdot \left( w_t \frac{\partial l^*}{\partial \sigma} - \frac{\partial c^*}{\partial \sigma} + \varepsilon_{t+1} \right) \right] d\sigma, \quad (\text{A4})$$

where the arguments of  $u_1$ ,  $u_2$ , and  $V_1$  are suppressed to reduce notation. Optimality of  $c_t^*$  and  $l_t^*$  implies that the terms involving  $\partial c^*/\partial \sigma$  and  $\partial l^*/\partial \sigma$  in (A4) cancel, as in the usual envelope theorem (these derivatives vanish at  $\sigma = 0$  anyway, for the reasons discussed below). Moreover,  $E_t V_1(a_{t+1}^*; \theta_{t+1}) \varepsilon_{t+1} = 0$  because  $\varepsilon_{t+1}$  is independent of  $\theta_{t+1}$  and  $a_{t+1}^*$ , evaluating the latter at  $\sigma = 0$ . Thus, the first-order cost of the gamble is zero, as in Arrow (1964) and Pratt (1965).

To second order, the effect of the gamble on household welfare is:

$$\begin{aligned} & \left[ u_{11} \left( \frac{\partial c^*}{\partial \sigma} \right)^2 + 2u_{12} \frac{\partial c^*}{\partial \sigma} \frac{\partial l^*}{\partial \sigma} + u_{22} \left( \frac{\partial l^*}{\partial \sigma} \right)^2 + u_1 \frac{\partial^2 c^*}{\partial \sigma^2} + u_2 \frac{\partial^2 l^*}{\partial \sigma^2} \right. \\ & \left. + \beta E_t V_{11} \cdot \left( w_t \frac{\partial l^*}{\partial \sigma} - \frac{\partial c^*}{\partial \sigma} + \varepsilon_{t+1} \right)^2 + \beta E_t V_1 \cdot \left( w_t \frac{\partial^2 l^*}{\partial \sigma^2} - \frac{\partial^2 c^*}{\partial \sigma^2} \right) \right] \frac{d\sigma^2}{2}. \quad (\text{A5}) \end{aligned}$$

The terms involving  $\partial^2 c^*/\partial \sigma^2$  and  $\partial^2 l^*/\partial \sigma^2$  cancel due to the optimality of  $c_t^*$  and  $l_t^*$ . The derivatives  $\partial c^*/\partial \sigma$  and  $\partial l^*/\partial \sigma$  vanish at  $\sigma = 0$  (there are two ways to see this: first, the linearized version of the model is certainty equivalent; alternatively, the gamble in (6) is isomorphic for positive and negative  $\sigma$ , hence  $c^*$  and  $l^*$  must be symmetric about  $\sigma = 0$ , implying the derivatives vanish). Thus, for infinitesimal gambles, (A5) simplifies to:

$$\beta E_t V_{11}(a_{t+1}^*; \theta_{t+1}) \varepsilon_{t+1}^2 \frac{d\sigma^2}{2}. \quad (\text{A6})$$

Finally,  $\varepsilon_{t+1}$  is independent of  $\theta_{t+1}$  and  $a_{t+1}^*$ , evaluating the latter at  $\sigma = 0$ . Since  $\varepsilon_{t+1}$  has unit variance, (A6) reduces to:

$$\beta E_t V_{11}(a_{t+1}^*; \theta_{t+1}) \frac{d\sigma^2}{2}. \quad (\text{A7})$$

Equating (A2) to (A7) allows us to solve for  $d\mu$  as a function of  $d\sigma^2$ . Thus, the limit  $\lim_{\sigma \rightarrow 0} 2\mu(\sigma)/\sigma^2$  exists and is given by:

$$\frac{-E_t V_{11}(a_{t+1}^*; \theta_{t+1})}{E_t V_1(a_{t+1}^*; \theta_{t+1})}. \quad (\text{A8})$$

To evaluate (A8) at the nonstochastic steady state, set  $a_{t+1} = a$  and  $\theta_{t+1} = \theta$  to get:

$$\frac{-V_{11}(a; \theta)}{V_1(a; \theta)}. \quad (\text{A9})$$

### Derivation of Risk Aversion, the Stochastic Discount Factor, and Risk Premia

Differentiating the household's Euler equation (15) and evaluating at steady state yields:

$$u_{11}(dc_t^* - E_t dc_{t+1}^*) + u_{12}(dl_t^* - E_t dl_{t+1}^*) = \beta E_t u_1 dr_{t+1}, \quad (\text{A10})$$

which, applying (36), becomes:

$$(u_{11} - \lambda u_{12})(dc_t^* - E_t dc_{t+1}^*) - \frac{u_1 u_{12}}{u_{22} + w u_{12}}(dw_t - E_t dw_{t+1}) = \beta E_t u_1 dr_{t+1}. \quad (\text{A11})$$

Note that (A11) implies, for each  $k = 1, 2, \dots$ ,

$$E_t dc_{t+k}^* = dc_t^* - \frac{u_1 u_{12}}{u_{11} u_{22} - u_{12}^2}(dw_t - E_t dw_{t+k}) - \frac{\beta u_1}{u_{11} - \lambda u_{12}} E_t \sum_{i=1}^k dr_{t+i}. \quad (\text{A12})$$

Combining (2)–(3), differentiating, and evaluating at steady state yields:

$$E_t \sum_{k=0}^{\infty} \frac{1}{(1+r)^k} (dc_{t+k}^* - w dl_{t+k}^* - l dw_{t+k} - dd_{t+k} - adr_{t+k}) = (1+r) da_t. \quad (\text{A13})$$

Substituting (36) and (A12) into (A13), and solving for  $dc_t^*$ , yields:

$$\begin{aligned} dc_t^* &= \frac{r}{1+r} \frac{1}{1+w\lambda} \left[ (1+r) da_t + E_t \sum_{k=0}^{\infty} \frac{1}{(1+r)^k} (l dw_{t+k} + dd_{t+k} + adr_{t+k}) \right] \\ &+ \frac{u_1 u_{12}}{u_{11} u_{22} - u_{12}^2} dw_t + \frac{r}{1+r} \frac{-u_1}{u_{11} - \lambda u_{12}} E_t \sum_{k=0}^{\infty} \frac{1}{(1+r)^k} \left[ \frac{\lambda}{1+w\lambda} dw_{t+k} - d \log R_{t,t+k} \right], \end{aligned} \quad (\text{A14})$$

where  $R_{t,t+k} \equiv \prod_{i=1}^k (1+r_{t+i})$ . Combining (35), (36), and (A14) gives:

$$\begin{aligned} dm_{t+1} &= \beta r \frac{u_{11} - \lambda u_{12}}{u_1} \frac{1}{1+w\lambda} \left[ da_{t+1} + E_{t+1} \sum_{k=1}^{\infty} \frac{1}{(1+r)^k} (l dw_{t+k} + dd_{t+k} + adr_{t+k}) \right] \\ &- \beta r E_{t+1} \sum_{k=1}^{\infty} \frac{1}{(1+r)^k} \left[ \frac{\lambda}{1+w\lambda} dw_{t+k} - d \log R_{t+1,t+k} \right], \end{aligned} \quad (\text{A15})$$

as in the main text. Equation (A15) also holds for the case of external habits (cf. Section 4.1).

For generalized recursive preferences, equations (A10)–(A14) still hold, but  $dm_{t+1}$  has extra terms related to  $dV_{t+1}$ . In this case, we get the more general expression:

$$dm_{t+1} = \beta r \left( \frac{u_{11} - \lambda u_{12}}{u_1} \frac{1}{1 + w\lambda} - \frac{\alpha u_1}{u} \right) \left[ da_{t+1} + E_{t+1} \sum_{k=1}^{\infty} \frac{1}{(1+r)^k} (l dw_{t+k} + dd_{t+k} + adr_{t+k}) \right] - \beta r E_{t+1} \sum_{k=1}^{\infty} \frac{1}{(1+r)^k} \left[ \frac{\lambda}{1 + w\lambda} dw_{t+k} - d \log R_{t+1,t+k} \right]. \quad (\text{A16})$$

### Proof of Proposition 6

For generalized recursive preferences, the hypothetical one-shot gamble and one-time fee faced by the household are the same as for the case of expected utility. However, the household's optimality conditions for  $c_t^*$  and  $l_t^*$  (and, implicitly,  $a_{t+1}^*$ ) are slightly more complicated:

$$u_1(c_t^*, l_t^*) = \beta (E_t V(a_{t+1}^*; \theta_{t+1})^{1-\alpha})^{\alpha/(1-\alpha)} E_t V(a_{t+1}^*; \theta_{t+1})^{-\alpha} V_1(a_{t+1}^*; \theta_{t+1}), \quad (\text{A17})$$

$$u_2(c_t^*, l_t^*) = -\beta w_t (E_t V(a_{t+1}^*; \theta_{t+1})^{1-\alpha})^{\alpha/(1-\alpha)} E_t V(a_{t+1}^*; \theta_{t+1})^{-\alpha} V_1(a_{t+1}^*; \theta_{t+1}). \quad (\text{A18})$$

Note that (A17) and (A18) are related by the usual  $u_2(c_t^*, l_t^*) = -w_t u_1(c_t^*, l_t^*)$ , and when  $\alpha = 0$ , (A17) and (A18) reduce to the standard optimality conditions for expected utility.

For an infinitesimal fee  $d\mu$  in (7), the change in welfare for the household with generalized recursive preferences is:

$$-V_1(a_t; \theta_t) \frac{d\mu}{1+r_t} = -\beta (E_t V(a_{t+1}^*; \theta_{t+1})^{1-\alpha})^{\alpha/(1-\alpha)} E_t V(a_{t+1}^*; \theta_{t+1})^{-\alpha} V_1(a_{t+1}^*; \theta_{t+1}) d\mu, \quad (\text{A19})$$

where the right-hand side of (A19) follows from the envelope theorem.

Turning now to the gamble in (6), the first-order effect of the gamble on household welfare is:

$$\left[ u_1 \frac{\partial c^*}{\partial \sigma} + u_2 \frac{\partial l^*}{\partial \sigma} + \beta (E_t V^{1-\alpha})^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_1 \cdot \left( w_t \frac{\partial l^*}{\partial \sigma} - \frac{\partial c^*}{\partial \sigma} + \varepsilon_{t+1} \right) \right] d\sigma, \quad (\text{A20})$$

where we have dropped the arguments of  $u_1$ ,  $u_2$ ,  $V$ , and  $V_1$  to simplify notation. As before, optimality of  $c_t^*$  and  $l_t^*$  implies that the terms involving  $\partial c^*/\partial \sigma$  and  $\partial l^*/\partial \sigma$  cancel, and  $E_t V^{-\alpha} V_1 \varepsilon_{t+1} = 0$  because  $\varepsilon_{t+1}$  is independent of  $\theta_{t+1}$  and  $a_{t+1}^*$ , evaluating the latter at  $\sigma = 0$ . Thus, the first-order cost of the gamble is zero.

To second order, the effect of the gamble on household welfare is:

$$\left\{ u_{11} \left( \frac{\partial c^*}{\partial \sigma} \right)^2 + 2u_{12} \frac{\partial c^*}{\partial \sigma} \frac{\partial l^*}{\partial \sigma} + u_{22} \left( \frac{\partial l^*}{\partial \sigma} \right)^2 + u_1 \frac{\partial^2 c^*}{\partial \sigma^2} + u_2 \frac{\partial^2 l^*}{\partial \sigma^2} + \alpha \beta (E_t V^{1-\alpha})^{(2\alpha-1)/(1-\alpha)} \left[ E_t V^{-\alpha} V_1 \cdot \left( w_t \frac{\partial l^*}{\partial \sigma} - \frac{\partial c^*}{\partial \sigma} + \varepsilon_{t+1} \right) \right]^2 - \alpha \beta (E_t V^{1-\alpha})^{\alpha/(1-\alpha)} E_t V^{-\alpha-1} \left[ V_1 \cdot \left( w_t \frac{\partial l^*}{\partial \sigma} - \frac{\partial c^*}{\partial \sigma} + \varepsilon_{t+1} \right) \right]^2 + \beta (E_t V^{1-\alpha})^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_{11} \cdot \left( w_t \frac{\partial l^*}{\partial \sigma} - \frac{\partial c^*}{\partial \sigma} + \varepsilon_{t+1} \right)^2 + \beta (E_t V^{1-\alpha})^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_1 \cdot \left( w_t \frac{\partial^2 l^*}{\partial \sigma^2} - \frac{\partial^2 c^*}{\partial \sigma^2} \right) \right\} \frac{d\sigma^2}{2}. \quad (\text{A21})$$

The derivatives  $\partial c^*/\partial\sigma$  and  $\partial l^*/\partial\sigma$  vanish at  $\sigma = 0$ , the terms involving  $\partial^2 c^*/\partial\sigma^2$  and  $\partial^2 l^*/\partial\sigma^2$  cancel due to the optimality of  $c_t^*$  and  $l_t^*$ , and  $\varepsilon_{t+1}$  is independent of  $\theta_{t+1}$  and  $a_{t+1}^*$  (evaluating the latter at  $\sigma = 0$ ). Thus, (A21) simplifies to:

$$\beta(E_t V^{1-\alpha})^{\alpha/(1-\alpha)} (E_t V^{-\alpha} V_{11} - \alpha E_t V^{-\alpha-1} V_1^2) \frac{d\sigma^2}{2}. \quad (\text{A22})$$

Equating (A19) to (A22), the Arrow-Pratt coefficient of absolute risk aversion is:

$$\frac{-E_t V^{-\alpha} V_{11} + \alpha E_t V^{-\alpha-1} V_1^2}{E_t V^{-\alpha} V_1}. \quad (\text{A23})$$

Since (A23) is already evaluated at  $\sigma = 0$ , to evaluate it at the nonstochastic steady state, set  $a_{t+1} = a$ ,  $\theta_{t+1} = \theta$  to get:

$$\frac{-V_{11}(a; \theta)}{V_1(a; \theta)} + \alpha \frac{V_1(a; \theta)}{V(a; \theta)}. \quad (\text{A24})$$

### Derivation of Risk Aversion with Long-Memory Internal Habits and EZ Preferences

We consider here the case of generalized recursive preferences:

$$V(a_t, h_t; \theta_t) = u(c_t^* - h_t, l_t^*) + \beta (E_t V(a_{t+1}^*, h_{t+1}^*; \theta_{t+1})^{1-\alpha})^{1/(1-\alpha)}, \quad (\text{A25})$$

and a longer-memory specification for habits:

$$h_t = \rho h_{t-1} + b c_{t-1}, \quad (\text{A26})$$

with  $|\rho| < 1$ , and we assume  $\rho + b < 1$  in order to ensure  $h < c$ .

We wish to compute  $V_1$  and  $V_{11}$ . The household's first-order conditions for (A25) with respect to consumption and labor are given by:

$$u_1 = \beta (E_t V^{1-\alpha})^{\alpha/(1-\alpha)} E_t V^{-\alpha} [V_1 - b V_2], \quad (\text{A27})$$

$$u_2 = -\beta w_t (E_t V^{1-\alpha})^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_1, \quad (\text{A28})$$

where we drop the arguments of  $u$  and  $V$  to reduce notation. Equations (A27) and (A28) are the same as in the main text except that the discounting of future periods involves the value function  $V$  when  $\alpha \neq 0$ .

Differentiating (A25) with respect to its first two arguments and applying the envelope theorem yields:

$$V_1 = \beta(1 + r_t) (E_t V^{1-\alpha})^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_1, \quad (\text{A29})$$

$$V_2 = -u_1 + \rho \beta (E_t V^{1-\alpha})^{\alpha/(1-\alpha)} E_t V^{-\alpha} V_2. \quad (\text{A30})$$

As in the main text, (A28) and (A29) can be used to solve for  $V_1$  in terms of current-period utility:

$$V_1(a_t, h_t; \theta_t) = -\frac{(1 + r_t)}{w_t} u_2(c_t^* - h_t, l_t^*). \quad (\text{A31})$$

To solve for  $V_{11}$ , differentiate (A31) with respect to  $a_t$  to yield:

$$V_{11}(a_t, h_t; \theta_t) = -\frac{(1 + r_t)}{w_t} \left( u_{12} \frac{\partial c_t^*}{\partial a_t} + u_{22} \frac{\partial l_t^*}{\partial a_t} \right), \quad (\text{A32})$$

It remains to solve for  $\partial c_t^*/\partial a_t$  and  $\partial l_t^*/\partial a_t$ . As in the main text, we solve for  $\partial c_\tau^*/\partial a_t$  and  $\partial l_\tau^*/\partial a_t$  for all dates  $\tau \geq t$  at the same time. We henceforth let a time subscript  $\tau \geq t$  denote a generic future date and reserve the subscript  $t$  to denote the date of the current period—the period in which the household faces the hypothetical one-shot gamble.

We solve for  $\partial l_\tau^*/\partial a_t$  in terms of  $\partial c_\tau^*/\partial a_t$  in the same manner as in the main text, except that the expressions are more complicated due to the persistence of habits and the household's more complicated discounting of future periods. Note first that (A30) can be used to solve for  $V_2$  in terms of current and future marginal utility:

$$V_2 = -(1 - \rho\beta F)^{-1} u_1, \quad (\text{A33})$$

where now  $F$  denotes the “generalized recursive” forward operator; that is,

$$F x_\tau \equiv (E_\tau V^{1-\alpha})^{\alpha/(1-\alpha)} E_\tau V^{-\alpha} x_{\tau+1}. \quad (\text{A34})$$

The household's intratemporal optimality condition ((A29) combined with (A30)) implies:

$$-u_2(c_\tau^* - h_\tau^*, l_\tau^*) = w_\tau [u_1(c_\tau^* - h_\tau^*, l_\tau^*) + b\beta E_\tau V_2(a_{\tau+1}^*, h_{\tau+1}^*; \theta_{\tau+1})]. \quad (\text{A35})$$

$$= w_\tau (1 - \beta b F (1 - \beta \rho F)^{-1}) u_1(c_\tau^* - h_\tau^*, l_\tau^*), \quad (\text{A36})$$

Differentiating (A36) with respect to  $a_t$  and evaluating at steady state yields:

$$-u_{12} \left( \frac{\partial c_\tau^*}{\partial a_t} - \frac{\partial h_\tau^*}{\partial a_t} \right) - u_{22} \frac{\partial l_\tau^*}{\partial a_t} = w (1 - \beta b F (1 - \beta \rho F)^{-1}) \left[ u_{11} \left( \frac{\partial c_\tau^*}{\partial a_t} - \frac{\partial h_\tau^*}{\partial a_t} \right) + u_{12} \frac{\partial l_\tau^*}{\partial a_t} \right], \quad (\text{A37})$$

where we have used the fact that:

$$\frac{\partial}{\partial a_t} F x_\tau = F \frac{\partial x_\tau}{\partial a_t}, \quad (\text{A38})$$

when the derivative is evaluated at steady state. Solving (A37) for  $\partial l_\tau^*/\partial a_t$  yields:

$$\begin{aligned} \frac{\partial l_\tau^*}{\partial a_t} = & - \frac{u_{12} + w u_{11} - \beta(\rho u_{12} + (\rho + b) w u_{11}) F}{u_{22} + w u_{12}} \times \\ & \left[ 1 - \frac{\beta(\rho u_{22} + (\rho + b) w u_{12})}{u_{22} + w u_{12}} F \right]^{-1} (1 - bL(1 - \rho L)^{-1}) \frac{\partial c_\tau^*}{\partial a_t}. \end{aligned} \quad (\text{A39})$$

where we've used  $h_\tau = bL(1 - \rho L)^{-1} c_\tau$  and we assume  $|\beta(\rho u_{22} + (\rho + b) w u_{12}) / (u_{22} + w u_{12})| < 1$  to ensure convergence. This solves for  $\partial l_t^*/\partial a_t$  in terms of (current and future)  $\partial c_\tau^*/\partial a_t$ .

We now turn to solving for  $\partial c_\tau^*/\partial a_t$ . The household's intertemporal optimality (Euler) condition is given by:

$$\frac{1}{w_\tau} u_2(c_\tau^* - h_\tau^*, l_\tau^*) = \beta F \frac{1 + r_\tau}{w_\tau} u_2(c_\tau^* - h_\tau^*, l_\tau^*). \quad (\text{A40})$$

Differentiating (A40) with respect to  $a_t$  and evaluating at steady state yields:

$$u_{12}(1 - F) [1 - bL(1 - \rho L)^{-1}] \frac{\partial c_\tau^*}{\partial a_t} = -u_{22}(1 - F) \frac{\partial l_\tau^*}{\partial a_t}. \quad (\text{A41})$$

Using (A39) and noting  $FL = 1$  at steady state, (A41) simplifies to:

$$[1 - \beta(\rho + b)F] (1 - F) [1 - bL(1 - \rho L)^{-1}] \frac{\partial c_\tau^*}{\partial a_t} = 0, \quad (\text{A42})$$

which, from (A41), also implies:

$$[1 - \beta(\rho + b)F](1 - F) \frac{\partial l_\tau^*}{\partial a_t} = 0. \quad (\text{A43})$$

Equations (A42) and (A43) hold for all  $\tau \geq t$ , hence we can invert the  $[1 - \beta(\rho + b)F]$  operator forward to get:

$$(1 - F) [1 - bL(1 - \rho L)^{-1}] \frac{\partial c_\tau^*}{\partial a_t} = 0, \quad (\text{A44})$$

$$(1 - F) \frac{\partial l_\tau^*}{\partial a_t} = 0. \quad (\text{A45})$$

Finally, we can apply  $(1 - \rho L)$  to both sides of (A44) to get:

$$(1 - F) [1 - (\rho + b)L] \frac{\partial c_\tau^*}{\partial a_t} = 0, \quad (\text{A46})$$

which then holds for all  $\tau \geq t + 1$ . Thus, whatever the initial responses  $\partial c_t^*/\partial a_t$  and  $\partial l_t^*/\partial a_t$ , we must have:

$$E_t \frac{\partial c_{t+1}^*}{\partial a_t} = (1 + b) \frac{\partial c_t^*}{\partial a_t},$$

$$E_t \frac{\partial c_{t+k}^*}{\partial a_t} = (1 + b(\rho + b)^{k-1}) \frac{\partial c_t^*}{\partial a_t}, \quad (\text{A47})$$

$$\text{and } E_t \frac{\partial l_{t+k}^*}{\partial a_t} = \frac{\partial l_t^*}{\partial a_t}, \quad k = 1, 2, \dots \quad (\text{A48})$$

Consumption responds gradually to a surprise change in wealth, while labor moves immediately to its new steady-state level.

From (A47), we can now solve (A41) to get:

$$\frac{\partial l_t^*}{\partial a_t} = -\lambda \frac{\partial c_t^*}{\partial a_t}. \quad (\text{A49})$$

where

$$\lambda \equiv \frac{w(1 - \beta(\rho + b))u_{11} + (1 - \beta\rho)u_{12}}{(1 - \beta\rho)u_{22} + w(1 - \beta(\rho + b))u_{12}} = \frac{u_1 u_{12} - u_2 u_{11}}{u_1 u_{22} - u_2 u_{12}}, \quad (\text{A50})$$

where the latter equality follows because  $w = -\frac{u_2}{u_1} \frac{1 - \beta\rho}{1 - \beta(\rho + b)}$  in steady state.

It remains to solve for  $\partial c_t^*/\partial a_t$ . The household's intertemporal budget constraint implies:

$$E_t \sum_{\tau=t}^{\infty} (1 + r)^{-(\tau-t)} \frac{\partial c_\tau^*}{\partial a_t} = (1 + r) + w \frac{1 + r}{r} \frac{\partial l_t^*}{\partial a_t}. \quad (\text{A51})$$

Substituting (A47) and (A49) into (A51) and solving for  $\partial c_t^*/\partial a_t$  yields:

$$\frac{\partial c_t^*}{\partial a_t} = \frac{(1 - \frac{\beta b}{1 - \beta\rho})r}{1 + (1 - \frac{\beta b}{1 - \beta\rho})w\lambda}. \quad (\text{A52})$$

Without habits or labor, an increase in assets would cause consumption to rise by the amount of the income flow from the change in assets—the first term on the right-hand side of (A52). The presence of habits attenuates this change by the amount  $\beta b/(1 - \beta\rho)$  in the numerator of the second term, and the consumption response is further attenuated by the household's change in hours worked, which is accounted for by the denominator of the second term in (A52).

Together, (A49) and (A52) allow us to compute the household's coefficient of absolute risk aversion (63) in Proposition 7:<sup>25</sup>

$$\frac{-V_{11}}{V_1} + \alpha \frac{V_1}{V} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{(1 - \frac{\beta b}{1 - \beta \rho}) r}{1 + (1 - \frac{\beta b}{1 - \beta \rho}) w \lambda} + \alpha \frac{r u_1}{u} \left(1 - \frac{\beta b}{1 - \beta \rho}\right). \quad (\text{A53})$$

The consumption-based coefficient of relative risk aversion is given by:

$$\frac{-AV_{11}}{V_1} + \alpha \frac{AV_1}{V} = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{(1 - \frac{\beta b}{1 - \beta \rho}) c}{1 + (1 - \frac{\beta b}{1 - \beta \rho}) w \lambda} + \alpha \frac{c u_1}{u} \left(1 - \frac{\beta b}{1 - \beta \rho}\right). \quad (\text{A54})$$

Equations (A53) and (A54) have obvious similarities to the corresponding expressions without habits and with expected utility preferences.

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<sup>25</sup>In order to express (A53) in terms of  $u_1$  and  $u_{11}$  instead of  $u_2$  and  $u_{22}$ , we use  $V_1 = (1 - \beta(\rho + b))u_1 / (\beta(1 - \beta\rho))$  and differentiate the first-order condition:

$$V_1(a_t, h_t; \theta_t) = (1 + r_t) (1 - \beta b F (1 - \beta \rho F)^{-1}) u_1(c_\tau^* - h_\tau, l_\tau^*),$$

with respect to  $a_t$  to solve for  $V_{11}$ .



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