A Shadow Rate New Keynesian Model*

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Abstract

We develop a New Keynesian model with a shadow rate, which is the federal funds rate during normal times. We show the mapping between a negative shadow rate and large-scale asset purchases or lending facilities when the policy rate is constrained at its zero lower bound. We find two anomalies at the zero lower bound in the standard New Keynesian model disappear once we introduce unconventional monetary policy through the shadow rate: a negative supply shock has a negative impact on the economy, and the government multiplier is under 1.

Keywords: shadow rate, New Keynesian model, unconventional monetary policy, zero lower bound, QE, lending facilities

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1 Introduction

After a decade of Japan's experience of zero interest rates, the Great Recession brought the US economy the same problem, followed by the UK and Euro area. The zero lower bound (ZLB) poses issues for the economy and consequently economic research. The ZLB invalidates the traditional monetary policy tool because the policy rate is bounded by the lower bound. Subsequently, central banks around the world have introduced unconventional policy tools such as large-scale asset purchases (or QE), lending facilities, and forward guidance. This paper proposes a novel New Keynesian model to accommodate the ZLB and unconventional monetary policy with a shadow rate.

A related issue economic researchers face is that the standard New Keyesian model is associated with some distinctive modeling implications at the ZLB, some of which are counterfactual or puzzling. We focus on two such implications that are often discussed in the literature. First, a negative supply shock stimulates the economy. Second, the government multiplier is much larger than usual.

We show both of these abnormal implications are due to the lack of policy interventions. The basic transmission mechanism during normal times works as follows: in response to higher inflation, the interest rate increases more than one-for-one, implying a higher real interest rate, which in turn suppresses the demand. The direction reverses when the ZLB prevails: a constant policy rate implies a lower real interest rate instead in the standard New Keynesian model, which then stimulates private consumption, investment, and hence the overall economy.

In contrast to the implications of the standard New Keynesian model, empirical evidence from Wieland (2015) and Garín, Lester, and Sims (2016) demonstrate a similar response of output to a supply shock during normal times and at the zero lower bound. In addition, the debate about the fiscal multiplier is ongoing. For example, Braun, Körber, and Wake (2012) and Mertens and Ravn (2014) find little difference in this multiplier at the ZLB and during normal times.

More broadly, Bullard (2012), Powell (2013), Blanchard (2016), and Wu and Xia (2016) argue a similarity exists between the conventional and unconventional monetary policies in various contexts. Furthermore, Wu and Xia (2016) show that both policies can be summarized coherently with a shadow federal funds rate. The shadow rate is the federal funds rate when the ZLB is not binding; otherwise, it is negative to account for unconventional policy tools. Wu and Xia (2016) have demonstrated the non-linear relationship between the federal funds rate and macroeconomy induced by the ZLB can be represented by a linear relationship using the shadow rate in a structural vector autoregression (VAR). Altig (2014) of the Atlanta Fed, Hakkio and Kahn (2014) of the Kansas City Fed, and others have subsequently used Wu and Xia (2016)'s shadow rate for policy analyses.

One main contribution of this paper is to introduce the shadow rate into a New Keynesian model. We first empirically establish that during the ZLB period, the shadow rate co-moves with private borrowing/lending rates and financial conditions almost perfectly, with correlations at about 0.8. Moreover, it moves almost one-for-one with private rates, captured by high-yield corporate yields or spreads. Therefore, private rates factor in the additional stimulus from the Fed's unconventional policy tools, which are summarized by the shadow rate, although the fed funds rate is a constant. If we let the private rates that are relevant for the households and firms instead of the fed funds rate enter the New Keynesian model, we can model their dynamics with the shadow rate. This constitutes the premise of our new model, which is a standard New Keynesian model during normal times.

At the ZLB, the two anomalies associated with the standard New Keyesian model do not exist in our new model with the shadow rate. The transmission mechanism works similarly to how it does in normal times: in response to higher inflation, the private interest rate increases more than one-for-one, despite a constant policy rate, implying a higher real interest rate, which in turn suppresses private demand. Therefore, the model implication that a negative supply shock causes a lower aggregate demand is data-consistent. Moreover, the fiscal multiplier is the same as its normal size.

We map two major unconventional policy tools into this shadow rate New Keynesian model. First, the negative shadow rate can be implemented through QE programs. When the government purchases bonds, a lower bond supply implies a higher price and hence lower yield. Therefore, QE is able to further lower the bond yield without changing the policy rate, which works through reducing the risk premium. This risk-premium channel of QE is consistent with the empirical findings of Hamilton and Wu (2012) and Gagnon, Raskin, Remache, and Sack (2011).

We then map lending facilities, which inject liquidity into the economy, into the shadow rate framework. Examples of this policy include the Federal Reserve's Term Asset-Backed Securities Loan Facility, the Eurosystem's valuation haircuts as well as the UK's credit controls. We model lending facilities by allowing the government to extend extra credit directly to the private sector; that is, the government can vary the loan-to-value ratio the borrowers face as a policy tool. The lending facilities are coupled with a tax policy on interest rate payments, which, according to Waller (2016) of the St. Louis Fed, is the nature of the recent negative interest rate policy in Europe and Japan.

The rest of the paper proceeds as follows. Section 2 demonstrates two anomalies in the standard New Keynesian model at the ZLB. Section 3 proposes a shadow rate New Keynesian model, and subsequently, Section 4 maps QE and lending facilities into this model. Section 5 analyzes the two anomalies quantitatively at the ZLB, and Section 6 concludes.

2 The standard New Keynesian model

This section lays out a standard New Keynesian model and two anomalies associated with it at the ZLB. They form the basis for our discussions. Later, we will show that, in our model presented in Section 3 - 4, these anomalies disappear once we allow unconventional monetary policy to play a role. We then make the quantitative comparisons between our model and the standard New Keynesian model in section 5.

Consider a linearized system of the New Keynesian model; see, for example, Galí (2008) and Walsh (2010). The system consists of the dynamic-IS curve, the New Keynesian Phillips curve, and a monetary policy rule. We explicitly accommodate the zero lower bound.

$$
\begin{aligned}
c_t &= -\frac{1}{\sigma}(r_t - \mathbb{E}_t\pi_{t+1} - r) + \mathbb{E}_t c_{t+1} && (2.1)\\
\pi_t &= \beta\mathbb{E}_t\pi_{t+1} + \lambda(mc_t - mc) && (2.2)\\
s_t &= (1+\phi_\pi)\pi_t + \phi_y(y_t - y) + r && (2.3)\\
r_t &= \max\{s_t, 0\}, && (2.4)
\end{aligned}
$$

where $\mathbb{E}$ is the expectation operator, lowercase letters are logs, and letters without t subscripts are either coefficients or steady-state values. c_t is consumption, r_t is the nominal policy interest rate, π_t is inflation, mc_t is the marginal cost, s_t is the shadow policy rate, and y_t is output. r, mc, and y are the steady-state policy rate, marginal cost, and output, respectively. All the coefficients are positive. Equation (2.1) is the dynamic-IS (DIS) curve, and it describes that demand is a decreasing function of the real interest rate $rr_t = r_t - \mathbb{E}_t\pi_{t+1}$, where σ is the reciprocal of the intertemporal elasticity of substitution. Equation (2.2) is the New Keynesian Phillips Curve, characterizing the relationship between inflation and the marginal cost. β is the discount factor, and λ depends on the degree of nominal rigidity and other preference parameters. Both equations are forward looking.

Equations (2.3) and (2.4) describe the interest rate rule. The shadow policy rate s_t always follows a Taylor rule, whereas the actual policy rate r_t is subject to a zero lower bound. The restriction $\phi_\pi > 0$ guarantees a unique, non-explosive equilibrium. When $s_t \geq 0$, the central bank changes the policy rate $r_t = s_t$ according to the Taylor rule to smooth out economic fluctuations as its conventional policy instrument. When $s_t < 0$, the zero lower bound binds and $r_t = 0$. Although the shadow rate still has its own dynamics reflecting the current economic situation, it does not pass through the economy in this system.

To close the model, we specify marginal cost mc_t and the market-clearing condition for the goods market as follows:

$$mc_t = \sigma c_t + \eta y_t \tag{2.5}$$

$$y_t = c_t, \tag{2.6}$$

where η is the Frisch elasticity of labor. (2.1)-(2.6) constitutes a simple New Keynesian model. Details of the model and derivations are in Appendix A.

2.1 Anomalies at the ZLB

We will describe two anomalies associated with the ZLB in a standard New Keyesian model, and use the lack of monetary policy intervention to explain them.

Stimulative negative supply shock We introduce a shock to technology as a supply shock

$$a_t = \rho_a a_{t-1} + e_{a,t}, \tag{2.7}$$

where a_t is technology, ρ_a describes how persistent it is, and $e_{a,t}$ is a zero-mean white-noise process. Then, the marginal cost becomes

$$mc_t = \sigma c_t + \eta y_t - (1+\eta) a_t, \tag{2.8}$$

which augments (2.5) with technology.

A negative TFP shock on productivity causes a higher marginal cost and inflation. At the zero lower bound, the standard New Keynesian model does not allow the central bank to respond to this change in economic condition. Consequently, a constant nominal interest rate together with a higher inflation expectation implies a lower real rate. The drop in the real rate increases consumption.

This model implication is counterfactual. Wieland (2015) tests this implication using the Great East Japan earthquake and global oil supply shocks, and finds the data imply a negative effect of such a shock at the ZLB. Similarly, Garín, Lester, and Sims (2016) use utilization-adjusted TFP series and find that positive TFP shocks are as expansionary at the ZLB as they are in normal times.

The responsive monetary policy during normal times allows the negative supply shock to have the data-consistent negative effect on the economy. The Taylor rule increases the nominal interest rate in response to a higher inflation more than one for one. This increase implies a higher real rate. Consequently, consumption and output decrease.

The empirical evidence suggests a similarity between normal times and the zero lower bound. The model's implication for the zero lower bound would be consistent with the data if the model were to incorporate unconventional monetary policy operating similarly to the conventional Taylor rule; that is, the monetary policy response to lower inflation or output stimulates the real economy.

Larger government-spending multiplier We introduce government spending into the model, which is specified as follows:

$$g_t = (1-\rho_g)g + \rho_g g_{t-1} + e_{g,t}, \tag{2.9}$$

where $\rho_g \in (0,1)$ and $\epsilon_{g,t}$ is a zero-mean white-noise process. We then replace the goods market-clearing condition in (2.6) with the following:

$$y_t = c_y c_t + g_y g_t, \tag{2.10}$$

where c_y is the steady-state consumption-output ratio, and g_y is the steady-state government-spending-output ratio.

The government-spending multiplier measures the total change in output in response to a one-unit increase in government spending. In response to a government-spending shock, if consumption does not move, the multiplier is 1. If consumption increases, it is bigger than 1, and vice versa. The size of the government-spending multiplier is a heavily debated topic, especially when the ZLB prevails.

When the nominal rate follows a Taylor rule, the multiplier is often considered to be smaller than 1. A positive government-spending shock increases output, which in turn increases inflation through a higher marginal cost. The Taylor rule raises the interest rate in response to higher inflation and higher output. This increase in the nominal rate implies a higher real rate, which crowds out private consumption. Hence, the overall increase in output is less than the shock itself.

The government-spending multiplier becomes unusually large at the zero lower bound when the nominal rate does not respond to this shock. A positive government-spending shock pushes up aggregate demand, which leads to a rising pressure on inflation. Without a corresponding increase in the nominal rate, the real rate decreases instead. The lower real rate further boosts the economy: consumption increases, and hence output increases more than the shock itself. The same mechanism has been found in Christiano, Eichenbaum, and Rebelo (2011) and Eggertsson (2010).

On the other hand, Braun, Körber, and Wake (2012) do not find much difference between the size of the multiplier at the ZLB and during normal times when solving the model analytically at the ZLB. Mertens and Ravn (2014) find that when a liquidity trap is caused by a self-fulfilling state of low consumer confidence, a government-spending stimulus has deflationary effects and becomes relatively ineffective at the ZLB. Similarly, Wieland (2015) shows that once he introduces a borrowing constraint to have a data-consistent implication for the negative supply shock, the fiscal multiplier is back to below 1. Swanson and Williams (2014) suggest that the fiscal policy had similar effects during the 2008-2010 period as before on the medium- and longer-term interest rates.

Unconventional monetary policy The above-mentioned anomalies at the ZLB rise because the benchmark New Keynesian model does not allow unconventional monetary policy to play a role. Woodford (2011) makes a related observation in terms of the fiscal multiplier. Unconventional monetary policy tools, such as large-scale asset purchases or forward guidance, are invented to continue stimulating the economy when the traditional policy tool comes to a halt. For example, QE programs purchase assets, which reduces the amount of assets that are available to the private agents, and hence increases the prices of these assets. At the same time, prices of other related assets experience a similar movement due to the no-arbitrage argument. Higher prices imply lower interest rate payments for these bonds, meaning households and firms face lower borrowing or lending rates, which subsequently boost the aggregate demand. These channels work similarly to how they would if the Fed were able to lower the short-term interest rate further. We introduce this mechanism in a tractable framework in Section 3.

Other issues at the ZLB Besides economic anomalies, the ZLB also introduces nonlinearity in the policy rule as in equation (2.4). This nonlinearity invalidates traditional methods' ability to solve the model regardless of whether a researcher uses the log-linearization or higher-order approximations; see, for example, Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015). In addition, a problem with multiple equilibria raised by the ZLB complicates the analysis further; see, for example, Aruoba, Cuba-Borda, and Schorfheide (2016) and Braun, Körber, and Wake (2012). For econometric implementation, focusing only on the ZLB subsample suffers from having too few observations.

3 A shadow rate New Keynesian model (SRNKM)

We propose a novel shadow rate New Keynesian model in this section, which overcomes the anomalies associated with the standard New Keynesian model discussed in Section 2. The key element is the shadow rate. We start by reviewing the shadow rate of Wu and Xia (2016) in Section 3.1, and then present some new evidence about this shadow rate in Section 3.2. In Section 3.3, we incorporate the shadow rate into a linear New Keynesian model, which is then micro-founded in Section 4 with unconventional policy tools.

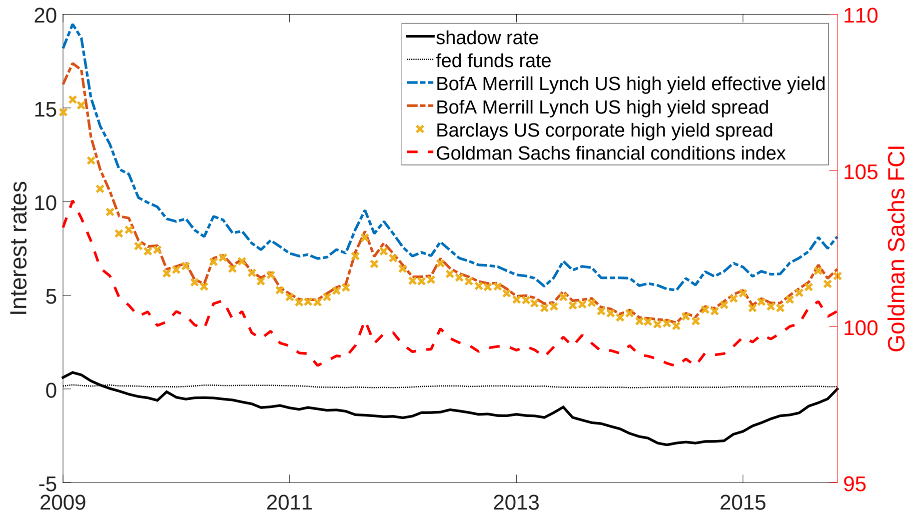
3.1 Background on the Wu and Xia (2016) shadow rate

Bullard (2012), Powell (2013), Blanchard (2016), and Wu and Xia (2016) argue a similarity exists between the conventional and unconventional monetary policy. The recent study by Wu and Xia (2016) proposes a tractable and empirically appealing framework to capture this similarity: simply replace the policy rate r_t with the shadow rate s_t. This action extends the model with only the conventional monetary policy to also accommodate for unconventional monetary policy at the ZLB. During normal times, the shadow rate is equal to the policy rate. Interestingly, unlike the policy rate, the shadow rate is not constrained by the ZLB: it can be negative, reflecting additional monetary policy stimulus, and it does not impose a structural break as the policy rate does in (2.4). Therefore, the shadow rate is a potential remedy for many problems raised by the ZLB. In particular, Wu and Xia (2016) show that in a VAR environment, the shadow rate has dynamic correlations with macro quantities, such as inflation and output, at the ZLB that are similar to those that the policy rate had previously.

The shadow rate can also be related to unconventional policy tools directly. For example, the QE operations are associated with sizable drops in the shadow rate. The shadow rate dropped about 1.5% for both the first and third rounds of asset purchases. Both of them were larger operations, and surprised the market. QE2, which was mostly expected ahead of time, only related to about a 0.12% drop in the shadow rate. Wu and Xia (2016)'s calculation also suggests that a one-year increase in the expected ZLB duration had roughly the same effect on the macroeconomy as a 15 basis-point decrease in the shadow rate.

The Wu and Xia (2016) shadow rate has since been used as a summary for unconventional monetary policy for policy analyses by Altig (2014) of the Atlanta Fed, Hakkio and Kahn

Figure 1: Yields



Notes: black solid line: the Wu-Xia shadow rate; black dotted line: the effective fed funds rate; blue dashed line: the BofA Merrill Lynch US high yield effective yield; orange dashed line: BofA Merrill Lynch US high yield spread; yellow crosses: Barclays US corporate high yield index spread; red dashed line: Goldman Sachs financial conditions index. Left scale: interest rates in percentage points; right scale: Goldman Sachs financial conditions index. The zero lower bound sample spans from January 2009 to November 2015.

(2014) of the Kansas City Fed, and others.

3.2 New evidence on the shadow rate

As we argued earlier, the lack of unconventional monetary policy is the reason that the standard New Keynesian model generates several unusual implications. Unconventional policy tools are intended to stimulate the economy by further lowering the private borrowing/lending rate or easing borrowing constraints, when the policy rate is stuck at a constant. A lower private borrowing/lending rate disincentives households from saving, and motivates firms to borrow and invest more, which leads to a higher aggregate demand. This works through the same mechanism as if the Fed were to lower the policy rate during normal times, which eventually transmits into the private borrowing/lending rate to take effect.

Next, we show the comovement between the shadow rate and various private interest rates or financial conditions in the data to demonstrate the choice of the shadow rate as the summary for the effects of unconventional policy tools. During the zero lower bound episode of US history, the effective fed funds rate does not move; see the black dotted line in Figure 1. However, the shadow rate in solid black still displays variation tracking unconventional monetary policy. It dropped 3% from the onset of the ZLB until mid-2014, representing an easing stance of the Fed. Subsequently, a 3% tightening was implemented between then and the time of the liftoff from the ZLB in November 2015. At the same time, various borrowing/lending rates that private agents face comove with the shadow rate. The blue line is the effective yield of the BofA Merrill Lynch US High Yield Master II Index. The orange line is the option-adjusted spread of the BofA Merrill Lynch High Yield Master II Index over the Treasury curve. The yellow crosses are the Barclays US corporate high yield spread. All these corporate borrowing rates (or spreads) display the same U shape as the shadow rate. Consequently, they are highly correlated with the shadow rate, with correlations of about 0.8 for all these indexes. The red line is the Goldman Sachs Financial Conditions Index, which tracks the broad financial markets including equity prices, the US dollar, Treasury yields, and credit spreads. It depicts the same story, and also has a high correlation with the shadow rate at 0.8. To obtain these correlations, the shadow rate's role is instrumental, and it cannot be replaced by, for example, the 10-year Treasury rate, whose correlations with these indexes ranges from 0.2 - 0.35. Other private rates have a similar feature: both the 30-year fixed mortgage rate and 5/1-year adjustable rate comove with the shadow rate, and the correlations are 0.51 and 0.73, respectively, for the ZLB period.

3.3 Incorporating the shadow rate in the New Keynesian model

The shadow rate is an attractive concept, and is a potential solution to all the issues discussed in Section 2.1. Next, we accommodate this idea into the linearized New Keynesian model in Section 2. Figure 1 shows a wedge exists between private borrowing/lending rates agents face and the shadow rate. This wedge allows private rates to drop further with the shadow rate in response to an easing policy, yet remain positive when the ZLB prevails for the conventional policy rate r_t. The resemblance of the dynamics between private rates and the shadow rate suggests a constant difference between them. We call the constant wedge the risk premium rp. Therefore, the private borrowing/lending rate is

$$r_t^B = s_t + rp.$$

To capture the effect of unconventional monetary policy, we let agents face the private rate r_t^B instead of the conventional policy rate r_t. Hence, the DIS curve in equation (2.1) becomes

$$c_t = -\frac{1}{\sigma}(s_t - \mathbb{E}_t\pi_{t+1} - r) + \mathbb{E}_t c_{t+1}. \tag{3.1}$$

(3.1) is the same as (2.1) except that r_t is replaced by the shadow rate s_t, and the constant rp is canceled out.

This new DIS curve uses the shadow rate to summarize the conventional monetary policy $r_t = s_t$ when $s_t \geq 0$, and the effects of unconventional policy tools on privates rates when $s_t < 0$. Section 4 will micro-found (3.1) by implementing two major unconventional policy tools. Wu and Xia (2016) have empirically demonstrated that a system consisting of a reduced-form VAR representation of (2.2)-(2.3), (2.5)-(2.6), and (3.1) does not incur a structural break.

For the example of a negative supply shock, the unconventional monetary policy, which is summarized by the shadow rate, reacts positively. This reaction is the same as what the central bank would do with a conventional monetary policy. Such a reaction increases the real rate private agents face, and implies a lower output in the model, which is consistent with the data. For the example of a positive government-spending shock, the shadow rate increases as well. This increase leads to a higher real rate and lower private consumption. Therefore, the government-spending multiplier is less than 1 in this case, as in normal times.

Besides the benefit of sensible economic implications, the shadow rate representation salvages the DSGE model from the structural break introduced by the occasionally binding ZLB on the policy rate. More specifically, the shadow rate rule responds to inflation and the output gap in the same manner, whether in normal times or at the ZLB, and therefore, it restores traditional estimation methods' validity. Also, the Blanchard-Kahn condition holds uniformly, and we regain the unique model solution. For empirical researchers, the shadow rate rule does not impose a structural break. Hence, it does not require a special treatment for the ZLB period, and can make use of the full sample of historical post-war data. Overall, researchers can rely on the same model and solution method to update their theoretical analyses and empirical studies, and the only change is to replace the policy rate with the shadow rate to represent unconventional monetary policy for the ZLB subsample.

4 Mapping unconventional policy tools into SRNKM

We have shown the relationship between the shadow rate and unconventional monetary policy empirically in Section 3. In this section, we formalize this link: we micro-found (3.1) using two major programs: QE and lending facilities.

4.1 QE

The first policy tool is large-scale asset purchases (or QE). Government purchases of outstanding loans lower the overall supply to private sectors, which increases prices and decreases interest rates of these assets. We build a simple model to capture this mechanism. The government issues a constant amount of bonds B. At the ZLB, it implements QE programs to purchase back part of the outstanding bond. Consequently, the overall supply of bonds at the ZLB is less than B and time varying. The QE program works through a risk premium channel: less bond supply lowers the risk compensation agents require to hold bonds. This channel is motivated by the empirical literature; see, for example, Gagnon, Raskin, Remache, and Sack (2011) and Hamilton and Wu (2012). We also include bond holding in households' utility. Bonds enter the utility similarly to money in the utility.

Households' period utility is

$$U\left(C_t, L_t, B_t^H\right) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{{L_t}^{1+\eta}}{1+\eta} + \chi \log B_t^H, \tag{4.1}$$

where C_t is consumption, L_t is labor supply, and B_t^H is the bonds purchased by households. All the coefficients are positive. They maximize their lifetime utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, L_t, B_t^H\right), \tag{4.2}$$

subject to the budget constraint

$$C_t + B_t^H = \frac{R_{t-1}^B B_{t-1}^H}{\Pi_t} + W_t L_t + T_t, \tag{4.3}$$

where R_{t-1}^B is the gross return of holding bonds from $t-1$ to t, known at $t-1$, and the superscript B stands for "bond." Π_t is inflation from $t-1$ to t, realized at t. W_t is the real wage, and T_t is net lump-sum transfer. The first-order condition with respect to B_t^H is

$$C_t^{-\sigma} - \frac{\chi}{B_t^H} = \beta R_t^B \mathbb{E}_t \left[\frac{C_{t+1}^{-\sigma}}{\Pi_{t+1}}\right]. \tag{4.4}$$

The linearized QE Euler equation is:

$$c_t = -\frac{\beta R^B}{\sigma} \left(r_t^B - \chi_b b_t^H - \mathbb{E}_t \pi_{t+1} - \kappa_C\right) + \beta R^B \mathbb{E}_t c_{t+1}, \tag{4.5}$$

where small letters are logs, letters without t subscripts are steady-state values, and $\kappa_C \equiv \log R^B - \left(1 - \frac{1}{\beta R^B}\right)\left(\log\left(1 - \beta R^B\right) - log \chi\right)$, $\chi_b \equiv \frac{\chi / B^H}{\beta R^B C^{-\sigma}}$ are constants. Equation (4.5) deviates from the benchmark Euler equation (2.1) in a standard New Keynesian model with the following terms: (1) it is the return on bonds rather than the fed funds rate that enters the households' Euler equation, and (2) bond holding b_t^H affects households' inter-temporal choice because it enters the utility function directly.

We decompose the bond return r_t^B into two components:

$$r_t^B(r_t, b_t) = r_t + rp_t(b_t), \tag{4.6}$$

where r_t is the policy rate and follows the Taylor rule during normal times as in (2.3) - (2.4). The wedge between the two rates rp_t is referred to as the risk premium, which depends on the total supply of bonds b_t. In equilibrium,

$$b_t = b_t^H, \tag{4.7}$$

and the steady-state values are also equal, $B = B^H$. During normal times, $b_t = b$, and $r_t^B(r_t, b) = r_t + rp$, where $rp = rp_t(b)$. In other words, the borrowing rate comoves with the policy rate with a constant wedge.

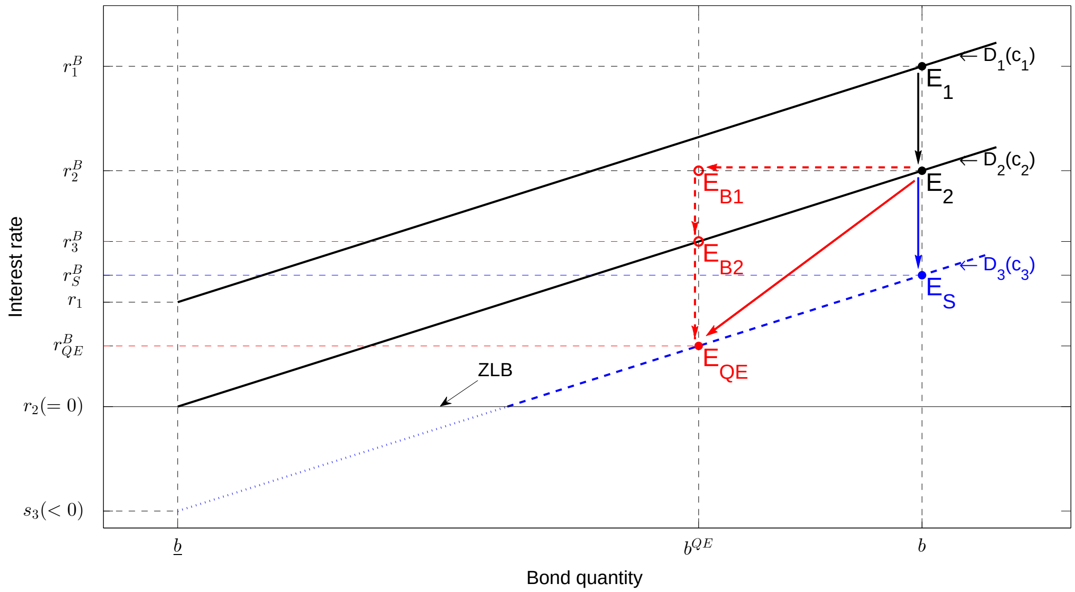
When the ZLB binds $r_t = 0$, the central bank turns to large-scale asset purchases to lower bond supply b_t. This has two effects. First, b_t enters the Euler equation directly. Second, it works through the risk premium $rp_t(b_t)$ indirectly. Empirical research, for example, Gagnon, Raskin, Remache, and Sack (2011) and Hamilton and Wu (2012), finds a lower bond supply through QE operations is associated with a lower risk premium, which suggests rp_t is an increasing function of b_t: $rp_t'(b_t) > 0$. Without making assumptions about the functional form relating $RP_t = exp(rp_t)$ to $B_t = exp(b_t)$, we can obtain the linear relationship for the log variables:

$$rp_t - rp = A_1(b_t - b), \tag{4.8}$$

where $A_1 > 0$. Moreover, the overall marginal effect of QE on consumption $-dc_t/db_t = A_1 - \chi_b$ is stimulus, suggesting $A_1 > \chi_b$.

The linearized model incorporating QE is captured by the new Euler equation (4.5), the yield decomposition (4.6), the risk premium channel of bond purchase (4.8), the bond market clearing condition (4.7), and together with the usual (2.2) - (2.6).

Figure 2: Bond demand and supply



Notes: Each demand curve D_i holds consumption and inflation constant. The vertical lines at b and b^{QE} are bond supplies during normal times and at the ZLB. The intercepts of demand curves D_i and the vertical line at $\underline{b} = b - rp/A_1$ are the policy rates r_i. $r_2 = 0$ marks the zero lower bound.

We illustrate the effects of conventional and unconventional monetary policy in Figure 2. Each demand curve D_i represents a relation between the return on bonds and the quantity demanded when other variables are kept constant. The demand curve is upward sloping, because a higher bond return is associated with a lower price of the same asset. The consumption-inflation bundles of all points on the same demand curve are the same, and the consumption on a lower demand curve is higher because $\partial c_t/\partial r_t^B < 0$. For convenience, we define $\underline{b}$ such that $rp_t(\underline{b}) = 0$ and $r_t^B(r_t, \underline{b}) = r_t$, where $\underline{b} = b - rp/A_1$. Therefore, we can read r_t off the demand curve at $b_t = \underline{b}$. The policy rate for D_1 is $r_1 > 0$, and the zero lower bound is not binding. The policy rate for D_2 is $r_2 = 0$, and the ZLB is just binding. Demand curves lower than D_2, for example, D_3, intersect $b_t = \underline{b}$ below zero, and the ZLB implies the policy rate is bounded at zero.

During normal times, government bond supply is constant at b. The central bank stimulates the economy by lowering the policy rate, for example, from r_1 to r_2, moving the economy from E_1 to E_2, and consumption from c_1 to a higher level c_2. However, if the central bank wishes to stimulate the economy further, it can no longer directly lower the policy rate, because it is at the ZLB. Instead, it implements a QE operation by lowering bond supply from b to b^{QE}. This operation moves the equilibrium from E_2 on D_2 to E_{QE} on D_3, decreases the bond return from r_2^B to r_{QE}^B, and increases consumption from c_2 to c_3. This movement can be decomposed into three steps. First, the direct effect of bond purchases, keeping the risk premium or bond return unchanged, lowers bond supply, moves from E_2 to E_{B1}, and lowers consumption. A lower bond supply has a negative effect on consumption. Both the second step from E_{B1} to E_{B2} and third step from E_{B2} to E_{QE} are indirectly through risk premium. The second step lowers the interest rate from r_2^B to r_3^B, and this change cancels out the negative direct effect on consumption. The third step moves the interest rate from r_3^B to r_{QE}^B, providing the actual stimulus to the economy.

The overall effect of QE on the economy from E_2 to E_{QE} is equivalent to moving the equilibrium from E_2 to E_S. The latter can be summarized by lowering the policy rate further from 0 to a negative shadow rate s_3, which is not subject to the ZLB constraint.

Next, we formalize the idea of equivalence. Monetary policy enters the Euler equation (4.5) through

$$r_t^B - \chi_b b_t = r_t + rp - \chi_b b + (A_1 - \chi_b)(b_t - b). \tag{4.9}$$

During normal times, $b_t = b$, (4.9) amounts to $r_t + rp - \chi_b b$, and monetary policy operates through the usual Taylor rule on r_t, which is equal to the shadow rate s_t. At the zero lower bound, the policy rate no longer moves, $r_t = 0$, and the overall effect of monetary policy is $rp - \chi_b b + (A_1 - \chi_b)(b_t - b)$. If $s_t = (A_1 - \chi_b)(b_t - b)$ at the ZLB, and then

$$r_t^B - \chi_b b_t = s_t + rp - \chi_b b \tag{4.10}$$

can capture both conventional and unconventional policies. Although the policy variable in (4.9) deviates from the conventional policy rate r_t with a time-varying wedge, the difference between the policy variable in (4.10) and s_t is a constant. This leads to the following proposition. Let us first define $\tilde{\kappa}_C \equiv \kappa_C - rp$.

Proposition 1 *The shadow rate New Keynesian model represented by the shadow rate Euler equation,*

$$c_t = -\frac{\beta R^B}{\sigma}\left(s_t - \chi_b b - \mathbb{E}_t \pi_{t+1} - \tilde{\kappa}_C\right) + \beta R^B \mathbb{E}_t c_{t+1}, \tag{4.11}$$

New Keynesian Phillips Curve (2.2) and (2.5), shadow rate Taylor rule (2.3), and market-clearing condition (2.6) nests both the conventional Taylor interest rate rule and QE operation that changes risk premium through (4.8) if

$$\begin{cases} r_t = s_t, b_t = b & \text{for } s_t \geq 0 \\ r_t = 0, b_t = b + \frac{s_t}{A_1 - \chi_b} & \text{for } s_t < 0. \end{cases}$$

Proof: See Appendix B.

Equation (4.11) maintains key features of (3.1). In fact, they are identical when bonds do not enter utility $\chi = 0$. Therefore, the QE channel provides one micro-foundation for (3.1). Under the same condition $\chi = 0$, this equivalence can be extend to corporate bonds as well. An approximate equivalence for corporate bonds can be established when $\chi \neq 0$ and the direct effect χ_b is small. We calibrate χ_b at 0.0026[1], meaning a 1% decrease in bond supply causes a 0.0026% increase in the shadow rate. The data available as of August 2016 suggest that the Fed holds 2.46 trillion of the 18.15 trillion total outstanding Treasury debt, which amounts to about 13.6%. The direct effect of overall purchase from zero holding causes a 0.035% increase in the shadow rate, which is two orders of magnitude smaller than the indirect effect through risk premium, which moves the shadow rate in the opposite direction.

[1]We set the discount factor $\beta = 0.98$. The steady-state debt-to-GDP ratio, measured as the amount of Treasury securities relative to GDP, is calibrated at $\frac{B}{Y} = 54.8\%$ as in Del Negro, Eggertsson, Ferrero, and Kiyotaki (2016). Real bond return is calibrated using BofA Merrill Lynch US high yield effective yield data and US CPI inflation rate. The parameter governing marginal utility of bonds is set to $\chi = 0.0014$ to ensure the implied steady-state real annual bond return of 7.1%.

4.2 Lending facilities

We can also map the lending facilities into our shadow rate framework. These facilities inject liquidity into the economy by extending loans to the private sectors. One prominent example is the Federal Reserve's Term Asset-Backed Securities Loan Facility. This channel has been assessed by, for example, Del Negro, Eggertsson, Ferrero, and Kiyotaki (2016) and Ashcraft, Gârleanu, and Pedersen (2010). Policies similar to lending facilities have been implemented by other central banks as well. For example, the Eurosystem's valuation haircuts vary the haircut schedule as a risk-management tool post financial crisis. The UK also has three decades of experience of using credit controls, which is discussed in Aikman, Bush, and Taylor (2016).

We extend the model in (2.1)-(2.6) in the following respects. First, we introduce entrepreneurs to produce intermediate goods using capital and labor and to then sell them in a competitive market to the retailers. Entrepreneurs maximize their lifetime utility. They have a lower discount factor and are less patient than households. They borrow from households using capital as collateral up to a constant loan-to-value ratio allowed by the households. Second, we allow the government to have two additional policy tools at the ZLB. First, it can loosen the borrowing constraint by directly lending to entrepreneurs through lending facilities, effectively making the loan-to-value ratio higher and time varying. Another policy the government employs at the zero lower bound is a tax on the interest rate income of households and a subsidy to entrepreneurs. Taxing interest rate income can be motivated by the recent phenomenon of negative interest rates in Europe and Japan. Their equivalence is suggested by Waller (2016) of the St. Louis Fed. The pre-tax/subsidy private borrowing interest rate imposes a constant markup over the policy rate $R_t^B = R_t RP$, similar to the setup in Section 4.1.

Entrepreneurs (denoted by a superscript E) produce intermediate good Y_t^E according to a Cobb-Douglas function,

$$Y_t^E = AK_{t-1}^{\alpha}(L_t)^{1-\alpha}, \tag{4.12}$$

where A is technology and K_{t-1} is physical capital used at period t and determined at $t-1$. Capital accumulates following the law of motion: $K_t = I_t + (1-\delta)K_{t-1}$, where δ is the depreciation rate, and I_t is investment. Entrepreneurs sell the intermediate goods to retailers at price P_t^E, and the markup for the retailers is $X_t \equiv P_t/P_t^E$.

Entrepreneurs choose consumption C_t^E, investment on capital stock I_t, and labor input L_t to maximize their utility $\mathbb{E}_0 \sum_{t=0}^{\infty} \gamma^t \log C_t^E$, where the entrepreneurs' discount factor γ is smaller than β. The borrowing constraint is

$$B_t \leq M_t \mathbb{E}_t (K_t \Pi_{t+1}/R_t^B), \tag{4.13}$$

where B_t is the amount of corporate bonds issued by the entrepreneurs and M_t is the loan-to-value ratio. The entrepreneurs' budget constraint is

$$Y_t^E/X_t + B_t = \frac{R_{t-1}^B B_{t-1}}{\mathcal{T}_{t-1}\Pi_t} + W_t L_t + I_t + C_t^E, \tag{4.14}$$

where the tax schedule $\mathcal{T}_{t-1}$ is posted at $t-1$ and levied at t. The first-order conditions are labor demand and the consumption Euler equation:

$$W_t = \frac{(1-\alpha)AK_{t-1}^{\alpha}L_t^{-\alpha}}{X_t}, \tag{4.15}$$

$$\frac{1}{C_t^E}\left(1 - \frac{M_t \mathbb{E}_t \Pi_{t+1}}{R_t^B}\right) = \gamma \mathbb{E}_t \left[\frac{1}{C_{t+1}^E}\left(\frac{\alpha Y_{t+1}^E}{X_{t+1}K_t} - \frac{M_t}{\mathcal{T}_t} + 1 - \delta\right)\right]. \tag{4.16}$$

Households maximize their utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\eta}}{1+\eta}\right)$$

subject to the budget constraint

$$C_t + B_t^H = \frac{R_{t-1}^B B_{t-1}^H}{\mathcal{T}_{t-1}\Pi_t} + W_t L_t + T_t. \tag{4.17}$$

Hence, households' consumption Euler equation is:

$$1 = \beta \mathbb{E}_t \left(R_t^B \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma} \Pi_{t+1} \mathcal{T}_t} \right), \tag{4.18}$$

and their labor supply satisfies:

$$W_t = C_t^{\sigma} L_t^{\eta}. \tag{4.19}$$

Households are willing to lend the firm B_t^H with a constant loan-to-value ratio M:

$$B_t^H \leq M \mathbb{E}_t (K_t \Pi_{t+1} / R_t^B). \tag{4.20}$$

During normal times, $B_t = B_t^H$ and $M_t = M$. At the ZLB, the government can provide extra credit to firms through lending facilities, which take the form

$$B_t^G = (M_t - M) \mathbb{E}_t (K_t \Pi_{t+1} / R_t^B). \tag{4.21}$$

Consequently, the total credit obtained by firms equals the households' bond holdings plus the government's bond holdings $B_t = B_t^H + B_t^G$.

The monopolistically competitive final goods producers, who face Calvo-stickiness, behave the same as in the benchmark model. Details can be found in Appendix A.3. The government still implements the Taylor rule during normal times. The goods market clears if

$$Y_t = C_t + C_t^E + I_t. \tag{4.22}$$

The unconventional policy variables, M_t and $\mathcal{T}_t$, appear in the households' consumption Euler equation (4.18) and budget constraint (4.17), and in the entrepreneurs' borrowing constraint (4.13), budget constraint (4.14), and first-order condition (4.16) in equilibrium conditions. They stimulate the economy through several channels. First, a looser borrowing constraint allows entrepreneurs to secure more loans. Second, the tax benefit for entrepreneurs' interest rate payment effectively lowers their borrowing cost, encouraging them to borrow, consume, invest, and produce more. Third, higher taxation on interest rate income reduces households' net return from saving, and hence incentivizes them to consume more. All together, these channels help the economy get out of the "liquidity trap," and boost the aggregate demand and hence output.

The unconventional policy variables appear in pairs with the conventional monetary policy R_t. In households' consumption Euler equation and households' and entrepreneurs' budget constraints, government policy appears in the form $R_t/\mathcal{T}_t$. In the entrepreneurs' borrowing constraint and first-order condition, it appears in the form R_t/M_t. Hence, to lower $R_t/\mathcal{T}_t$ and R_t/M_t, the government can operate through the conventional monetary policy by lowering R_t, or through unconventional policy tools by losing the credit constraint (increasing M_t) and increasing tax $\mathcal{T}_t$. Moreover, $M_t/\mathcal{T}_t$ enters entrepreneurs' Euler equation, and moving both proportionally keeps this ratio constant. The following proposition formalizes this equivalence.

Lemma 1 *If*

$$\begin{cases} R_t = S_t, \mathcal{T}_t = 1, M_t = M & \text{for } S_t \geq 1 \\ \mathcal{T}_t = M_t/M = 1/S_t & \text{for } S_t < 1, \end{cases}$$

then $R_t/\mathcal{T}_t = S_t,\ R_t/M_t = S_t/M,\ M_t/\mathcal{T}_t = M\ \forall S_t.$

Proof: See Appendix B.

Lemma 1 suggests the dynamics of $R_t/\mathcal{T}_t$ and R_t/M_t can be captured by the single variable S_t.

The linear system describing the equilibrium allocation $\{c_t, c_t^E, y_t, k_t, i_t, b_t\}_{t=0}^{\infty}$ and prices

$\{x_t, \pi_t, r_t\}_{t=0}^{\infty}$ consists of (2.3), (2.4), and

$$
\begin{aligned}
c_t &= -\frac{1}{\sigma}(r_t^B - \tau_t - \mathbb{E}_t\pi_{t+1} - r - rp) + \mathbb{E}_t c_{t+1}, && (4.23)\\
C^E c_t^E &= \alpha\frac{Y}{X}(y_t - x_t) + Bb_t - R^B B(r_{t-1}^B + b_{t-1} - \tau_{t-1} - \pi_{t-1}) - Ii_t + \Lambda_1, && (4.24)\\
b_t &= \mathbb{E}_t(k_t + \pi_{t+1} + m_t - r_t^B), && (4.25)\\
0 &= \left(1 - \frac{M}{R^B}\right)(c_t^E - \mathbb{E}_t c_{t+1}^E) + \frac{\gamma\alpha Y}{XK}\mathbb{E}_t(y_{t+1} - x_{t+1} - k_t) \\
&+ \frac{M}{R^B}\mathbb{E}_t(\pi_{t+1} - r_t^B + m_t) + \gamma M(\tau_t - m_t) + \Lambda_2, && (4.26)\\
y_t &= \frac{\alpha(1+\eta)}{\alpha+\eta}k_{t-1} - \frac{1-\alpha}{\alpha+\eta}(x_t + \sigma c_t) + \frac{1+\eta}{\alpha+\eta}a + \frac{1-\alpha}{\alpha+\eta}\log(1-\alpha), && (4.27)\\
k_t &= (1-\delta)k_{t-1} + \delta i_t - \delta\log\delta, && (4.28)\\
\pi_t &= \beta\mathbb{E}_t\pi_{t+1} - \lambda\left(x_t - x\right), && (4.29)\\
y_t &= \frac{C}{Y}c_t + \frac{C^E}{Y}c_t^E + \left(1 - \frac{C}{Y} - \frac{C^E}{Y}\right)i_t, && (4.30)
\end{aligned}
$$

where Λ_1 and Λ_2 are constants related to steady-state values.[2] (4.23) linearizes the households' consumption Euler equation (4.18), and it differs from the standard Euler equation (2.1) mainly because we have a tax in households' budget constraint. (4.24) is from the entrepreneurs' budget constraint (4.14) and labor demand first-order condition (4.15). (4.25) is the linear expression for the borrowing constraint (4.13). (4.26) linearizes the entrepreneurs' consumption Euler equation (4.16). (4.27) combines the production function (4.12) and labor supply first-order condition (4.19). (4.28) is the linearized capital accumulation process. (4.29) is the New Keynesian Phillips Curve expressed with the price markup, which is equivalent to (2.2) because $mc_t = -x_t$. (4.30) is the linearized version of the goods market-clearing condition (4.22).

Finally, the following proposition builds the equivalence between the shadow rate policy and lending facility / tax policy in the linear model:

[2] $\Lambda_1 = C^E\log C^E - \alpha\frac{Y}{X}\log\frac{Y}{X} - B\log B + R^B B\log R^B B + I\log I$, and $\Lambda_2 = -\frac{\gamma\alpha Y}{XK}\log\frac{Y}{XK} - \left(\frac{1}{R^B} - \gamma\right)M\log M + \frac{M}{R^B}\log R^B$.

Proposition 2 *The shadow rate New Keynesian model represented by*

$$
\begin{aligned}
c_t &= -\frac{1}{\sigma}(s_t - \mathbb{E}_t\pi_{t+1} - r) + \mathbb{E}_t c_{t+1}, && (4.31)\\
C^E c_t^E &= \alpha\frac{Y}{X}(y_t - x_t) + Bb_t - R^B B(s_{t-1} + rp + b_{t-1} - \pi_{t-1}) - Ii_t + \Lambda_1, && (4.32)\\
b_t &= \mathbb{E}_t(k_t + \pi_{t+1} + m - s_t - rp), && (4.33)\\
0 &= \left(1 - \frac{M}{R^B}\right)(c_t^E - \mathbb{E}_t c_{t+1}^E) + \frac{\gamma\alpha Y}{XK}\mathbb{E}_t(y_{t+1} - x_{t+1} - k_t) \\
&+ \frac{M}{R^B}\mathbb{E}_t(\pi_{t+1} - s_t - rp + m) - \gamma M m + \Lambda_2, && (4.34)
\end{aligned}
$$

the shadow rate Taylor rule (2.3) together with (4.27) - (4.30) nests both the conventional Taylor interest rate rule and lending facility / tax policy in the model summarized by (2.3)-(2.4) and (4.23)-(4.30) if

$$
\begin{cases}
r_t = s_t, \tau_t = 0, m_t = m & \textit{for } s_t \geq 0 \\
\tau_t = m_t - m = -s_t & \textit{for } s_t < 0.
\end{cases}
$$

Proof: See Appendix B.

Note that the Euler equation equation (4.31) is identical to (3.1); hence, the lending facility / tax policy channel provides another micro-foundation for (3.1).

5 Quantitative analyses

In this section, we demonstrate the quantitative implications of our model with the two anomalies associated with the ZLB discussed in Section 2.1. We first extend the model to include more ingredients to match some empirical economic moments, and explain our methodology.

5.1 Extended model and methodology

Model The mechanism for how the shadow rate New Keynesian model works has been demonstrated qualitatively with the baseline model in Section 3. This section introduces additional modeling ingredients to better match some empirical moments of the economy for quantitative analysis.

Many components are from Iacoviello (2005)'s full model, including five sectors, of which two are household sectors. Both of them work, consume, and hold housing stocks. The difference is their discount factors. Patient households have a higher discount factor and save. Impatient households have lower discount factors and borrow from patient households using their existing housing as collateral. Entrepreneurs also have a lower discount factor than patient households, and hence borrow from them with collateral as well. Entrepreneurs consume, invest, and hold houses. They use housing, capital, and labor as inputs to produce identical intermediate goods and sell them in a competitive market to retailers. Retailers are monopolistically competitive. They differentiate intermediate goods into final goods, and set prices with Calvo-type price stickiness. The government implements a Taylor rule.

We analyze the model with and without unconventional monetary policy. For the model without unconventional monetary policy, it is $r_t = 0$ that enters the Euler equation, budget constraint, and borrowing constraint, and so on at the ZLB. For the model with unconventional monetary policy, it is the negative shadow rate s_t instead.

We have shown how this shadow rate Taylor rule in (2.3) can be implemented through various unconventional policy tools in Section 4. These unconventional tools set our model apart from Iacoviello (2005)'s. Without making further assumptions about parameters, we use a time-varying risk premium to capture QE discussed in Section 4.1.[3] Second, we allow the loan-to-value ratio to be time-varying to model lending facilities. Additionally, lenders' (borrowers') bond returns (payments) are subject to a time-varying tax (subsidy) at the ZLB. These two policies together constitute the channel discussed in Section 4.2. We also

[3] In this case, $\chi = 0$.

differ from his model by allowing the government to adjust the aggregate demand through changing its expenditure so that we can study the government-spending multiplier. Full details of the model can be found in Appendix C.

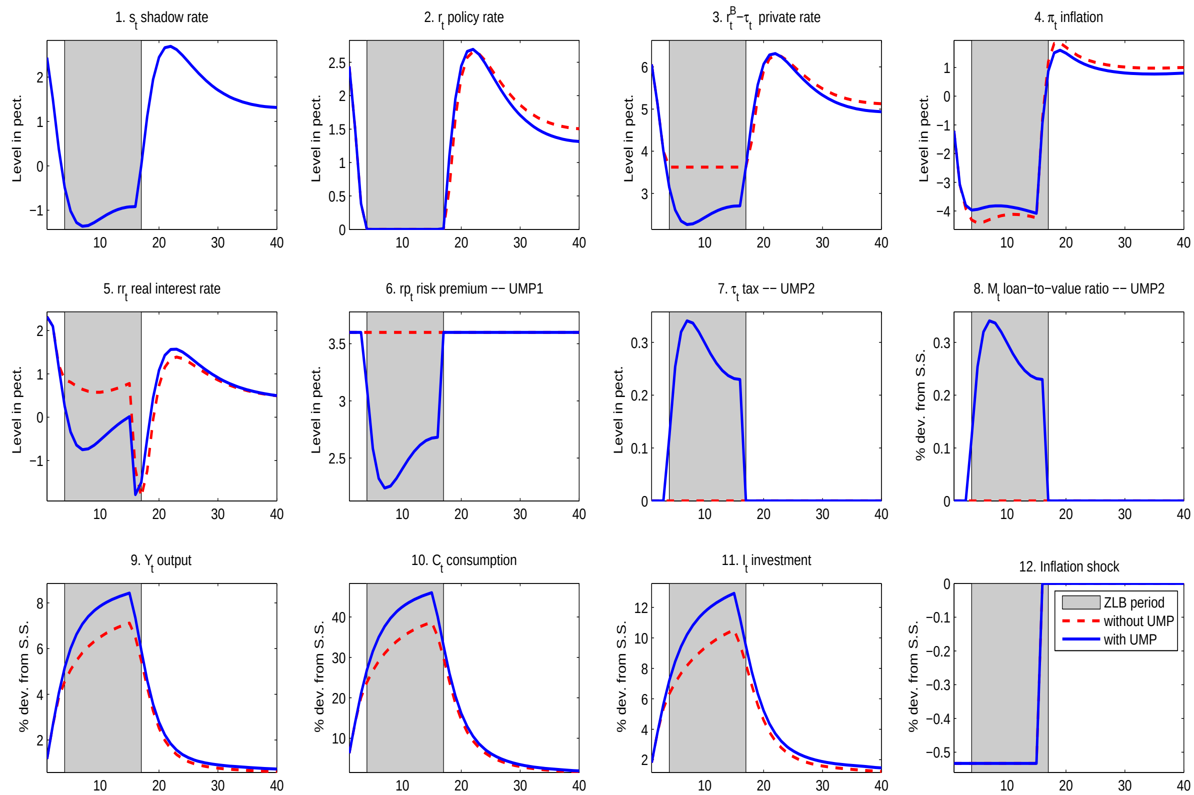
Methodology For our model with unconventional monetary policy, which is represented by a shadow rate Taylor rule, we work with a log-linear approximation. In this case, the constraint of the ZLB for the policy rate does not affect how we solve the model. We use the same method as if no lower bound were present. As a comparison, when we analyze the model with the ZLB constraint and no unconventional monetary policy, we apply the piecewise linear method of Guerrieri and Iacoviello (2015).

5.2 Negative supply shock at the ZLB

According to the standard New Keynesian model, during normal times, a negative supply shock produces a negative effect on output. By contrast, at the ZLB, the same shock produces a positive effect. The latter is counterfactual; for example, see Wieland (2015) and Garín, Lester, and Sims (2016). See more detailed discussion in Section 2.1. Our model with the shadow rate in Section 3.3 reconciles the similarity between normal times and the ZLB found in the data and the contrast implied by the New Keynesian model. Although the policy rate still has a ZLB, a coherent shadow rate Taylor rule summarizes both the conventional and unconventional policy tools. Hence, it is able to produce the right implications for both time periods. We have demonstrated these implications in a qualitative analysis in Section 3.3. This section focuses on quantitative implications.

To create a ZLB environment, we follow Eggertsson (2012) to impose a series of negative inflation shocks on the economy from period 1 to 15, with a total size of 8% (see plot 12 of Figure 3). These shocks push the nominal policy rate r_t in plot 2 to zero at period 4 and keeps it there until about period 17. Red dashed lines are impulse responses when no unconventional monetary policy is introduced at the ZLB. The unconventional policy tools

Figure 3: Negative inflation shocks



Notes: We hit the economy with a series of negative inflation shocks, which occurs in periods 1 - 15, and the total shock size is 8%. The red lines represent the case in which, when the policy rate is bounded by zero, no unconventional monetary policy is implemented. The blue lines represent the case in which unconventional monetary policy is implemented through a negative shadow rate. Plots 1-6 are in annualized percentage points. The shadow rate in plot 1 is an overall measure of monetary policy. In normal times, it is implemented through the policy rate in plot 2. At the ZLB, it can be implemented through QE in plot 6 (per Section 4.1) or the lending facilities in plots 7 and 8 (per Section 4.2). The shaded area marks the ZLB period from 4-17.

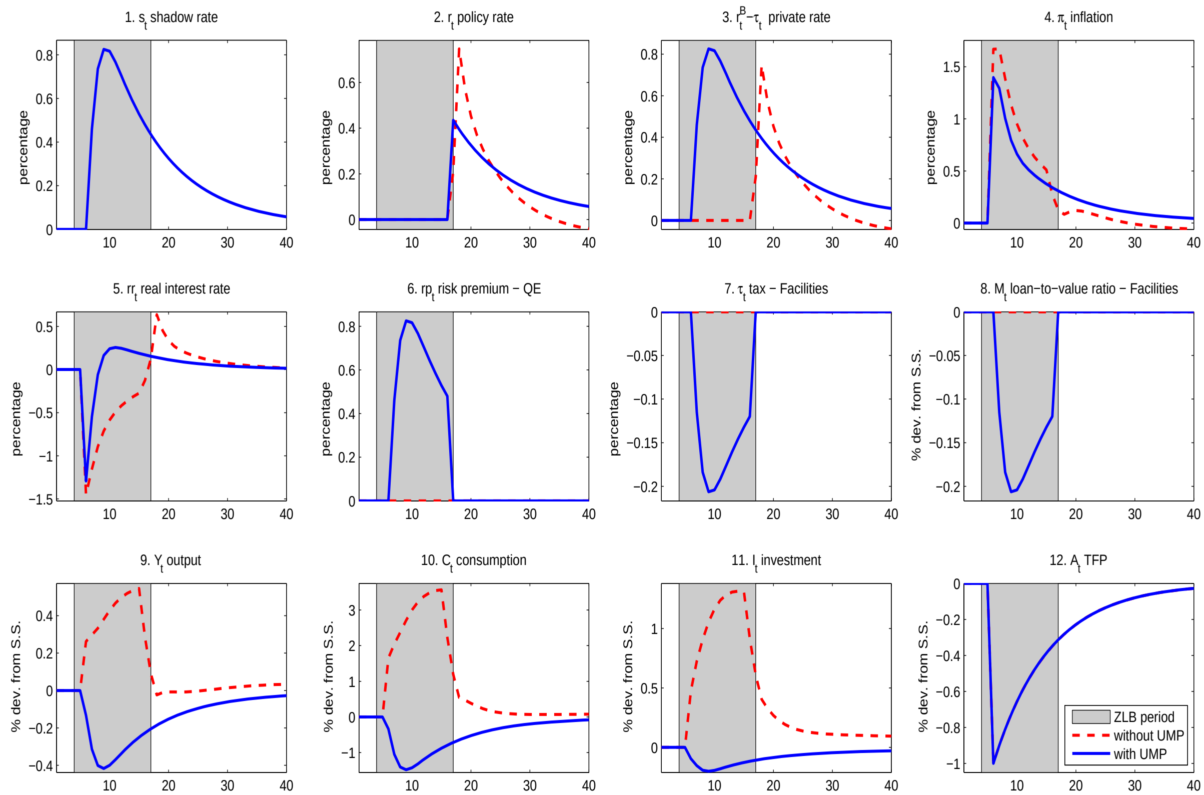
– risk premium in plot 6 for QE, the loan-to-value ratio in plot 8 for lending facilities, and the tax rate in plot 7 – remain constant, as does the private borrowing/lending rate (plot 3). Blue lines allow unconventional monetary policy to be implemented to be consistent with the shadow rate Taylor rule in (2.3). The access to unconventional monetary policy can decrease the private borrowing rate (blue line in plot 3) further when the policy rate is at zero. The shadow rate (plot 1), which summarizes unconventional monetary policy, drops to as low as -1.36%. A negative shadow rate can be implemented either through a QE channel (blue line in plot 6) or a lending facility/fiscal policy (blue lines in plots 7 and 8) as soon as the ZLB hits in period 4. The drop in the risk premium from the steady-state level 3.6% to 2.24% can explain the 1.36% decrease in the shadow rate from zero. Alternatively, the loan-to-value ratio goes up by 0.34%, and the tax rate goes from 0 to 0.34%. Translating these numbers into the annual rate, $0.34\% \times 4 = 1.36\%$, can explain the same amount of change in the shadow rate. Note the tax is levied on total proceeds. Negative inflation shocks lower inflation in plot 4. Unconventional monetary policy mitigates some of this effect; that is, the blue line is above the red. A lower nominal rate and higher inflation expectation imply the real interest rate in plot 5 is lower in blue. A lower real rate stimulates demand. Therefore, output, consumption, and investment (plots 9-11) increase. The ZLB constraint in red dampens this effect due to the lack of response of the nominal interest rate.

We then add an additional negative TFP innovation of the size 1% at period 6. To investigate its impact on the economy, we take the difference between the total effect of both shocks and the effect of only inflation shocks, and plot the difference in Figure 4. The blue lines represent the marginal impact of the TFP shock when unconventional monetary policy is present. The red lines show the impact when unconventional monetary policy is not available.

The red line in plot 9 shows a negative supply shock increases output at the ZLB. This finding is consistent with the implication of a standard New Keynesian model, and contradicts the empirical findings. By contrast, the blue line, where we introduce unconventional monetary policy through our shadow rate policy rule, produces a negative impact of such a shock. This result is data-consistent. The same contrast can be further extended to other real variables, consumption, and investment.

Unconventional monetary policy can explain this difference. The central bank (the government, more broadly) implements unconventional monetary policy (blue lines in Figure 4). The shadow rate is higher. The private borrowing rate increases by the same amount as the

Figure 4: Negative TFP shock



Notes: We hit the economy with two types of shocks. First, a series of negative inflation shocks occurs in periods 1-15, and the total shock size is 8%. Second, a negative TFP happens in period 6 with a size of 1% . We difference out the effect of inflation shocks in Figure 3 and only plot the additional effect the TFP shock has. The red lines represent the case in which, when the policy rate is bounded by zero, no unconventional monetary policy is implemented. The blue lines represent the case in which unconventional monetary policy is implemented through a negative shadow rate. Plots 1-6 are in annualized percentage points. The shadow rate in plot 1 is an overall measure of monetary policy. In normal times, it is implemented through the policy rate in plot 2. At the ZLB, it can be implemented through QE in plot 6 (per Section 4.1) or the lending facilities in plots 7 and 8 (per Section 4.2). The shaded area marks periods 4-17.

shadow rate. The inflation rate increases less than the case without any monetary intervention. Consequently, the real rate decreases less and for a shorter period of time. The decrease in the real rate is not enough to offset the negative impact of the TFP shock on output. Hence, output, consumption, and investment still decrease.

Because the negative TFP shock pushes up future inflation, the policy rate gets out of the ZLB faster, and the shadow rate is less negative compared to the experiment in Figure 3. Therefore, a smaller-scale unconventional monetary policy is needed.

5.3 Government spending multiplier at the ZLB

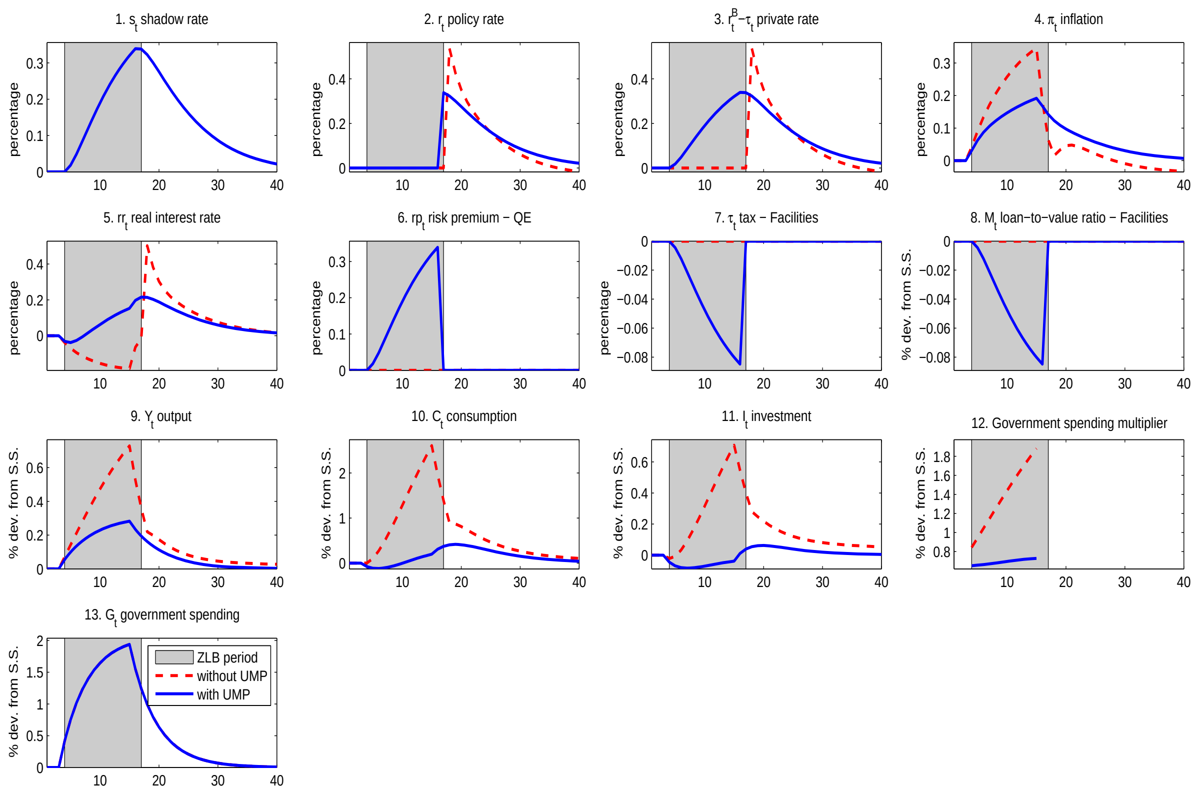
The government-spending multiplier is generally considered to be less than 1 during normal times. Whether this is the case at the ZLB is a heavily debated topic. Many studies, such as Christiano, Eichenbaum, and Rebelo (2011) and Eggertsson (2010), argue that at the ZLB, the multiplier is larger than 1. This is a standard result of the New Keynesian model, and we have demonstrated the intuition in Section 2.1. By contrast, Braun, Körber, and Wake (2012) and Mertens and Ravn (2014) do not find much difference between the fiscal multiplier at the ZLB and during normal times. We have shown in Section 3.3 that their finding is consistent with a New Keynesian model accommodating unconventional monetary policy.

This section further provides some numerical evidence for this contrast. Our analyses are in Figure 5. In addition to the 15-period negative inflation shocks that create the ZLB environment, we introduce another source of shocks that increase government spending from period 4 to 15 with a total size of 5%. The red lines capture the additional impact of government-spending shocks without UMP. The blue lines represent the differences these additional shocks make when UMP is present.

The red line in plot 12 shows the government-spending multiplier is mostly above 1 and peaks at around 1.9 when the policy rate is bounded at zero and the central bank takes no additional measures to smooth the economy. By contrast, the number is less than 0.8 in blue when the central bank monitors and adjusts the shadow rate through implementing unconventional monetary policy.

A positive government shock pushes up the aggregate demand, which leads to a rising pressure on inflation; that is, both the red and blue lines are above zero in plot 4. Without a corresponding increase in the nominal rate (red line in plot 3), the real rate decreases. The

Figure 5: Positive government spending shocks



Notes: We hit the economy with two types of shocks. First, a series of negative inflation shocks occurs in periods 1 - 15, and the total shock size is 8%. Second, government-spending shocks occur from periods 4-15 with a total size of 5%. We difference out the effect of inflation shocks in Figure 3, and only plot the additional effect of the government-spending shock. The red lines represent the case in which, when the policy rate is bounded by zero, no unconventional monetary policy is implemented. The blue lines represent the case in which unconventional monetary policy is implemented through a negative shadow rate. Plots 1-6 are in annualized percentage points. The shadow rate in plot 1 is an overall measure of monetary policy. In normal times, it is implemented through the policy rate in plot 2. At the ZLB, it can be implemented through QE in plot 6 (per Section 4.1) or the lending facilities in plots 7 and 8 (per Section 4.2). The shaded area marks periods 4-17.

lower real rate boosts the economy: output, consumption, and investment are higher.

If the central bank implements unconventional monetary policy, the nominal private borrowing/lending rate increases without any constraint. The persistent and large increase mitigates the rise in inflation; that is, the blue line is lower than the red line in plot 4. This in turn implies a higher real interest rate, which depresses private investment and consumption

– the blue lines in plots 10 and 11 are below zero for the periods when the government-spending shocks occur. Therefore, government spending crowds out private spending as usual, and the overall increase in output is less than the total government spending.

6 Conclusion

We build a New Keynesian model with the shadow rate, which coherently captures the conventional interest rate rule in normal times, and unconventional monetary policy at the ZLB. The shadow rate is the policy rate when the latter is above zero, whereas it is negative when the latter is constrained by the ZLB. At the ZLB, the central bank continues to monitor and adjust the shadow rate following the historical Taylor rule. This shadow rate Taylor rule can be implemented, for example, by QE and/or lending facilities.

We use our model to investigate two anomalies in the New Keynesian literature at the ZLB. First, a standard New Keynesian model implies a counterfactually positive impact of a negative supply shock. Second, the government-spending multiplier is larger than usual. The continuity of the shadow rate between normal times and the ZLB demonstrates these anomalies are not the case, because unconventional monetary policy works fundamentally the same as the usual Taylor rule. These results reconcile the New Keynesian model with the empirical evidence at the ZLB.

References

Aikman, David, Oliver Bush, and Alan M. Taylor (2016) "Monetary versus macroprudential policies – causal impacts of interest rates and credit controls in the era of the UK radcliffe report." Working paper, Bank of England.

Altig, Dave (2014) "What is the stance of monetary policy?" *Macroblog, Federal Reserve Bank of Atlanta*.

Aruoba, S. Boragan, Pablo Cuba-Borda, and Frank Schorfheide (2016) "Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries" Working paper.

Ashcraft, Adam, Nicolae Gârleanu, and Lasse Heje Pedersen (2010) "Two monetary tools: interest rates and haircuts" *NBER Macroeconomics Annual 2010* 25, 143–180.

Blanchard, Olivier (2016) "Currency wars, coordination, and capital controls" Working paper, MIT.

Braun, R. Anton, Lena Mareen Körber, and Yuichiro Wake (2012) "Some Unpleasant Properties of Log-Linearized Solutions When the Nominal Rate is Zero." Federal Reserve Bank of Atlanta Working Paper Series, 2015-5a.

Bullard, James (2012) "Shadow interest rates and the stance of U.S. monetary policy" Annual corporate finance conference.

Christiano, Lawrence, Martin Eichenbaum, and Sergio Rebelo (2011) "When is the Government Spending Multiplier Large?" *Journal of Political Economy* 119, 78–121.

Del Negro, Marco, Gauti Eggertsson, Andrea Ferrero, and Nobuhiro Kiyotaki (2016) "The great escape? A quantitative evaluation of the Fed's liquidity facilities" Federal Reserve Bank of New York Staff Reports, No. 520.

Eggertsson, Gauti (2010) "What fiscal policy is effective at zero interest rates?" *NBER Macroecnomic Annual*, 59–112.

Eggertsson, Gauti (2012) “Was the New Deal contractionary?” *American Economic Review* 102, 524–555.

Fernández-Villaverde, Jesus, Grey Gordon, Guerrón-Quintana, and Juan F. Rubio-Ramírez (2015) “Nonlilnear Adventures at the Zero Lower Bound” *Journal of Economic Dynamics and Control* 57, 182–204.

Gagnon, Joseph, Matthew Raskin, Julie Remache, and Brian Sack (2011) “The financial market effects of the Federal Reserve’s large-scale asset purchases” *International Journal of Central Banking* 7, 3–43.

Galí, Jordi (2008) *Monetary policy, inflation, and the business cycle: an introduction to the New Keynesian framework* Princeton University Press, New Jersey.

Garín, Julio, Robert Lester, and Eric Sims (2016) “Are supply shocks contractionary at the ZLB? Evidence from utilization-adjusted TFP data.” NBER working paper No. 22311.

Guerrieri, Luca, and Matteo Iacoviello (2015) “OccBin: A toolkit for solving dynamic models with occasionally binding constraints easily.” *Journal of Monetary Economics* 70, 22–38.

Hakkio, Craig S., and George A. Kahn (2014) “Evaluating monetary policy at the zero lower bound” *The Macro Bulletin, Federal Reserve Bank of Kansas City.*

Hamilton, James D., and Jing Cynthia Wu (2012) “Identification and estimation of Gaussian affine term structure models.” *Journal of Econometrics* 168, 315–331.

Iacoviello, Matteo (2005) “House Price, Borrowing Constraints, and Monetary Policy in the Business Cycle.” *American Economic Review* 95, 739–764.

Mertens, Karel R. S. M., and Morten O. Ravn (2014) “Fiscal policy in an expectations-driven liquidity trap.” *The Review of Economic Studies* 81, 1637–1667.

Powell, Jerme H. (2013) “Advanced economy monetary policy and emerging market economies.” *Speeches of Federal Reserve Officials.*

Smets, Frank, and Rafael Wouters (2007) "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach" *American Economic Review* 97, 586–606.

Swanson, Eric T., and John C. Williams (2014) "Measuring the effect of the zero lower bound on medium- and longer-term interest rates" *American Economic Review* 104, 3154–3185(32).

Waller, Christopher J. (2016) "Negative interest rate: a tax in sheep's clothing" *On the Economy, Federal Reserve Bank of St. Louis.*

Walsh, Carl (2010) *Monetary theory and policy* The MIT Press, Cambridge, Massachusetts 3 edition.

Wieland, Johannes F. (2015) "Are Negative Supply Shocks Expansionary at the Zero Lower Bound?" Working paper, University of California, San Diego.

Woodford, Michael (2011) "Simple analytics of the government expenditure multiplier" *American Economic Journal: Macroeconomics* 3, 1–35.

Wu, Jing Cynthia, and Dora Fan Xia (2016) "Measuring the macroeconomic impact of monetary policy at the zero lower bound." *Journal of Money, Credit, and Banking* 48, 253–291.

Appendix A Baseline model

Appendix A.1 Households

A representative infinitely-living household seeks to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\eta}}{1+\eta} \right) \tag{A.1}$$

subject to the budget constraint

$$C_t + B_t \leq \frac{R_{t-1} B_{t-1}}{\Pi_t} + W_t L_t + T_t, \tag{A.2}$$

where $\mathbb{E}$ is the expectation operator, C_t and L_t denote time t consumption and hours worked, respectively. The variable B_t denotes the quantity of one-period bonds purchased by the household at time t. The nominal gross interest rate R_{t-1} pays for bonds carried from $t-1$ to t, determined at time $t-1$. $\Pi_t \equiv P_t / P_{t-1}$ is inflation from $t-1$ to t, where P_t is the price level. W_t and T_t denote the real wage rate and firms' profits net of lump-sum taxes.

The optimal consumption-saving and labor supply decisions are given by the two first-order conditions below:

$$1 = \beta \mathbb{E}_t \left(R_t \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma} \Pi_{t+1}} \right) \tag{A.3}$$

$$W_t = \frac{L_t^{\eta}}{C_t^{-\sigma}}. \tag{A.4}$$

Appendix A.2 Wholesale firms

A continuum of wholesale firms exist, producing identical intermediate goods and selling them in a competitive market. All firms have the same production function:

$$Y_t^E = A L_t, \tag{A.5}$$

where A is the technology and is normalized to 1. The price for intermediate goods is P_t^E, and we define the price markup as $X_t = P_t / P_t^E$.

Firms maximize their profit by choosing labor:

$$\begin{aligned} \max_{L_t} \quad & Y_t^E / X_t - W_t L_t \\ s.t. \quad & Y_t^E = A L_t. \end{aligned}$$

The first-order condition is

$$\frac{1}{X_t} = \frac{W_t}{A}. \tag{A.6}$$

Appendix A.3 Retailers

A continuum of monopolistically competitive retailers of mass 1, indexed by z, differentiate one unit of intermediate goods into one unit of retail goods $Y_t(z)$ at no cost, and sell it at price $P_t(z)$. The final good Y_t is a CES aggregation of the differentiated goods, $Y_t = (\int_0^1 Y_t(z)^{1-\frac{1}{\epsilon}} dz)^{\frac{\epsilon}{\epsilon-1}}$.[4] We also refer to Y_t as output. Each firm may reset its price with probability $1-\theta$ in any given period, independent of when the last adjustment happened. The remaining θ fraction of firms keep their prices unchanged. A retailer that

[4] $Y_t = (\int_0^1 Y_t(z)^{1-\frac{1}{\epsilon}} dz)^{\frac{\epsilon}{\epsilon-1}} \approx \int_0^1 Y_t(z) dz = Y_t^E$. The approximation is done through linearization in the neighborhood of the zero-inflation steady state. In what follows, we do not differentiate between Y_t and Y_t^E.

can reset its price will choose price $P_t^*(z)$ to maximize the present value of profits while that price remains effective:

$$\max_{P_t^*(z)} \sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \left(\frac{P_t}{P_{t+k}}\right) [P_t^*(z) Y_{t+k|t}(z) - P_{t+k}^E Y_{t+k|t}(z))], \tag{A.7}$$

where $Y_{t+k|t}(z)$ is the demand for goods z at time $t+k$ when the price of the good is set at time t at $P_t^*(z)$, which satisfies

$$Y_{t+k|t}(z) = \left(\frac{P_t^*(z)}{P_{t+k}}\right)^{-\epsilon} Y_{t+k}. \tag{A.8}$$

Because every firm faces the same optimization problem, we eliminate the index z. The first-order condition associated with the firm's optimization problem is:

$$\sum_{k=0}^{\infty} \theta^k \beta^k \mathbb{E}_t \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \left(\frac{P_t}{P_{t+k}}\right) Y_{t+k|t} \left(P^* - \frac{\epsilon}{\epsilon - 1} P_{t+k}^E\right) = 0, \tag{A.9}$$

where the nominal marginal cost at t is P_t^E. Therefore, the real marginal cost is

$$MC_t \equiv \frac{P_t^E}{P_t} = \frac{1}{X_t} = \frac{W_t}{A}. \tag{A.10}$$

We combine this with the households' labor supply condition (A.4) and production function (A.5), and obtain

$$MC_t = \frac{W_t}{A} = L_t^{\eta} C_t^{\sigma} = Y_t^{\eta} C_t^{\sigma}. \tag{A.11}$$

The aggregate price dynamics follow the equation:

$$\Pi_t^{1-\epsilon} = \theta + (1-\theta) \left(\frac{P_t^*}{P_{t-1}}\right)^{1-\epsilon}. \tag{A.12}$$

Appendix A.4 Government

Monetary policy follows a standard Taylor rule subject to the ZLB:

$$S_t = R \Pi_t^{1+\phi_\pi} \left(\frac{Y_t}{Y}\right)^{\phi_y}, \tag{A.13}$$

where R and Y are the steady-state policy interest rate and output, and

$$R_t = \max\{S_t, 1\}. \tag{A.14}$$

Appendix A.5 Equilibrium

The goods market clears if

$$Y_t = C_t. \tag{A.15}$$

(A.3), (A.10), and (A.5) imply the following relationship between steady-state variables:

$$R = 1/\beta \tag{A.16}$$
$$MC = 1/X = \epsilon/(\epsilon - 1) \tag{A.17}$$
$$Y = AL = L. \tag{A.18}$$

(2.1) is the linear version of (A.3). Log-linearizing (A.9) and (A.12) yields to (2.2). Taking logs of (A.13) and (A.14) givess us (2.3) and (2.4), respectively. (A.11) and (A.15) imply (2.5) and (2.6), respectively.

Appendix A.6 Two extensions

Appendix A.6.1 Negative supply shock

We augment the production function (A.5) to have a time-varying technology $Y_t = A_t L_t$, where technology follows the process $A_t/A = (A_{t-1}/A)^{\rho_a} \varepsilon_{a,t}$ with the steady-state value $A = 1$ still. The process for technology implies (2.7). The marginal cost (A.11) changes to accommodate the time-varying technology:

$$MC_t = \frac{W_t}{A_t} = \frac{L_t^\eta C_t^\sigma}{A_t} = \frac{(Y_t/A_t)^\eta C_t^\sigma}{A_t} = Y_t^\eta C_t^\sigma A_t^{-(1+\eta)}, \tag{A.19}$$

which corresponds to (2.8).

Appendix A.6.2 Government-spending multiplier

We introduce government spending G_t, and it follows the process:

$$\frac{G_t}{G} = \left(\frac{G_{t-1}}{G}\right)^{\rho_g} \varepsilon_{g,t}, \tag{A.20}$$

and the log version is in (2.9). For simplicity, we assume government spending is financed by lump-sum taxes. The goods market clearing gives:

$$Y_t = C_t + G_t, \tag{A.21}$$

which implies (2.10).

Appendix B Proof of Propositions

Proof for Proposition 1 During normal times $b_t^H = b_t = b, r_t^B = r_t + rp, r_t = s_t$, the Euler equation (4.5) becomes

$$\begin{aligned} c_t &= -\frac{\beta R^B}{\sigma}\left(r_t + rp - \chi_b b - \mathbb{E}_t\pi_{t+1} - \kappa_C\right) + \beta R^B \mathbb{E}_t c_{t+1} \\ &= -\frac{\beta R^B}{\sigma}\left(s_t - \chi_b b - \mathbb{E}_t\pi_{t+1} - \tilde{\kappa}_C\right) + \beta R^B \mathbb{E}_t c_{t+1}. \end{aligned} \tag{B.1}$$

At the ZLB $r_t = 0, b_t = b + \frac{s_t}{A_1 - \chi_b}$, use the unconventional monetary policy in (4.8) and market-clearing condition (4.7), and (4.5) becomes

$$\begin{aligned} c_t &= -\frac{\beta R^B}{\sigma}\left(0 + rp(b_t) - \chi_b b_t - \mathbb{E}_t\pi_{t+1} - \kappa_C\right) + \beta R^B \mathbb{E}_t c_{t+1} \\ &= -\frac{\beta R^B}{\sigma}\left(rp + A_1(b_t - b) - \chi_b b_t - \mathbb{E}_t\pi_{t+1} - \kappa_C\right) + \beta R^B \mathbb{E}_t c_{t+1} \\ &= -\frac{\beta R^B}{\sigma}\left(s_t - \chi_b b - \mathbb{E}_t\pi_{t+1} - \tilde{\kappa}_C\right) + \beta R^B \mathbb{E}_t c_{t+1}. \end{aligned} \tag{B.2}$$

Both (B.1) and (B.2) are the same as (4.11).

Proof for Lemma 1 During normal times, $R_t = S_t$, $\mathcal{T}_t = 1$, and $M_t = M$ imply $R_t/\mathcal{T}_t = S_t$, $R_t/M_t = S_t/M$, and $M_t/\mathcal{T}_t = M$. At the ZLB, $\mathcal{T}_t = M_t/M = 1/S_t$, and $R_t = 1$ imply $R_t/\mathcal{T}_t = S_t$, $R_t/M_t = S_t/M$, and $M_t/\mathcal{T}_t = M$.

Proof for Proposition 2 $r_t - \tau_t$ enters (4.23) and (4.24), and Lemma 1 have shown $r_t - \tau_t = log(R_t/\mathcal{T}_t) = s_t$. $r_t - m_t$ enters (4.25) and (4.26), and Lemma 1 have shown $r_t - m_t = log(R_t/M_t) = s_t - m$. $\tau_t - m_t$ enters (4.26), and Lemma 1 have shown $m_t - \tau_t = log(M_t/\mathcal{T}_t) = m$. Therefore, equations (4.23)-(4.26) can be expressed with the shadow rate as in (4.31) - (4.34).

Appendix C Extended model

Appendix C.1 Setup

Appendix C.1.1 Patient households

Patient households (denoted with a superscript P) maximize their lifetime utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (\log C_t^P + j \log H_t^P - (L_t^P)^{1+\eta}/(1+\eta) + \chi_M \log M_t^P/P_t),$$

where β is the discount factor, C_t^P is consumption, j indicates the marginal utility of housing, H_t^P is the holdings of housing, L_t^P is hours of work, and M_t^P/P_t is the real money balance.

Assume households lend in nominal terms at time $t-1$ with the amount of loan B_{t-1}^P, and receive $R_{t-1}^B R_{t-1}^B$ at time t. The bond return R_{t-1}^B is determined at time $t-1$ for bond-carrying between $t-1$ and t. The bond return is higher than the policy rate R_t by a risk premium RP_t and $R_t^B = R_t RP_t$. The gross tax rate on bond return $\mathcal{T} - t - 1$ is assumed to be known $t-1$. The budget constraint of households follows:

$$C_t^P + Q_t \Delta H_t^P + B_t^P = \frac{R_{t-1}^B B_{t-1}^P}{\mathcal{T}_{t-1}\Pi_t} + W_t^P L_t^P + \mathcal{D}_t + T_t^P - \Delta M_t^P/P_t, \tag{C.1}$$

where Δ is the first difference operator. Q_t denotes the real housing price, W_t^P is the real wage, and $\Pi_t \equiv P_t/P_{t-1}$ is the gross inflation rate. $\mathcal{D}_t$ is the lump-sump profits received from the retailer, and T_t^P is the net government transfer.

The first-order conditions for consumption, labor supply, and housing demand are

$$\begin{aligned} \frac{1}{C_t^P} &= \beta \mathbb{E}_t \left(\frac{R_t^B}{\mathcal{T}_t \Pi_{t+1} C_{t+1}^P} \right) && \text{(C.2)} \\ W_t^P &= (L_t^P)^\eta C_t^P && \text{(C.3)} \\ \frac{Q_t}{C_t^P} &= \frac{j}{H_t^P} + \beta \mathbb{E}_t \left(\frac{Q_{t+1}}{C_{t+1}^P} \right). && \text{(C.4)} \end{aligned}$$

Appendix C.1.2 Impatient households

Impatient households (denoted with a superscript I) have a lower discount factor β^I than the patient ones, which guarantees the borrowing constraint for the impatient households binds in equilibrium. They choose consumption C_t^I, housing service H_t^I, and labor supply L_t^I to maximize lifetime utility given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} (\beta^I)^t (\log C_t^I + j \log H_t^I - (L_t^I)^{1+\eta}/(1+\eta) + \chi_M \log M_t^I/P_t).$$

The budget constraint and borrowing constraint are

$$\begin{aligned} & C_t^I + Q_t \Delta H_t^I + \frac{R_{t-1}^B B_{t-1}^I}{\mathcal{T}_{t-1}\Pi_t} = B_t^I + W_t^I L_t^I + T_t^I - \Delta M_t^I/P_t && \text{(C.5)} \\ & B_t^I \leq M_t^I \mathbb{E}_t (Q_{t+1} H_t^I \Pi_{t+1}/R_t^B). && \text{(C.6)} \end{aligned}$$

The first-order conditions for labor supply and housing service can be summarized as follows:

$$W_t^I = (L_t^I)^\eta C_t^I \tag{C.7}$$

$$\frac{Q_t}{C_t^I} = \frac{j}{H_t^I} + \mathbb{E}_t\left[\beta^I \frac{Q_{t+1}}{C_{t+1}^I}\left(1 - \frac{M_t^I}{\mathcal{T}_t}\right) + \frac{M_t^I Q_{t+1}\Pi_{t+1}}{C_t^I R_t^B}\right]. \tag{C.8}$$

Appendix C.1.3 Entrepreneurs

Entrepreneurs (denoted by superscript E) produce intermediate good Y_t^E according to a Cobb-Douglas function:

$$Y_t^E = A_t K_{t-1}^\mu (H_{t-1}^E)^\nu (L_t^P)^{\alpha(1-\mu-\nu)}(L_t^I)^{(1-\alpha)(1-\mu-\nu)}, \tag{C.9}$$

where the technology A_t has a random shock $A_t/A = (A_{t-1}/A)^{\rho_a}\,\varepsilon_{a,t}$. Both the housing input H_{t-1}^E and physical capital K_{t-1} used for the period t production are determined at time $t-1$. Capital accumulates following the law of motion: $K_t = I_t + (1-\delta)K_{t-1}$, where δ is the depreciation rate, and I_t is investment. Capital installation entails an adjustment cost: $\xi_{K,t} = \psi(I_t/K_{t-1} - \delta)^2 K_{t-1}/(2\delta)$. Entrepreneurs sell the intermediate goods to retailers at price P_t^E. The markup for the retailers is $X_t \equiv P_t/P_t^E$.

Entrepreneurs choose consumption c_t, investment on capital stock I_t, housing service H_t^E, and labor input L_t^P and L_t^I to maximize their utility $\mathbb{E}_0 \sum_{t=0}^\infty \gamma^t \log C_t^E$, where the entrepreneurs' discount factor γ is smaller than β. The borrowing constraint entrepreneurs face is

$$B_t^E \leq M_t^E \mathbb{E}_t(Q_{t+1}H_t^E\Pi_{t+1}/R_t^B). \tag{C.10}$$

The budget constraint is

$$\frac{Y_t^E}{X_t} + B_t^E = C_t^E + Q_t\Delta H_t^E + \frac{R_{t-1}^B B_{t-1}^E}{\mathcal{T}_{t-1}\Pi_t} + W_t^P L_t^P + W_t^I L_t^I + I_t + \xi_{K,t}. \tag{C.11}$$

The first-order conditions can be expressed in four equations:

$$\frac{Q_t}{C_t^E} = \mathbb{E}_t\left\{\frac{\gamma}{C_{t+1}^E}\left[\frac{\nu Y_{t+1}^E}{X_{t+1}H_t^E} + \left(1 - \frac{M_t^E}{\mathcal{T}_t}\right)Q_{t+1}\right] + \frac{1}{C_t^E}\frac{M_t^E Q_{t+1}\Pi_{t+1}}{R_t^B}\right\} \tag{C.12}$$

$$W_t^P = \frac{\alpha(1-\mu-\nu)Y_t^E}{X_t L_t^P} \tag{C.13}$$

$$W_t^I = \frac{(1-\alpha)(1-\mu-\nu)Y_t^E}{X_t L_t^I} \tag{C.14}$$

$$\frac{1}{C_t^E}\left[1 + \frac{\psi}{\delta}\left(\frac{I_t}{K_{t-1}} - \delta\right)\right] = \gamma\mathbb{E}_t\left\{\frac{1}{C_{t+1}^E}\left[\frac{\mu Y_{t+1}^E}{X_{t+1}K_t} + (1-\delta) - \frac{\psi}{2\delta}(\delta - \frac{I_{t+1}}{K_t})(2 - \delta + \frac{I_{t+1}}{K_t})\right]\right\} \tag{C.15}$$

Appendix C.1.4 Retailers

A continuum of retailers of mass 1, indexed by z, buy intermediate goods Y_t^E from entrepreneurs at P_t^E in a competitive market, differentiate one unit of goods at no cost into one unit of retail goods $Y_t(z)$, and sell it at the price $P_t(z)$. Final goods Y_t are from a CES aggregation of the differentiated goods produced by retailers, $Y_t = (\int_0^1 Y_t(z)^{1-\frac{1}{\epsilon_t}}dz)^{\frac{\epsilon_t}{\epsilon_t - 1}}$, the aggregate price index is $P_t = (\int_0^1 P_t(z)^{1-\epsilon_t}dz)^{\frac{1}{1-\epsilon_t}}$, and the individual demand curve is $Y(z) = \left(\frac{P_t(z)}{P_t}\right)^{-\epsilon_t} Y_t$, where ϵ_t is the elasticity of substitution for the CES aggregation, and bears an inflation shock: $\epsilon_t = \epsilon_{t-1}^{\rho_\epsilon}\varepsilon_{\epsilon,t}$.

They face Calvo-stickiness: the sales price can be updated every period with a probability of $1-\theta$. When retailers can optimize the price with a probability θ, they reset it at $P_t^*(z)$; otherwise, the price is partially indexed to the past inflation; that is,

$$P_t(z) = \begin{cases} P_{t-1}(z)\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\xi_p}\Pi^{1-\xi_p}, \\ P_t^*(z) \end{cases} \tag{C.16}$$

where Π is the steady-state inflation.

The optimal price $P_t^*(z)$ set by retailers that can change price at time t solves:

$$\max_{P_t^*(z)} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[\Lambda_{t,k}(P_t/P_{t+k}) \left(P_t^*(z)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\xi_p} \Pi^{(1-\xi_p)k} Y_{t+k}(z) - P_{t+k}^E Y_{t+k|t}(z)\right)\right],$$

where $\Lambda_{t,k} \equiv \beta^k (C_t^P / C_{t+k}^P)$ is the patient households' real stochastic discount factor between t and $t+k$, and subject to

$$Y_{t+k|t}(z) = \left(\frac{P_t^*(z)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^{\xi_p}\Pi^{(1-\xi_p)k}}{P_{t+k}}\right)^{-\epsilon_{t+k}} Y_{t+k}.$$

The first-order condition for the retailer's problem takes the form

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left[\Lambda_{t,k} P_t \left(\frac{(\epsilon_{t+k}-1)P_t^*(z)(P_{t+k-1}/P_{t-1})^{\xi_p}\Pi^{(1-\xi_p)k}}{P_{t+k}} - \frac{\epsilon_{t+k}}{X_{t+k}}\right) Y_{t+k|t}(z)\right] = 0. \tag{C.17}$$

The aggregate price level evolves as follows:

$$P_t = \left\{\theta\left[P_{t-1}\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\xi_p}\Pi^{1-\xi_p}\right] + (1-\theta)(P_t^*)^{1-\epsilon_t}\right\}^{1/(1-\epsilon_t)}. \tag{C.18}$$

Appendix C.1.5 Government

The central bank adjusts policy rates following a Taylor rule bounded by 0:

$$\frac{S_t}{R} = \left(\frac{S_{t-1}}{R}\right)^{r_R} \left[(\Pi_{t-1}/\Pi)^{1+r_\Pi}(Y_{t-1}/Y)^{r_Y}\right]^{1-r_R} \varepsilon_{R,t}, \tag{C.19}$$

$$R_t = \max\{S_t, 1\}, \tag{C.20}$$

where R, Π, and Y are steady-state policy rate, inflation, and output, respectively. The monetary policy shock is represented by $\varepsilon_{R,t}$.

The net government transfer in households' sectors consists of two parts: one is to balance the change in real money balance, and the other is lump-sum taxes to finance government spending, bond purchases (QE), and lending to private sectors (lending facilities):

$$T_t^P = T_t^{P,1} + T_t^{P,2} \tag{C.21}$$

$$T_t^{P,1} = \Delta M_t^P / P_t \tag{C.22}$$

$$T_t^{P,2} = -\alpha(G_t + B_t^G) \tag{C.23}$$

$$T_t^I = T_t^{I,1} + T_t^{I,2} \tag{C.24}$$

$$T_t^{I,1} = \Delta M_t^I / P_t \tag{C.25}$$

$$T_t^{I,2} = -(1-\alpha)(G_t + B_t^G). \tag{C.26}$$

where $T_t^{P,1}(T_t^{I,1})$ is the transfer to patient (impatient) households to balance their changes in real money balance, and $T_t^{P,2}(T_t^{I,2})$ is a negative transfer or a lump-sum tax to patient (impatient) households to cover government spending and UMP. The share of lump-sum tax of each sector is determined by their labor share, respectively. The government budget constraint is of the form

$$G_t + B_t^G - \frac{R_{t-1}^B B_{t-1}^G}{\mathcal{T}_{t-1}\Pi_t} - T_t^{P,2} - T_t^{I,2} = 0, \tag{C.27}$$

where G_t is government spending, and follows the process:

$$\frac{G_t}{G} = \left(\frac{G_{t-1}}{G}\right)^{\rho_g} \varepsilon_{g,t}, \tag{C.28}$$

where $\varepsilon_{g,t}$ is the government-spending shock.

Appendix C.1.6 Equilibrium

The equilibrium consists of an allocation,

$$\{H_t^E, H_t^P, H_t^I, L_t^E, L_t^P, L_t^I, Y_t, C_t^E, C_t^P, C_t^I, B_t^E, B_t^P, B_t^I, B_t^G, G_t\}_{t=0}^{\infty},$$

and a sequence of prices,

$$\{W_t^P, W_t^I, S_t, P_t, P_t^*, X_t, Q_t\}_{t=0}^{\infty},$$

that solves the household and firm problems and market-clearing conditions: $H_t^E + H_t^P + H_t^I = H$, $C_t^E + C_t^P + C_t^I + I_t + G_t = Y_t$, $B_t^P + B_t^G = B_t^E + B_t^I$, $\Delta M_t/P_t = \Delta M_t^P/P_t + \Delta M_t^I/P_t$.

Appendix C.2 Calibration

Table 1 presents the calibrated parameters in the extended model. Most of the parameter values follow the calibration and estimation results of Iacoviello (2005) when they also appear in his model. The only exception is the autocorrelation parameter of TFP shocks: the estimate for ρ_a in Iacoviello (2005) is 0.03, which is much lower than the conventional value used in the literature. We follow the literature and set it to 0.95. For the parameters that do not exist in Iacoviello (2005), we set their values to match certain moments in the data. For example, the steady-state gross inflation is set to 1.005, which implies a 2% annual inflation rate. The net quarterly risk premium is set to 0.9% to match the 3.6% average historical annual risk premium. The steady-state tax on gross interest rate income is set to 1.001 to imply the 10.1% tax on net interest rate income. We follow Christiano, Eichenbaum, and Rebelo (2011) to assume the steady-state government spending is 20% of output, and the autocorrelation of the government-spending shock is 0.80. The standard error of the government-spending shock is 0.52, following the literature, for example, Smets and Wouters (2007). For simplicity, we assume at the steady state, the government does not hold any private bonds.

Table 1: Calibrated parameters in the extended model

β	discount factor of patient households	0.99
β''	discount factor of impatient households	0.95
γ	discount factor of entrepreneurs	0.98
j	steady-state weight on housing services	0.1
η	labor supply aversion	0.01
μ	capital share in production	0.3
ν	housing share in production	0.03
δ	capital depreciation rate	0.03
X	steady state gross markup	1.05
θ	probability that cannot re-optimize	0.75
α	patient households' wage share	0.64
m	loan-to-value ratio for entrepreneurs	0.89
m''	loan-to-value ratio for impatient households	0.55
Π	steady-state inflation	1.005
$\frac{G}{Y}$	steady-state government-spending-to-output ratio	0.20
B^G	steady-state government bond holdings	0
ξ_p	price indexation	0.24
r_R	interest rate persistence	0.73
r_Y	interest rate response to output	0.27
r_π	interest rate response to inflation	0.13
$\mathcal{T}$	steady-state tax/subsidy on interest rate income/payment	1
rp	steady-state risk premium	1.009
ρ_a	autocorrelation of technology shock	0.95
ρ_u	autocorrelation of price shock	0.59
ρ_g	autocorrelation of government-spending shock	0.80
σ_a	standard deviation of technology shock	2.24
σ_u	standard deviation of price shock	0.17
σ_g	standard deviation of government-spending shock	0.52

Appendix C.3 Steady state

The patient households' Euler equation gives us the steady-state private borrowing rate, shadow rate, and the real private borrowing rate:

$$R^B = \Pi/\beta \tag{C.29}$$
$$SR = R^B/RP \tag{C.30}$$
$$RR^B = 1/\beta. \tag{C.31}$$

Capital accumulation and entrepreneurs' first-order condition on investment together result in the investment-output ratio:

$$\frac{I}{Y} = \frac{\gamma\mu\delta}{[1-\gamma(1-\delta)]X}. \tag{C.32}$$

Entrepreneurs' first-order condition on housing, the borrowing constraint, and budget constraint give their real estate share, debt-to-output, and consumption-to-output ratio:

$$
\begin{aligned}
\frac{QH^E}{Y} &= \frac{\gamma\nu}{X(1-\gamma^e)} && \text{(C.33)}\\
\frac{B^E}{Y} &= \beta m \frac{QH^E}{Y} && \text{(C.34)}\\
\frac{C^E}{Y} &= \left[\mu + \nu - \frac{\delta\gamma\mu}{1-\gamma(1-\delta)} - (1-\beta)mX\frac{QH^E}{Y}\right]\frac{1}{X}, && \text{(C.35)}
\end{aligned}
$$

where $\gamma^e = \gamma - m\gamma + m\beta$.

Impatient households' budget constraint, borrowing constraint, and first-order condition on housing give their real estate share, debt-to-output, and consumption-to-output ratio:

$$
\begin{aligned}
\frac{QH^I}{C^I} &= j/[1-\beta''(1-M^I) - M^I/(RR^B)] && \text{(C.36)}\\
\frac{B^I}{QH^I} &= M^I\Pi/(R^B) && \text{(C.37)}\\
\frac{T^I - \Delta M^I/P}{Y} &= -(1-\alpha)\frac{G}{Y} && \text{(C.38)}\\
\frac{C^I}{Y} &= \frac{s^I + \frac{T^I - \Delta M^I/P}{Y}}{1 + \frac{QH^I}{C^I}(RR^B - 1)\frac{B^I}{QH^I}}, && \text{(C.39)}
\end{aligned}
$$

where $s^I = \frac{(1-\alpha)(1-\mu-\nu)}{X}$ is the income share of impatient households.

The bond-market-clearing condition, patient households' budget constraint ,and first-order condition with respect to housing imply

$$
\begin{aligned}
\frac{B^P}{Y} &= \frac{B^E}{Y} + \frac{B^I}{Y} && \text{(C.40)}\\
\frac{T^P - \Delta M^P/P}{Y} &= -\alpha\frac{G}{Y} && \text{(C.41)}\\
\frac{C^P}{Y} &= s^P + \frac{T^P - \Delta M^P/P}{Y} + (RR^B - 1)\frac{B^P}{Y} && \text{(C.42)}\\
\frac{QH^P}{C^P} &= j/(1-\beta) && \text{(C.43)}\\
\frac{QH^P}{Y} &= \frac{QH^P}{C^P}\frac{C^P}{Y}, && \text{(C.44)}
\end{aligned}
$$

where

$$
s^P = [\alpha(1-\mu-\nu) + X - 1]/X
$$

is the income shares of patient households.

Housing shares of different sectors follows:

$$
\begin{aligned}
\frac{H^E}{H^P} &= \frac{QH^E}{Y}/\frac{QH^P}{Y} && \text{(C.45)}\\
\frac{H^I}{H^P} &= \frac{QH^I}{Y}/\frac{QH^E}{Y}. && \text{(C.46)}
\end{aligned}
$$

Appendix C.4 Log-linear model

Propositions 1 - 2 describe the conditions under which the conventional and two unconventional policy tools are equivalent and can be coherently summarized by the shadow rate. We present the linear model with the shadow rate representation first in Appendix C.4.1. Then, we map it into specific policy tools in Appendix C.4.2 - Appendix C.4.3.

Appendix C.4.1 Shadow rate representation

In this representation, $R_t^B = S_t RP$, $M_t^I = M^I, M_t^E = M^E$, and $\mathcal{T}_t = \mathcal{T}$. Let hatted variables in lower case denote percentage changes from the steady state. The model can be expressed in the following blocks of equations:
1. Aggregate demand:

$$\begin{aligned}
\widehat{y}_t &= \frac{C^E}{y}\widehat{c}_t^E + \frac{C^P}{y}\widehat{c}_t^P + \frac{C^I}{y}\widehat{c}_t^I + \frac{I}{Y}\widehat{i}_t + \frac{G}{Y}\widehat{g}_t && \text{(C.47)}\\
\widehat{c}_t^P &= \mathbb{E}_t(\widehat{c}_{t+1}^P - \widehat{r}_t^B + \widehat{\pi}_{t+1}) && \text{(C.48)}\\
\hat{i}_t - \hat{k}_{t-1} &= \gamma\left(\mathbb{E}_t\hat{i}_{t+1} - \hat{k}_t\right) + \frac{1-\gamma(1-\delta)}{\psi}\left(\mathbb{E}_t\left[\hat{y}_{t+1} - \hat{x}_{t+1}\right] - \hat{k}_t\right) + \frac{1}{\psi}\left(\hat{c}_t^E - \mathbb{E}_t\hat{c}_{t+1}^E\right) && \text{(C.49)}
\end{aligned}$$

2. Housing/consumption margin:

$$\begin{aligned}
\hat{q}_t &= \gamma^e\mathbb{E}_t\hat{q}_{t+1} + (1-\gamma^e)\left(\mathbb{E}_t\hat{y}_{t+1} - \mathbb{E}_t\hat{x}_{t+1} - \hat{h}_t^E\right) + \left(1 - M^E\beta\right)\left(\hat{c}_t^E - \mathbb{E}_t\hat{c}_{t+1}^E\right)\\
&\quad + M^E\beta\left(\mathbb{E}_t\hat{\pi}_{t+1} - \hat{s}_t\right) && \text{(C.50)}\\
\hat{q}_t &= \gamma^h\mathbb{E}_t\hat{q}_{t+1} - \left(1-\gamma^h\right)\hat{h}_t^I + M^I\beta\left(\mathbb{E}_t\hat{\pi}_{t+1} - \hat{s}_t\right)\\
&\quad + \left(1 - \frac{M^I\beta}{\mathcal{T}}\right)\hat{c}_t^I - \beta^I\left(1 - M^I\right)\mathbb{E}_t\hat{c}_{t+1}^I && \text{(C.51)}\\
\hat{q}_t &= \beta\mathbb{E}_t\hat{q}_{t+1} + \left(\hat{c}_t^P - \beta\mathbb{E}_t\hat{c}_{t+1}^P\right) + (1-\beta)\frac{H^E}{H^P}\hat{h}_t^E - (1-\beta)\frac{H^I}{H^P}\hat{h}_t^I, && \text{(C.52)}
\end{aligned}$$

where

$$\begin{aligned}
\gamma^e &= M^E\beta + (1-M^E)\gamma\\
\gamma^h &= M^I\beta + \left(1 - M^I\right)\beta^I
\end{aligned}$$

3. Borrowing constraints:

$$\begin{aligned}
\widehat{b}_t &= \mathbb{E}_t\widehat{q}_{t+1} - (\widehat{s}_t - \mathbb{E}_t\widehat{\pi}_{t+1}) + \widehat{h}_t^E && \text{(C.53)}\\
\widehat{b}_t'' &= \mathbb{E}_t\widehat{q}_{t+1} - (\widehat{s}_t - \mathbb{E}_t\widehat{\pi}_{t+1}) + \widehat{h}_t^I && \text{(C.54)}
\end{aligned}$$

4. Aggregate supply:

$$\begin{aligned}
\widehat{y}_t &= \frac{1+\eta}{\eta+\nu+\mu}(\widehat{a}_t + \nu\widehat{h}_{t-1}^E + \mu\widehat{k}_{t-1}) - \frac{1-\nu-\mu}{\eta+\nu+\mu}(\widehat{x}_t + \alpha\widehat{c}_t^P + (1-\alpha)\widehat{c}_t^I) && \text{(C.55)}\\
\widehat{\pi}_t &= \frac{\beta}{1+\beta\xi_p}\mathbb{E}_t\widehat{\pi}_{t+1} + \frac{\xi_p}{1+\beta\xi_p}\widehat{\pi}_{t-1} - \frac{1}{1+\beta\xi_p}\kappa\widehat{x}_t + \widehat{\epsilon}_t, && \text{(C.56)}
\end{aligned}$$

where

$$\kappa = (1-\theta)(1-\beta\theta)/\theta$$

5. Flows of funds/evolution of state variables:

$$
\begin{aligned}
\widehat{k}_t &= \delta\widehat{i}_t + (1-\delta)\widehat{k}_{t-1} && \text{(C.57)}\\
\frac{B^E}{Y}\widehat{b}^E_t &= \frac{C^E}{Y}\widehat{c}^E_t + \frac{QH^E}{Y}(\widehat{h}^E_t - \widehat{h}^E_{t-1}) + \frac{I}{Y}\widehat{i}_t + RR^B\frac{B^E}{Y}(\widehat{s}_{t-1} - \widehat{\pi}_t + \widehat{b}^E_{t-1})\\
&\quad -(1-s^P-s^I)(\widehat{y}_t - \widehat{x}_t) && \text{(C.58)}\\
\frac{B^I}{Y}\widehat{b}^I_t &= \frac{C^I}{Y}\widehat{c}^I_t + \frac{QH^I}{Y}(\widehat{h}^I_t - \widehat{h}^I_{t-1}) + RR^e\frac{B^I}{Y}(\widehat{r}^B_{t-1} - \widehat{\pi}_t + \widehat{b}^I_{t-1}) - s^I(\widehat{y}_t - \widehat{x}_t) - \frac{(1-\alpha)G}{Y}\widehat{g}_t && \text{(C.59)}
\end{aligned}
$$

6. Monetary policy rule and shock processes:

$$
\begin{aligned}
\widehat{s}_t &= (1-r_R)[(1+r_\pi)\widehat{\pi}_{t-1} + r_y\widehat{y}_{t-1}] + r_R\widehat{s}_{t-1} + \widehat{e}_{r,t} && \text{(C.60)}\\
\widehat{a}_t &= \rho_a\widehat{a}_{t-1} + \widehat{e}_{a,t} && \text{(C.61)}\\
\widehat{u}_t &= \rho_u\widehat{u}_{t-1} + \widehat{e}_{\epsilon,t} && \text{(C.62)}\\
\widehat{g}_t &= \rho_g\widehat{g}_{t-1} + \widehat{e}_{g,t}. && \text{(C.63)}
\end{aligned}
$$

Appendix C.4.2 QE

Use the decomposition in (4.6),

$$
R^B_t = R_t RP_t. \qquad \text{(C.64)}
$$

During normal times, the central bank varies R_t, whereas at the ZLB, it lowers RP_t through purchasing bonds from impatient households' and entrepreneurs' to decrease the bond supply to patient households. Both actions can mimic the dynamics in the shadow rate S_t. In this case, we keep the following policy variables constant: $M^I_t = M^I$, $M^E_t = M^E$, and $\mathcal{T}_t = \mathcal{T}$.

Proposition 1 implies[5]

$$
\begin{cases}
\widehat{r}_t = \widehat{s}_t, \widehat{rp}_t = 0 \rightarrow \widehat{r}^B_t = \widehat{s}_t & \text{for } s_t \geq 0\\
\widehat{rp}_t = \widehat{s}_t + s \rightarrow \widehat{r}^B_t = \widehat{s}_t & \text{for } s_t < 0.
\end{cases}
$$

Appendix C.4.3 Lending facilities

In this case, risk premium is kept at a constant $R^B_t = R_t RP$. At the ZLB, the government can increase the loan-to-value ratio so that impatient households and entrepreneurs can borrow more money for consumption and production, whereas the patient households still lend according to the borrowing constraints with constant loan-to-value ratios. Moreover, a tax is placed on interest rate income, which is then transferred to the borrowers.

Proposition 2 implies

$$
\begin{cases}
\widehat{r}_t = \widehat{s}_t, \widehat{\tau}_t = \widehat{m}^I_t = \widehat{m}^E_t = 0 \rightarrow \widehat{r}^B_t - \widehat{\tau}_t = \widehat{s}_t & \text{for } s_t \geq 0\\
\widehat{\tau}_t = \widehat{m}^I_t = \widehat{m}^E_t = -(\widehat{s}_t + s) \rightarrow \widehat{r}^B_t - \widehat{\tau}_t = \widehat{s}_t & \text{for } s_t < 0.
\end{cases}
$$

Appendix C.4.4 No unconventional monetary policy

For the model without unconventional monetary policy, replace $\widehat{s}_t$ with $\widehat{r}_t$ in (C.47) - (C.59), and augment the monetary policy in (C.60) with (2.4).

[5] $\chi_b = 0$.