Assessing DSGE Model Nonlinearities

S. Borağan Aruoba*
University of Maryland

Luigi Bocola
Northwestern University, NBER

Frank Schorfheide
University of Pennsylvania, CEPR, PIER, NBER

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Abstract

We develop a new class of time series models to identify nonlinearities in the data and to evaluate DSGE models. U.S. output growth and the federal funds rate display nonlinear conditional mean dynamics, while inflation and nominal wage growth feature conditional heteroskedasticity. We estimate a DSGE model with asymmetric wage and price adjustment costs and use predictive checks to assess its ability to account for nonlinearities. While it is able to match the nonlinear inflation and wage dynamics, thanks to the estimated downward wage and price rigidities, these do not spill over to output growth or the interest rate. (JEL C11, C32, C52, E32)

Key words: Asymmetric Adjustment Costs; Bayesian Analysis; Downward Rigidities; DSGE Models; Econometric Model Evaluation; Nonlinear Dynamics; Perturbation Solution; Predictive Checks; Quadratic Autoregressions

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* Correspondence: B. Aruoba: University of Maryland, Department of Economics, College Park, MD 20742. Email: aruoba@econ.umd.edu. L. Bocola: Northwestern University, Department of Economics, Evanston, IL 60208. Email: luigi.bocola@northwestern.edu. F. Schorfheide: University of Pennsylvania, Department of Economics, Philadelphia, PA 19104. Email: schorf@ssc.upenn.edu. This paper was presented as the JEDC Plenary Lecture during the *23rd International Conference on Computing in Economics and Finance* hosted by the Society for Computational Economics (New York, 2017). We thank Thomas Lubik (Editor-in-Chief), Jaroslav Borovicka, Frank Diebold, Gianni Lombardo, Karel Mertens, as well as participants at various seminars and conferences for helpful comments. Much of this paper was written while the authors visited the Federal Reserve Bank of Philadelphia, whose hospitality they are thankful for. Aruoba and Schorfheide gratefully acknowledge financial support from the National Science Foundation under Grant 1424843.

1 Introduction

Dynamic stochastic general equilibrium (DSGE) models are now widely used for empirical research in macroeconomics, as well as for forecasting and quantitative policy analysis in central banks. In these models, decision rules of economic agents are derived from assumptions about agents' preferences and production technologies utilizing some fundamental principles such as optimization, rational expectations, and competitive equilibrium. In practice, this means that the functional forms and parameters of equations that describe the behavior of economic agents are tightly restricted by the equilibrium conditions. Consequently, a careful evaluation of the DSGE model-implied restrictions is an important aspect of empirical research.

Until recently, much of the research that estimates DSGE models used first-order approximations to the equilibrium decision rules. This made linear models such as vector autoregressions (VARs) appropriate for evaluating DSGE model restrictions. The comparison of DSGE models and VARs can take different forms. First, VARs and DSGE models can be connected in a minimum-distance or indirect inference framework in which it is assumed that the DSGE model provides a realistic probabilistic representation of the data. The researcher then chooses the DSGE model parameters such that VAR coefficients or impulse response functions obtained from actual data match those obtained from DSGE model-simulated data as closely as possible. The magnitude of the minimized discrepancy provides a measure of fit. This idea was first formalized in Smith (1993) and Cogley and Nason (1994) and has been widely used in various forms since.

Second, rather than viewing the VAR as an auxiliary model, it could alternatively be regarded itself as a flexible realistic probabilistic representation of the data, whereas the DSGE model is *a priori* regarded as (potentially) misspecified. This notion underlies the evaluation methods in Schorfheide (2000), the DSGE-VAR framework in Del Negro and Schorfheide (2004) and Del Negro, Schorfheide, Smets, and Wouters (2007), the posterior odds and out-of-sample forecast comparisons between VARs and DSGE models in Smets and Wouters (2003) and Smets and Wouters (2007), as well as the type of impulse-response matching estimator used, for instance, in Christiano, Eichenbaum, and Evans (2005).

In addition to providing econometric techniques for the evaluation of existing DSGE models, informal or formal comparisons between VAR and DSGE model dynamics have provided researchers with valuable insights on how to modify the specification of DSGE models to improve their fit and achieve a more realistic propagation of shocks. More recently, building on work by Cogley and Sargent (2002), Primiceri (2005), and Sims and Zha (2006), constant-parameter VARs have been replaced by time-varying parameter (TVP) or regime-switching VARs to examine how the propagation of shocks has changed over the decades or how it differs across states of the business cycle (e.g., Benati and Lubik (2014), Hubrich and Tetlow (2015), Amir-Ahmadi, Matthes, and Wang (2016), Dahlhaus (2017)).

Underlying the popular linear approximation techniques are intrinsically nonlinear structures that arise from functional forms characterizing agents' preferences and technologies. However, for many traditional DSGE models, e.g., a standard stochastic growth or New Keynesian model, parameterized to match pre-Great-Recession business cycle fluctuations in developed market economies, the endogenous nonlinearities are small and only matter for the calculation of asset prices and welfare comparisons. In the past decade the literature has shifted toward models with explicit nonlinearities such as stochastic volatility (e.g., Justiniano and Primiceri (2008), Fernández-Villaverde, Guerrón-Quintana, Rubio-Ramírez, and Uribe (2011), Fernández-Villaverde, Guerrón-Quintana, and Rubio-Ramírez (2015), and Diebold, Schorfheide, and Shin (2017)), an effective lower bound on nominal interest rates (e.g., Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015), Maliar and Maliar (2015), Gust, Herbst, Lopez-Salido, and Smith (2017), Aruoba, Cuba-Borda, and Schorfheide (2017)), or financial frictions (e.g., Gertler, Kiyotaki, and Queralto (2012), Brunnermeier and Sannikov (2014), He and Krishnamurthy (2015), and Bocola (2016)).

Recent computational advances made it feasible to solve DSGE models with the above-described nonlinearities that are rich enough to be fitted to macroeconomic time series; see the handbook chapters of Maliar and Maliar (2014) and Fernández-Villaverde, Rubio-Ramírez, and Schorfheide (2016) for overviews. Moreover, beginning with the work of Fernández-Villaverde and Rubio-Ramírez (2007), researchers started to integrate the non-linear solution of DSGE models with likelihood-based estimation, culminating in the recent

Bayesian estimation of a Smets-Wouters-style DSGE model with an effective lower bound constraint on nominal interest rates by Gust, Herbst, Lopez-Salido, and Smith (2017). While there is a burgeoning literature on the estimation of nonlinear DSGE models and their applications to substantive macroeconomic questions, there does not seem to be an obvious nonlinear time series model that can serve as a benchmark for the evaluation of DSGE model nonlinearities.

The objective of this paper is to develop a class of time series models that mimic nonlinearities of DSGE models and to use these models as a benchmark for the evaluation of a nonlinear DSGE model. We will focus on "smooth" nonlinearities, leaving the evaluation of macro-financial models with occasionally-binding constraints as a topic for future research. Motivated by the popular second-order perturbation approximations of DSGE model dynamics, we consider autoregressive models that involve quadratic terms of lagged endogenous variables as well as interactions between current period innovations and lagged endogenous variables, which generate conditional heteroskedasticity. These time series models are derived from a perturbation solution to a nonlinear difference equation and have a recursively linear structure that makes it straightforward to characterize stability properties and derive moments. While multivariate extensions are possible, we focus in this paper on univariate specifications, that we refer to as QAR(p,q) models, where "Q" stands for quadratic.[1] In the empirical work, we use $p = q = 1$.

After documenting some of the theoretical properties of the QAR models, the first step of the empirical analysis is to fit QAR(1,1) models to growth of real gross domestic product (GDP), inflation, nominal wage growth, and interest rate data for the U.S. We start our sample in 1960 but consider various subsamples, using 1983 (the end of the Volcker era and the start of the Great Moderation) and 2007 (the end of the Great Moderation and the start of the Great Recession) as additional start and end points. We find three sets of important nonlinearities across the variables and samples we consider. First, GDP growth displays pronounced nonlinearities in the post-1983 samples with sharp output losses during recessions are relatively slow recoveries. Second, for inflation and wage growth the long

[1]The abbreviation QAR has previously been used for Quantile Autoregressions, see Koenker and Xiao (2006).

samples that start in 1960 and extend beyond the 1990s exhibit high volatility in times of high inflation and wage growth, which is mainly driven by the observations in the 1970s. Finally, QAR estimates for interest rates imply an asymmetric behavior by the Federal Reserve in the post-1983 era; interest rates increase more gradually than they fall.

The second step of the empirical analysis consists of the estimation of a DSGE model. In our application we focus on the estimation and evaluation of a New Keynesian DSGE model with asymmetric price and wage adjustment costs, building on Kim and Ruge-Murcia (2009). This model can generate downward nominal wage and price rigidity and is interesting for several reasons. First, it is well known that in the absence of the zero-lower-bound (ZLB) constraint on nominal interest rates, unrealistically large shocks or degrees of risk aversion, New Keynesian DSGE models do not generate significant nonlinearities (see, for instance, An (2007)). However, once one allows for asymmetric adjustment costs, agents' decision rules can become strongly nonlinear. Thus, ex ante, to the extent that there are nonlinearities in the data, the model may be able to deliver some of these.

Second, downward rigidity is a well-documented feature of nominal wage changes at the micro-level, e.g., Gottschalk (2005), Daly, Hobijn, and Lucking (2012), and Barattieri, Basu, and Gottschalk (2014). Third, there are a number of papers that have incorporated downward nominal wage rigidity into DSGE models to study its macroeconomic effects. For instance, Kim and Ruge-Murcia (2009) study optimal monetary policy in the presence of downward nominal wage rigidity. Schmitt-Grohe and Uribe (2017) use downward nominal wage rigidity to generate large output losses and a jobless recovery in a deflation (or liquidity-trap) equilibrium of a New Keynesian model with ZLB constraint. Thus, a careful evaluation of the nonlinearities that this mechanism generates is important.

In estimating the DSGE model, we use the same data set as in the estimation of the univariate QAR models and consider two samples, one long and one short, both of which end in 2007 to avoid using data where the ZLB starts to bind. By and large, the parameter estimates for the DSGE models are consistent with estimates that have been reported elsewhere in the literature. In particular, our estimates indicate asymmetries in the adjustment costs for both prices and nominal wages that make increases less costly than decreases.

The final, and most important step of the analysis is to conduct a posterior predictive check of the DSGE model that compares coefficient estimates obtained from data simulated from an estimated DSGE model to coefficient estimates obtained from actual data. The predictive check amounts to assessing how far the QAR estimates obtained from the actual data lie in the tails of the predictive distribution. The general conclusion is that the DSGE model does not generate very strong nonlinearities except for inflation and nominal wage growth, both of which show conditional heteroskedasiticity. This means that the asymmetric adjustment costs in prices and wages are able to deliver asymmetric behavior in inflation and nominal wage growth in line with the data but this asymmetry does not spill over to real GDP growth, or to the policy instrument of the Federal Reserve.

Our work is related to several branches of the literature. There exists a large body of work on nonlinear time series models. The proposed QAR family can be viewed as a set of tight restrictions on the coefficients of a Volterra (1930) representation of a nonlinear time series and is most closely related to generalized autoregressions (GAR) discussed in Mittnik (1990). GAR models represent the conditional mean of the dependent variable y_t as a polynomial function of its lags. Unfortunately, the GAR model has very undesirable instability properties. Thus, rather than simply augmenting a linear autoregressive model by quadratic terms and interactions between lagged endogenous variables and innovations, we derive its structure from a second-order perturbation approximation to the solution of a nonlinear difference equation along the lines of Holmes (1995).

Other related nonlinear time series models include bilinear models, e.g. Granger and Andersen (1978) and Rao (1981); threshold and smooth-transition autoregressive models, e.g. Tong and Lim (1980) and Teräsvirta (1994); (G)ARCH-M models, e.g. Engle, Lilien, and Robins (1987) and Grier and Perry (1996); linear autoregressive conditional heteroskedasticity (LARCH) models, e.g., Giraitis, Robinson, and Surgailis (2000); regime switching models, e.g. Hamilton (1989) and Sims and Zha (2006); and TVP models, e.g. Cogley and Sargent (2002) and Primiceri (2005). Unfortunately, none of the above-mentioned model classes seem to be directly usable for our purposes, because their nonlinearities do not match the nonlinearities of DSGE models solved with higher-order perturbation methods. We will discuss the precise relationship between the QAR and some of these models in more detail in Section 3.4.

There are two recent papers that construct nonlinear impulse response functions that can serve as a benchmark for the evaluation of nonlinear DSGE models. Barnichon and Matthes (2016) use a Gaussian mixture approximation of the moving average representation of a general nonlinear process to directly estimate impulse response functions which then could be compared to DSGE model response functions. Ruge-Murcia (2016) develop an indirect inference procedure that uses a GAR model and its impulse responses as an auxiliary model for the estimation of DSGE models solved with high order perturbation. His approach inherits the undesirable properties of GAR models but is able to exploit the fact that multivariate GAR models are easily estimable with equation-by-equation OLS.

In this paper we use so-called posterior predictive checks to evaluate a prototypical DSGE model. A general discussion of the role of predictive checks in Bayesian analysis can be found in Lancaster (2004) and Geweke (2005). Canova (1994) is the first paper that uses predictive checks to assess implications of a DSGE model. While Canova's (1994) checks were based on the prior predictive distribution, we use posterior predictive checks in this paper as, for instance, in Chang, Doh, and Schorfheide (2007). Finally, Abbritti and Fahr (2013) use a model with asymmetric wage adjustment costs and search and matching frictions to investigate the ability of the model to deliver nonlinearities, focusing on skewness and turning point statistics.

The remainder of the paper is organized as follows. In Section 2 we review the structure of second-order perturbation approximations of DSGE models. The QAR model is developed in Section 3. We discuss some of its theoretical properties as well as Bayesian inference. Estimates of the QAR model for U.S. data are presented in Section 4. The DSGE model with asymmetric price and wage adjustment costs is introduced in Section 5. The estimation and evaluation of the DSGE model is presented in Section 6. Finally, Section 7 concludes. The Online Appendix contains detailed derivations of the properties of the QAR model, as well as details of the Markov chain Monte Carlo (MCMC) methods employed in this paper.

2 DSGE Model Nonlinearities

Most estimated nonlinear DSGE models are solved with perturbation methods because they can be efficiently applied to models with a large state space. A DSGE model solved by second-order perturbation can be generically written as

$$\begin{aligned} c_{i,t} &= \psi_{1i}(\theta) + \psi_{2ij}(\theta)x_{j,t} + \psi_{3ij}(\theta)z_{j,t} \\ &\quad + \psi_{4ijk}(\theta)x_{j,t}x_{k,t} + \psi_{5ijk}(\theta)x_{j,t}z_{k,t} + \psi_{6ijk}(\theta)z_{j,t}z_{k,t} \\ x_{i,t+1} &= \zeta_{1i}(\theta) + \zeta_{2ij}(\theta)x_{j,t} + \zeta_{2ij}(\theta)z_{j,t} \\ &\quad + \zeta_{4ijk}(\theta)x_{j,t}x_{k,t} + \zeta_{5ijk}(\theta)x_{j,t}z_{k,t} + \zeta_{6ijk}(\theta)z_{j,t}z_{k,t} \\ z_{i,t+1} &= \xi_{2ij}(\theta)z_{j,t} + \xi_{3i}(\theta)\epsilon_{i,t+1}, \end{aligned} \tag{1}$$

where θ denotes the parameters of the model and the DSGE model variables are grouped into control variables $c_{i,t}$, e.g., consumption, endogenous state variables $x_{i,t}$, e.g., the capital stock, and exogenous state variables $z_{i,t}$, e.g., total factor productivity. The notation $a_{ijk}x_{j,t}x_{k,t}$ in (1) is shorthand for $\sum_{j=1}^{n}\sum_{k=1}^{n} a_{ijk}x_{j,t}x_{k,t}$. Since not all of the control and state variables are observable, it is common to augment the system by a measurement equation of the form

$$y_{i,t} = A_{1i}(\theta) + A_{2ij}(\theta)c_{j,t} + A_{3ij}(\theta)x_{j,t} + A_{4ij}(\theta)z_{j,t} + e_{i,t}, \tag{2}$$

where the $e_{i,t}$'s are measurement errors. Typically, the vector of observables y_t is composed of a subset of the state and control variables such that the A matrices are simple selection matrices.

Nonlinear features may arise endogenously or exogenously. Curvature in utility functions, adjustment cost function, and production functions can generate nonlinear decision rules of households and firms endogenously. An example of an exogenous nonlinearity is stochastic volatility in the exogenous shocks that generate business cycle fluctuations. In (1) the endogenous nonlinearity is captured by the quadratic functions of x_t and z_t that appear in the law of motion of the control variables $c_{i,t}$ and the endogenous state variables $x_{i,t+1}$. The representation assumes that there are no exogenous nonlinearities as the exogenous states $x_{i,t}^{exo}$ evolve according to a linear autoregressive process.

The objective of this paper is to propose an econometric method to assess the empirical adequacy of the nonlinear terms in the DSGE model solution (1). The DSGE model generates cross-coefficient restrictions between the first-order terms and the higher-order terms which may or may not be correctly specified. In principle, one could try to estimate two versions of the state-space model given by (1) and (2): a restricted version that imposes the functional relationship between the low-dimensional DSGE model parameter vector θ and the state-space coefficients $\psi(\cdot)$, $\zeta(\cdot)$, and $\xi(\cdot)$ and an unrestricted version in which the ψ's, ζ's, and ξ's are freely estimated. The discrepancy between the restricted and unrestricted estimates provides a measure of empirical adequacy. However, due to the large number of parameters and some inherent identification problems, the unrestricted estimation of the state-space system (1) and (2) is difficult to implement. In fact, even the literature that evaluates linearized DSGE models has by and large abstained from trying to estimate unrestricted state-space representations.

In the remainder of the paper we will develop a nonlinear time series model that mimics the nonlinearities of a DSGE model solved with a perturbation technique and then compare parameter estimates for this nonlinear time series model obtained from actual U.S. data and data simulated from a nonlinear DSGE model. If the DSGE model is well specified, then the estimates of the auxiliary models across these data sets ought to be very similar. This comparison is formalized as a Bayesian posterior predictive check. We proceed by providing a detailed description of the auxiliary time series model that is used for the DSGE model evaluation.

3 Quadratic Autoregressive Models

The most popular (and empirically successful) nonlinear time series models are those capturing time variation in the coefficients of linear time series models, e.g., Markov-switching models, time-varying coefficient models, GARCH models, stochastic volatility models. However, none of these models provide a good characterization of the nonlinearity generated endogenously by the DSGE model solution in (1). For this reason we develop a new class of nonlinear autoregressive time series models that are more closely tied to the DSGE model solution in (1).

We introduce the specification of a first-order quadratic autoregressive (QAR) model in Section 3.1. We subsequently characterize some of its important properties in Section 3.2 and describe the implementation of posterior inference in Section 3.3. Section 3.4 provides generalizations of the basic specification and discusses the relationship of our QAR models to other nonlinear time series models.

3.1 Specification of the QAR(1,1) Model

The starting point is a perturbation approximation of the solution of a nonlinear difference equation. Suppose that the difference equation takes the form[2]

$$y_t = f(y_{t-1}, \omega u_t), \quad u_t \stackrel{iid}{\sim} N(0,1). \tag{3}$$

We assume that the process characterized by (3) has a unique deterministic steady state that solves the equation $y_* = f(y_*, 0)$. Following the literature on perturbation methods, e.g. Holmes (1995) and Lombardo (2010), we construct an approximate solution of the form

$$y_t^* = y_t^{(0)} + \omega y_t^{(1)} + \omega^2 y_t^{(2)}. \tag{4}$$

It turns out that this solution is second-order accurate in the sense that

$$y_t = y_t^* + \mathcal{O}_p(\omega^3) \tag{5}$$

as $\omega \longrightarrow 0$.

To obtain y_t^*, we take a second-order Taylor expansion of the function f around $y_t = y_*$ and $\omega = 0$:

$$\begin{aligned} y_t - y_* \;&=\; f_y(y_{t-1} - y_*) + f_u \omega u_t \\ &\quad + \frac{1}{2} f_{y,y}(y_{t-1} - y_*)^2 + f_{y,u}(y_{t-1} - y_*)\omega u_t \\ &\quad + \frac{1}{2} f_{u,u}(\omega u_t)^2 + \text{higher-order terms}, \end{aligned} \tag{6}$$

[2]To simplify the derivation we assume that it is possible to solve for y_t as a function of y_{t-1} and ωu_t. More generally, one could also start from an equation of the form $f(y_t, y_{t-1}, \omega u_t)$.

where $f_{x,y}$ denotes the (x,y)'th derivative of f evaluated at the point $(y_t = y_*, \omega = 0)$. Substituting (4) into (6) and neglecting terms of order $\mathcal{O}_p(\omega^3)$, one obtains:

$$\begin{aligned} & y_t^{(0)} - y_* + \omega y_t^{(1)} + \omega^2 y_t^{(2)} \\ = \;& f_y \left(y_{t-1}^{(0)} - y_* + \omega y_{t-1}^{(1)} + \omega^2 y_{t-1}^{(2)} \right) + f_u \omega u_t \\ & + \frac{1}{2} f_{y,y} \left(y_{t-1}^{(0)} - y_* + \omega y_{t-1}^{(1)} + \omega^2 y_{t-1}^{(2)} \right)^2 \\ & + \frac{1}{2} f_{y,u} \left(y_t^{(0)} - y_* + \omega y_{t-1}^{(1)} + \omega^2 y_{t-1}^{(1)} \right) \omega u_t + \frac{1}{2} f_{u,u} \omega^2 u_t^2 + \mathcal{O}_p(\omega^3). \end{aligned} \tag{7}$$

We set $y_t^{(0)} = y_{t-1}^{(0)} = y_*$ and then match terms of the same ω-order on the left-hand-side and the right-hand-side of (7) to obtain the laws of motion for $y_t^{(1)}$ and $y_t^{(2)}$:

$$\begin{aligned} y_t^{(1)} &= f_y y_{t-1}^{(1)} + f_u u_t, \\ y_t^{(2)} &= f_y y_{t-1}^{(2)} + \frac{1}{2} f_{y,y} \left(y_{t-1}^{(1)} \right)^2 + \frac{1}{2} f_{y,u} y_{t-1}^{(1)} u_t + \frac{1}{2} f_{u,u} u_t^2. \end{aligned}$$

Notice that $y_t^{(1)}$ follows an AR(1) process and that conditional on $y_t^{(1)}$ the dynamics of $y_t^{(2)}$ are also linear. Substituting the expressions for $y_t^{(1)}$ and $y_t^{(2)}$ into (4) and collecting terms, we obtain that a second-order accurate perturbation approximation of the nonlinear difference equation (3) is given by the system:

$$\begin{aligned} y_t &= y_* + f_y (y_{t-1} - y_*) + \frac{1}{2} f_{y,y} \left(\omega y_{t-1}^{(1)} \right)^2 + \left(f_u + \frac{1}{2} f_{y,u} y_{t-1}^{(1)} \right) \omega u_t + \frac{1}{2} f_{u,u} \omega^2 u_t^2, \\ y_t^{(1)} &= f_y y_{t-1}^{(1)} + f_u u_t. \end{aligned} \tag{8}$$

We undertake a few additional modifications. We define $s_t \equiv \omega y_t^{(1)}$ and introduce the parameters

$$\phi_0 = y_*, \quad \phi_1 = f_y, \quad \tilde{\phi}_2 = \frac{1}{2} f_{y,y}, \quad \tilde{\gamma} = \frac{1}{2\omega} f_{y,u}, \quad \sigma = f_u \omega.$$

Moreover, we drop the term $\frac{1}{2} f_{u,u} u_t^2$ to obtain a conditional Normal distribution of y_t. Overall, this leads to the nonlinear state-space model:

$$\begin{aligned} y_t &= \phi_0 + \phi_1 (y_{t-1} - \phi_0) + \tilde{\phi}_2 \tilde{s}_{t-1}^2 + (1 + \tilde{\gamma} \tilde{s}_{t-1}) \sigma u_t \qquad (9) \\ \tilde{s}_t &= \phi_1 \tilde{s}_{t-1} + \sigma u_t, \quad u_t \overset{iid}{\sim} N(0,1). \end{aligned}$$

To complete the specification of the time series model we assume that the distribution of the initial values in period $t = -T_*$ have distribution F_{-T_*}, and that the innovations u_t are normally distributed:

$$(y_{-T_*}, \tilde{s}_{-T_*}) \sim F_{-T_*}, \quad u_t \stackrel{iid}{\sim} N(0,1). \tag{10}$$

We refer to (9) as the QAR(1,1) model. The first "1" indicates the number of lags in the conditional mean function and the second "1" denotes how many lags interact with the innovation u_t.

It is convenient to reparameterize the QAR(1,1) model as follows. Define ϕ_2, γ, and s_t such that

$$\phi_2 = \tilde{\phi}_2 \frac{\sigma^2}{1-\phi_1^2}, \quad \gamma = \frac{\sigma}{\sqrt{1-\phi_1^2}}\tilde{\gamma}, \quad \text{and} \quad s_t = \frac{\sqrt{1-\phi_1^2}}{\sigma}\tilde{s}_t. \tag{11}$$

Under the reparameterization the coefficients ϕ_2 and γ interact with standardized versions of s_{t-1}^2 and s_{t-1}, respectively. Thus, (9) becomes

$$\begin{aligned} y_t &= \phi_0 + \phi_1(y_{t-1} - \phi_0) + \phi_2 s_{t-1}^2 + (1 + \gamma s_{t-1})\sigma u_t, \\ s_t &= \phi_1 s_{t-1} + \sqrt{1-\phi_1^2}u_t, \quad u_t \stackrel{iid}{\sim} N(0,1). \end{aligned} \tag{12}$$

3.2 Important Properties of the QAR(1,1) Model

In order to appreciate two of the important implications of the recursively linear structure of the QAR(1,1) model given by (12) consider the alternative specification (omitting the constant term and the volatility dynamics) $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-1}^2 + u_t$, $0 < \phi_1 < 1$ and $\phi_2 > 0$. It is straightforward to verify that this specification has two steady states, namely, $y_*^{(1)} = 0$ and $y_*^{(2)} = (1-\phi_1)/\phi_2$. The second steady state arises as an artefact of the quadratic representation even if the underlying nonlinear model (3) only has a single steady state. Moreover, from writing $\Delta y_t = (-1 + \phi_1 + \phi_2 y_{t-1})y_{t-1} + u_t$ notice that the system becomes explosive if a large shock has pushed y_{t-1} above $y_*^{(2)}$. This explosiveness can arise regardless of the value of ϕ_1.

The multiplicity of steady states and the undesirable explosive dynamics have been pointed out in the context of second-order perturbation solutions of DSGE models by Kim, Kim, Schaumburg, and Sims (2008) who proposed an ex-post modification of quadratic autoregressive equations to ensure that unwanted higher-order terms do not propagate forward and generate explosive behavior not present in the underlying nonlinear model. This modification is called pruning in the literature.[3] Our derivation of the QAR model in Section 3.1 automatically generates a recursively linear structure with a unique steady state and non-explosive dynamics for suitably restricted values of ϕ_1. If the marginal distribution of s_{-T_*} is $N(0,1)$, then the process s_t, $t \geq -T_*$, is strictly stationary under the restriction $|\phi_1| < 1$. In turn, the vector process $z_t = [s_{t-1}, s_{t-1}^2, u_t]'$ is strictly stationary and we can rewrite the law of motion of y_t in (9) as

$$y_t = \phi_0 + \phi_1(y_{t-1} - \phi_0) + g(z_t) = \phi_0 + \sum_{j=0}^{\infty} \phi_1^j g(z_{t-j}). \tag{13}$$

This representation highlights that y_t is a stationary process. Since $g(z_t)$ is a nonlinear function of u_t and its history, the process is, however, not linear in u_t anymore. In fact, under the assumption that y_t was initialized in the infinite past ($T_* \longrightarrow -\infty$), we obtain the following representation:

$$y_t = \phi_0 + \sigma \sum_{j=0}^{\infty} \phi_1^j u_{t-j} + \sigma \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \left(\tilde{\gamma} \mathcal{I}\{l > j\} \phi_1^{l-j} + \tilde{\phi}_2 \sum_{k=0}^{\min\{j,l\}} \phi_1^{j+l-k} \right) u_{t-j} u_{t-l}. \tag{14}$$

(14) is a discrete-time Volterra series expansion, in which the Volterra kernels of order one and two are tightly restricted and the kernels of order larger than two are equal zero.[4] The recursively linear structure also facilitates the computation of higher-order moments of y_t. Further details are provided in the appendix.

Impulse responses defined as

$$IRF_t(h) = \mathbb{E}_t[y_{t+h} | u_t = 1] - \mathbb{E}_t[y_{t+h}]$$

are state dependent. For instance, for $h = 1$ we obtain

$$IRF_t(0) = \sigma(1 + \gamma s_{t-1}), \quad IRF_t(1) = \sigma \left(\phi_1(1 + \gamma s_{t-1}) + 2\phi_1\phi_2 \sqrt{1 - \phi_1^2}\, s_{t-1} \right). \tag{15}$$

[3] Lombardo (2010) constructs a pruned perturbation solution of a DSGE model directly rather than by ex-post adjustment. Lan and Meyer-Gohde (2013) solve DSGE models by constructing approximate second-order Volterra series expansions for the model variables, which also eliminates unwanted higher-order terms. Andreasen, Fernández-Villaverde, and Rubio-Ramírez (2017) derive the moments of observables from a general state-space representation for pruned DSGE models to facilitate moment-based estimation.

[4] The infinite sequences of coefficients on terms $\{u_{t-j}\}_{j \geq 0}$, $\{u_{t-j}u_{t-l}\}_{j \geq 0, l \geq 0}$, $\{u_{t-j}u_{t-l}u_{t-k}\}_{j \geq 0, l \geq 0, k \geq 0}$, etc. are called Volterra kernels.

Moreover, the model generates conditional heteroskedasticity. The conditional variance of y_t is given by

$$\mathbb{V}_{t-1}[y_t] = (1 + \gamma s_{t-1})^2 \sigma^2. \tag{16}$$

3.3 Posterior Inference for the QAR(1,1) Model

We estimate the QAR(1,1) model using Bayesian methods. Starting point is a joint distribution of data, parameters, and initial states:

$$p(Y_{0:T}, \theta, s_0) = p(Y_{1:T}|y_0, s_0, \theta)p(y_0, s_0|\theta)p(\theta),$$

where $p(Y_{1:T}|y_0, s_0, \theta)$ is a likelihood function that conditions on the initial values of y_0 and s_0, $p(y_0, s_0|\theta)$ characterizes the distribution of the initial values, and $p(\theta)$ is the prior density of the QAR(1,1) parameters, and $\theta = [\phi_0, \phi_1, \phi_2, \gamma, \sigma^2]'$. Since for large values of $|s_{t-1}|$ the term $(1 + \gamma s_{t-1})$ in (12) may become close to zero or switch signs, we replace it by

$$\left((1 - \vartheta) \exp\left[\frac{\gamma}{1 - \vartheta} s_{t-1} \right] + \vartheta \right), \tag{17}$$

where $1 + \gamma s_{t-1}$ is the first-order Taylor series expansion of (17). The exponential transformation guarantees non-negativity of the time-varying standard deviation and the constant ϑ provides some regularization that ensures that the shock standard deviation is strictly greater than $\sigma \exp(\vartheta)$ in all states of the world.

It is convenient to factorize the likelihood function into conditional densities as follows:

$$p(Y_{1:T}|y_0, s_0, \theta) = \prod_{t=1}^{T} p(y_t|y_{0:t-1}, s_0, \theta).$$

Given s_0 and θ it is straightforward to evaluate the likelihood function iteratively. Conditional on s_{t-1} the distribution of y_t is normal. The equation for y_t in (12) can be solved for u_t to determine s_t, which completes iteration t. In addition to the likelihood function, we need to specify an initial distribution $p(y_0, s_0|\theta)$. We assume that the system was in its steady state in period $t = -T_*$, that is, $y_{-T_*} = \phi_0$ and $s_{-T_*} = 0$. Based on iterating the original system (9) forward we compute a mean and variance for (y_0, s_0) and assume that the initial values are normally distributed. Further details of this initialization are provided in the Online Appendix. Since the dimension of θ is small, we use a single-block random-walk Metropolis-Hastings (RWMH) algorithm to generate draws from the posterior of θ.

3.4 Further Discussion

The QAR(1,1) model in (8) has a straightforward generalization which includes additional lag terms:

$$\begin{aligned} y_t &= \phi_0 + \sum_{l=1}^{p} \phi_{1,l}(y_{t-l} - \phi_0) + \sum_{l=1}^{p}\sum_{m=l}^{p} \tilde{\phi}_{2,lm} s_{t-l} s_{t-m} + \left(1 + \sum_{l=1}^{q} \tilde{\gamma}_l s_{t-l}\right) \sigma u_t, \qquad (18) \\ \tilde{s}_t &= \sum_{l=1}^{p} \phi_{1,l} s_{t-l} + \sigma u_t. \end{aligned}$$

We refer to (18) as QAR(p,q) model.[5] As in the standard AR(p) model, the stationarity of y_t is governed by the roots of the lag polynomial $1 - \sum_{l=1}^{p} \phi_{1,l} z^l$. The quadratic terms generate an additional $p(p+1)/2$ coefficients in the conditional mean equation for y_t. Since the number of coefficients grows at rate p^2, a shrinkage estimation method is required even for moderate values of p, in order to cope with the dimensionality problem. The QAR model can also be extended to the vector case, which is an extension that we are pursuing in ongoing research. The empirical analysis presented in Section 6 is based on the QAR(1,1) specification.

The QAR model is closely related, but not identical, to some of the existing nonlinear time series models. For $\gamma = 0$ the QAR(1,1) can be viewed as a pruned version of the generalized autoregressive model (GAR) discussed in Mittnik (1990), which augments the standard AR model by higher-order polynomials of the lagged variables. The conditional heteroskedasticity in (9) has a linear autoregressive structure. For $\phi_2 = 0$ our model is a special case of the LARCH model studied in Giraitis, Robinson, and Surgailis (2000). Since the conditional variance of y_t can get arbitrarily close to zero, likelihood-based estimation of LARCH models is intrinsically difficult. We circumvent these difficulties by introducing the exponential transformation in (17).

[5] If we start with a pth order nonlinear difference equation in (3), then we can arrive at a QAR(p,p) model.

Grier and Perry (1996, 2000) have estimated GARCH-M models on macroeconomic time series. GARCH-M models provide a generalization of the ARCH-M models proposed by Engle, Lilien, and Robins (1987) and can be written as

$$\begin{aligned} y_t &= \phi_0 + \phi_1(y_{t-1} - \phi_0) + \phi_2(\sigma_t^2 - \sigma^2) + \sigma_t u_t, \\ \sigma_t^2 - \sigma^2 &= \gamma_1(u_{t-1}^2 - \sigma^2) + \gamma_2(\sigma_{t-1}^2 - \sigma^2). \end{aligned}$$

Under suitable parameter restrictions y_t can be expressed as a nonlinear function of u_t and its lags. As in the case of the QAR model, y_t depends on the sequence $\{u_{t-j}^2\}$. In addition, the term $\sigma_t u_t$ introduces interactions between u_t and u_{t-j}^2, $j > 1$. However, coefficients on terms of the form $u_{t-j}u_{t-l}$, $j \neq l$ are restricted to be zero. From our perspective, the biggest drawback of the GARCH-M model is that nonlinear conditional mean dynamics are tied to the volatility dynamics: in the absence of conditional heteroskedasticity the dynamics of y_t are linear. The QAR model is much less restrictive in this regard: y_t can be conditionally homoskedastic ($\gamma = 0$) but at the same time have nonlinear conditional mean dynamics, that is, $\phi_2 \neq 0$.

As discussed in the introduction, there is a literature that uses TVP-VARs against which DSGE models can be evaluated. To illustrate the conceptual difference between TVP models and the QAR models introduced in our paper, consider a simplified version of the univariate process in (3) with additively separable error term:

$$y_t = f(y_{t-1}) + \sigma u_t. \tag{19}$$

Without loss of generality, we could write the conditional mean function as

$$f(y_{t-1}) = \phi(y_{t-1})y_{t-1} = \phi_t y_{t-1} \tag{20}$$

to express (19) as AR(1) model with a time-varying autoregressive parameter ϕ_t. In a TVP model the parameter ϕ_t evolves according to some exogenous law of motion, e.g.,

$$\phi_t = \phi_{t-1} + \sigma_\eta \eta_t, \quad \eta_t \sim N(0,1). \tag{21}$$

Because y_{t-1} does not feed back into ϕ_t, *a priori* the TVP nonlinearity is very different from the nonlinearity in the original model (19). *A posteriori* the estimates $\hat{\phi}_t$ may trace out the function $\phi(y_{t-1})$, but this estimate is fairly inefficient because it does not impose that $\phi(y_t) \approx \phi(y_{t+h})$ whenever $y_t \approx y_{t+h}$.

Rather than taking a Taylor approximation of $f(y_{t-1})$ around a steady state, as we did in the derivation of the QAR process, one could try to estimate the function $\phi(y_{t-1})$ – and hence $f(y_{t-1})$ – non-parametrically. In a Bayesian setting, this would involve specifying a prior distribution for the function $\phi(\cdot)$. In its most basic form, this prior could be generated by a Gaussian process with the following properties. Suppose $y_{(1)} < y_{(2)} < \cdots < y_{(K)}$. Then,

$$\phi(0) \sim N(\rho, \lambda), \quad \phi(y_{(k)}) - \phi(y_{(k-1)}) \sim N\big(0, y_{(k)} - y_{(k-1)}\big), \quad k = 2, \ldots, K$$

and the increments

$$\phi(y_{(2)}) - \phi(y_{(1)}), \quad \phi(y_{(3)}) - \phi(y_{(2)}), \quad \ldots \quad , \phi(y_{(K)}) - \phi(y_{(K-1)})$$

are independent. This prior could then be updated based on the sample $y_0, y_1, \ldots, y_T$. Such a non-parametric approach is less parsimonious than the local approximation approach pursued in this paper but could in principle capture the global dynamics of processes such as (3) or (19).

4 QAR Empirics

We begin the empirical analysis by fitting the QAR(1,1) model to per capita output growth, nominal wage growth, GDP deflator inflation, and federal funds rate data.[6] The choice of data is motivated by the DSGE model that is being evaluated subsequently. The DSGE model features potentially asymmetric wage and price adjustment costs and we will assess

[6]All series are quarterly and obtained from the FRED database of the Federal Reserve Bank of St. Louis. Output growth is the log difference of real GDP (GDPC96). We compute log differences of civilian noninstitutional population (CNP16OV) and remove a one-sided eight-quarter moving average to smooth population growth. The smoothed population growth series is used to obtain per capita GDP growth rates. Inflation is the log difference of the GDP deflator (GDPDEF). Nominal wage growth is the log difference of compensation per hour in the nonfarm business sector (COMPNFB). As interest rate we use quarterly averages of monthly effective federal funds rates (FEDFUNDS).

Table 1: Estimation Samples and Pre-Samples

Sample	Estimation Sample	Pre-Sample for Prior
1	1960:Q1 - 1983:Q4	1955:Q1 - 1959:Q4
2	1960:Q1 - 2007:Q4	1955:Q1 - 1959:Q4
3	1960:Q1 - 2012:Q4	1955:Q1 - 1959:Q4
4	1984:Q1 - 2007:Q4	1955:Q1 - 1983:Q4
5	1984:Q1 - 2012:Q4	1955:Q1 - 1983:Q4

whether the nonlinearities generated by this DSGE model are consistent with the nonlinearities in U.S. data. We report parameter estimates for the QAR model in Section 4.1 and explore the properties of the estimated models in Section 4.2.

4.1 Estimation of QAR(1,1) Model on U.S. Data

We estimate QAR(1,1) models for output growth, inflation, nominal wage growth, and interest rates using five different sample periods, which are summarized in Table 1. The longest sample spans the period from 1960:Q1 to 2012:Q4. This sample includes the high-inflation episode of the 1970s, the Volcker disinflation period, as well as the Great Recession of 2008-09 and the subsequent recovery. We then split this sample after 1983:Q4 into a pre-Great-Moderation sample that ranges from 1960:Q1 to 1983:Q4 and a post-Great-Moderation sample from 1984:Q1 to 2012:Q4. Since the 2008-09 recession involves large negative GDP growth rates which may be viewed as outliers, and federal funds rate has been at or near the lower bound of 0% since 2008, we consider two additional samples that exclude the Great Recession data and end in 2007:Q4.

To specify the prior distribution for the QAR parameters we use normal distributions for ϕ_0, ϕ_2, and γ. The autoregressive coefficient ϕ_1 is *a priori* also normally distributed, but the normal distribution is truncated to ensure stationarity of the QAR model. Finally, the prior distribution of σ is of the inverted gamma form. We use pre-sample information to parameterize the priors. The pre-sample periods for our five estimation samples are provided

Table 2: Prior Distribution for QAR(1,1) Models

	Pre-Sample	GDP Growth	Wage Growth	Inflation	Fed Funds Rate
ϕ_0	1955-59	$N\big(0.48, 2\big)$	$N\big(1.18, 2\big)$	$N\big(2.38, 2\big)$	$N\big(2.50, 2\big)$
	1955-83	$N\big(0.43, 2\big)$	$N\big(1.58, 2\big)$	$N\big(4.38, 2\big)$	$N\big(6.08, 2\big)$
ϕ_1	1955-59	$N^{\dagger}(0.36, 0.5)$	$N^{\dagger}(-0.02, 0.5)$	$N^{\dagger}(0.00, 0.5)$	$N^{\dagger}(0.66, 0.5)$
	1955-83	$N^{\dagger}(0.28, 0.5)$	$N^{\dagger}(0.34, 0.5)$	$N^{\dagger}(0.85, 0.5)$	$N^{\dagger}(0.94, 0.5)$
σ	1955-59	$IG(1.42, 4)$	$IG(0.82, 4)$	$IG(1.87, 4)$	$IG(0.58, 4)$
	1955-83	$IG(1.33, 4)$	$IG(0.88, 4)$	$IG(1.83, 4)$	$IG(1.45, 4)$
ϕ_2	1955-59	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$
	1955-83	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$
γ	1955-59	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$
	1955-83	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$	$N(0, 0.1)$

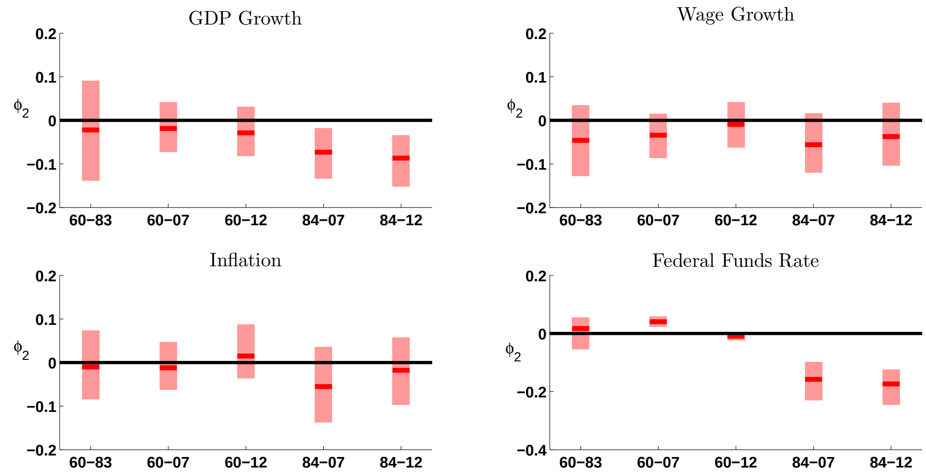
Notes: (†) The prior for ϕ_1 is truncated to ensure stationarity. The IG distribution is parameterized such that $p_{IG}(\sigma|\nu, s) \propto \sigma^{-\nu-1} e^{-\nu s^2/2\sigma^2}$.

in the last column of Table 1 and the marginal prior distributions for the QAR parameters are summarized in Table 2. The prior distributions for ϕ_1, the first-order autoregressive coefficient, are centered at the pre-sample first-order autocorrelations of the four time series. The inverse Gamma distribution of σ is centered at the residual standard deviation associated with the pre-sample estimation of an AR(1) model. Finally, the prior mean of ϕ_0 is specified such that the implied $\mathbb{E}[y_t]$ of the QAR(1,1) model corresponds to the pre-sample mean of the respective time series. The priors for ϕ_2 and γ are centered at zero and have a standard deviation of 0.1. Throughout the estimation the tuning constant ϑ in (17) is fixed at $\vartheta = 0.1$.

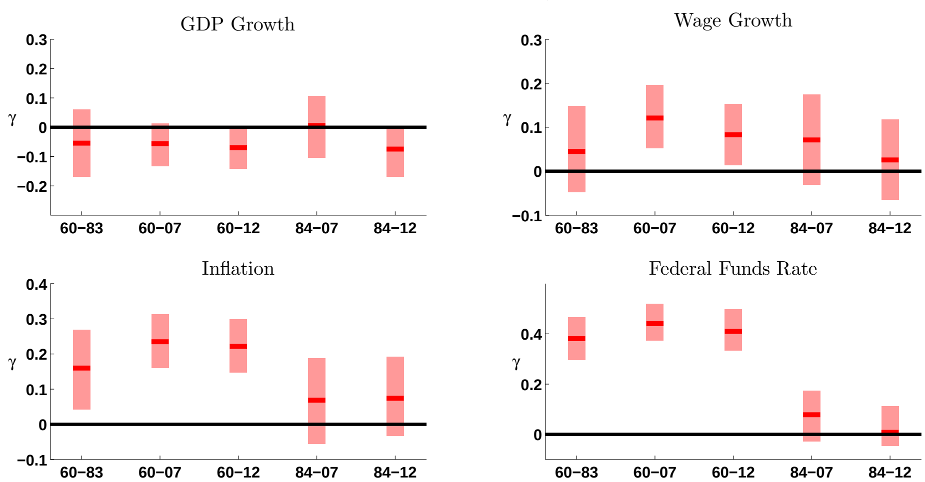
Figure 1 summarizes the posterior distributions of ϕ_2 and γ. The boxes represent the 90% credible intervals and the solid bars indicate posterior medians. Detailed estimation results for the remaining QAR(1,1) parameters are tabulated in the Online Appendix. The ϕ_2 posteriors for GDP growth using the three samples starting in 1960 are essentially centered at zero with the 90% credible interval covering both positive and negative values. The γ

Figure 1: Posterior Medians and Credible Intervals for QAR Parameters

Parameter ϕ_2



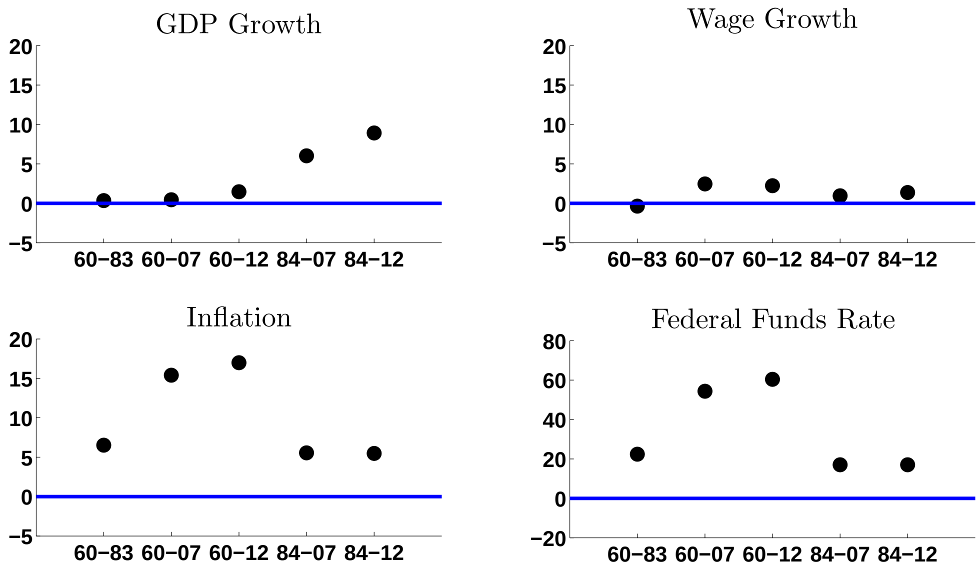
Parameter γ



Notes: The solid bars indicate posterior medians and the shaded boxes delimit 90% equal-tail-probability credible intervals.

posterior medians for the same samples are slightly negative, around -0.05, but the 90% credible sets also cover positive values, providing only some mild evidence for conditional variance dynamics. For the two post-Great Moderation samples the ϕ_2 estimates drop to about -0.1 and the credible set now excludes zero. The strongest evidence for nonlinearity in GDP growth is present in the 1984-2012 sample, which includes large negative growth rates of output during the Great Recession, in the form of $\phi_2 < 0$ and $\gamma < 0$. Nonlinearities in wages and inflation are reflected in positive estimates of γ. These nonlinearities are most

Figure 2: Log Marginal Data Density Differentials: QAR(1,1) versus AR(1)



Notes: The figure depicts log marginal data density differentials. A positive number provides evidence in favor of the QAR(1,1) specification.

pronounced for the 1960-2007 and the 1960-2012 samples. For the federal funds rate we obtain estimates of ϕ_2 near zero and estimates of γ of about 0.4 for samples that include the pre-1984 observations. For samples that start after the Great Moderation the pattern is reversed: the estimates of ϕ_2 are around -0.2 and the estimates of γ are close to zero. We will discuss the interpretation of these estimates in Section 4.2.

Figure 2 depicts log marginal likelihood differentials for the QAR(1,1) versus a linear autoregressive AR(1) model. The AR(1) models are estimated by setting $\phi_2 = \gamma = 0$ and using the same priors for ϕ_0, ϕ_1, and σ as in the estimation of the QAR(1,1) model. A positive value indicates evidence in favor for the nonlinear QAR(1,1). Under equal prior probabilities, the difference in log marginal data density between two models has the interpretation of log posterior odds. By and large, the marginal likelihood differentials favor the QAR(1,1) specification. The evidence in favor of the nonlinear specification is strongest for the federal funds rate. Marginal likelihood differentials range from 20 to 60. For output growth there is substantial evidence in favor of the QAR model for the post-Great Moderation samples, whereas for inflation large positive log marginal likelihood differentials are obtained for the

1960-2007 and the 1960-2012 samples. For wage growth the evidence in favor of the nonlinear specification is less strong: log marginal likelihood differentials are around 2.

4.2 Properties of the Estimated QAR Models

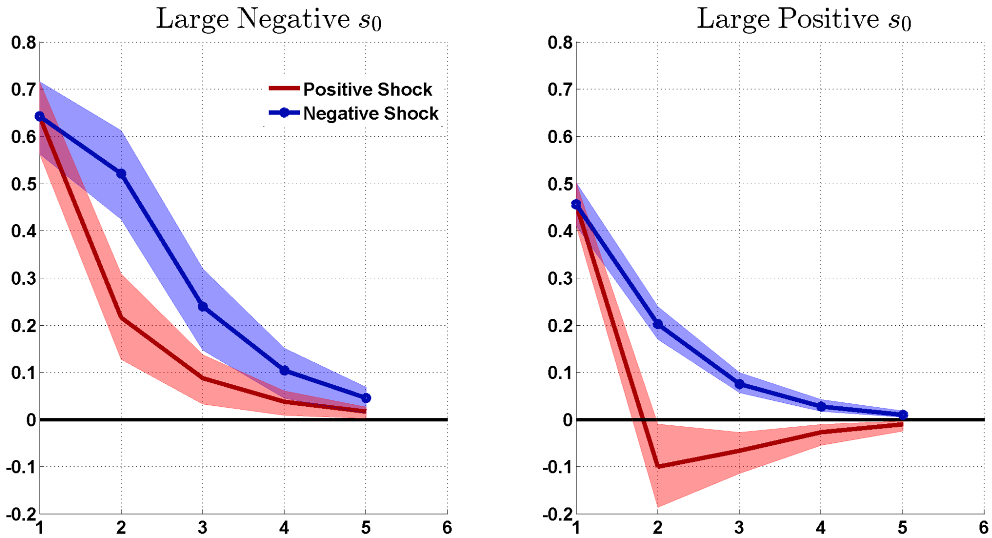
In this section we discuss what the nonlinearities we identified in the previous section mean for each variable. For ease of exposition, we focus on the subsample that "maximizes" the nonlinearities for each variable, which roughly corresponds to picking the subsample that has the largest marginal data density differential between the AR(1) and the QAR(1,1) models.

GDP Growth. Our results show that the posterior medians of ϕ_2 and γ for GDP growth are less than zero. The largest estimates (in absolute terms) are obtained for the 1984-2012 sample. As (16) shows, with a negative γ, the periods of below-mean growth (likely to be recessions) are also periods where volatility is higher, which is a well-known business cycle fact. A negative ϕ_2, along with a negative γ, implies that the response to shocks is a function of the initial state s_0. Using the formulas in (15), Figure 3 depicts the absolute responses of GDP growth to a negative and a positive one-standard deviation shock. In the left panel, we assume that the initial state s_0 takes on large negative values whereas the responses in the right panel condition on large positive s_0's.[7] This figure highlights that regardless of the initial state negative shocks are more persistent than positive shocks. Moreover, both shocks are more persistent in recessions. Combining these results, we deduce that multiple positive shocks are necessary to recover from a recession, while a small number of negative shocks can generate a recession. In other words output losses during recessions are sharp and recoveries are slow.

The impulse response findings are consistent with the time-series plot of GDP growth, which is provided in the top left panel of Figure 4. In this figure shaded areas indicate NBER recessions and the solid vertical line indicates the year 1984, which is the starting point of two of the five estimation samples. The unconditional mean of the variable is shown as a horizontal dashed line. Focusing on the post-1983 sample, the most extreme observations

[7]To obtain the s_0 for a given draw, we compute a two-period moving average to smooth the s_t series and use the minimum and the maximum values for this smoothed series.

Figure 3: Impulse Responses of GDP Growth (In Absolute Terms)



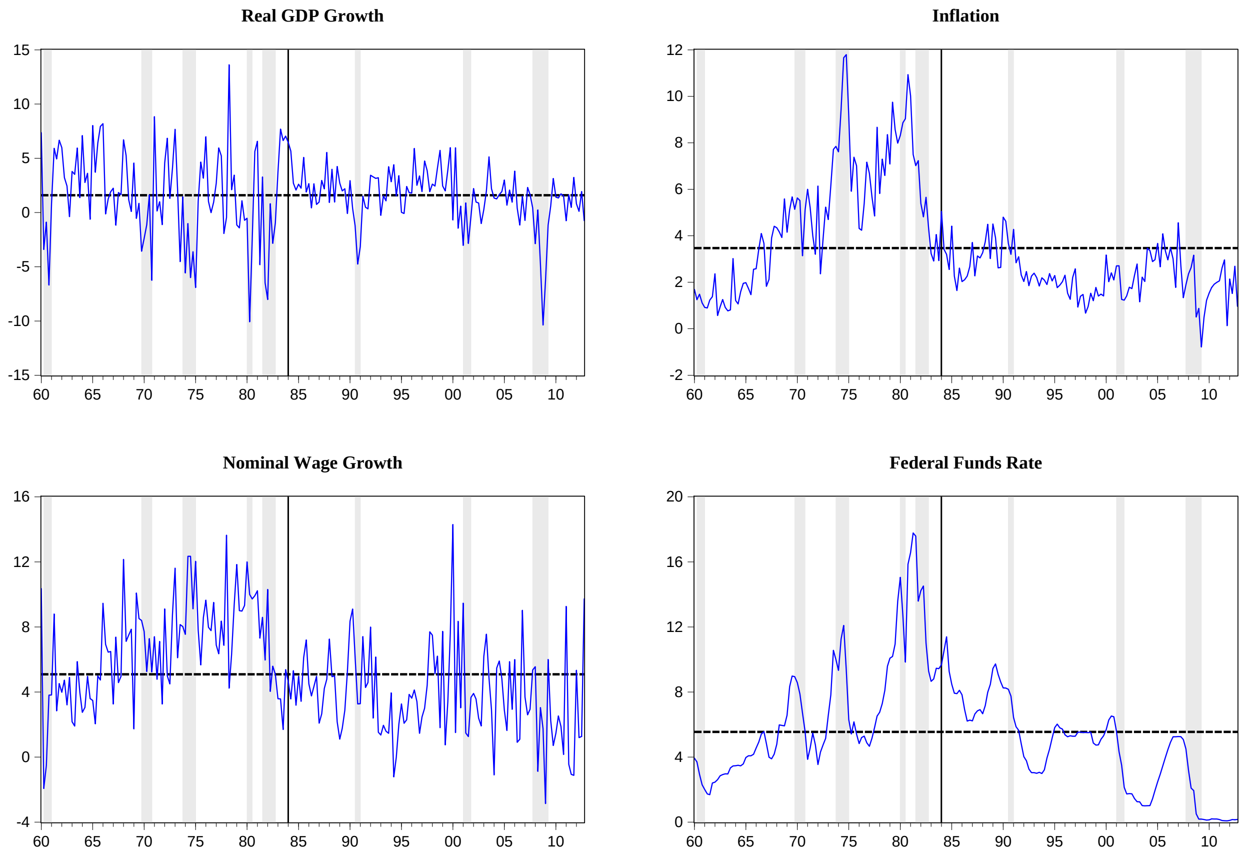
Notes: 1984-2012 sample. Solid and solid-dotted lines correspond to median impulse responses to one-standard-deviation shocks and shaded bands represent 60% credible intervals (equal tail probability). To initialize the latent state s_0 we compute two-quarter moving averages based on the states associated with the estimated QAR model and calculate the minimum and the maximum of the smoothed series. For the left panel (large negative s_0) the initialization is based on the minimum and in the right panel (large positive s_0) it is based on the maximum.

are all during recessions, confirming the effect of $\gamma < 0$. Looking at the quarters just prior to and just after NBER recessions, we see that the declines in GDP growth are always very sharp but the recoveries, defined as getting back to and staying at pre-recession level, take much longer.

It is easy to see why the nonlinearities identified in the samples starting in 1984 are not as pronounced in the samples that start in 1960. First, prior to 1984 there are more episodes of large positive GDP growth rates. These are, in absolute terms, as large as the negative growth rates observed between 1960 and 2012. Thus, recessions are not necessarily periods of higher volatility. Second, the recoveries from recessions are as sharp as the entries, not displaying the clear asymmetry in the later sample. These findings explain why a linear AR(1) is a good description of GDP growth pre-1984.[8]

[8]Qualitatively, our results for GDP growth are in line with findings by Brunner (1997), who estimated three nonlinear models for real Gross National Product (GNP). Based on a sample from 1947 to 1990 the author obtained strong evidence of countercyclical volatility, that is recessions are periods of high volatility.

Figure 4: Data



Notes: All variables are in annualized percentage units. Shaded areas indicate NBER recessions and the dashed horizontal line represents the sample mean of the series.

Inflation and Wage Growth. The nonlinearities in the inflation dynamics are most pronounced in the 1960-2007 sample with $\gamma > 0$ and $\phi_2 = 0$. Once again referring to the conditional variance formula in (16), we conclude that periods of above-mean inflation are associated with high volatility. In fact, the top right panel of Figure 4 shows that the period from 1970 to 1980 has high and volatile inflation. A similar conclusion can be reached in the post-1983 sample but to a lesser degree. The bottom left panel of Figure 4 shows that nominal wage growth displays properties similar to inflation. In the 1960-2007 sample, which

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Moreover, Brunner (1997) detects nonlinear conditional mean dynamics: according to the impulse responses the effects of a negative shock accumulate faster than those of a positive shock, in line with our findings. Similarly, McKay and Reis (2008) find that the brevity and violence of contractions and expansions are about equal in a sample that encompasses our longest sample, once again in line with our results.

is also the relevant one for nominal wage growth, volatility tends to be high when the level is high. Because nominal wage growth is more volatile than inflation, and there are many large negative observations, the estimate of γ is smaller for the former series.

Federal Funds Rate. The bottom right panel of Figure 4 shows the plot of the federal funds rate. Based on the QAR(1,1) estimation results, there are two samples with strong nonlinearities. In the 1960-2007 sample, we find a positive γ. As was the case for inflation and nominal wage growth, this is due to the observations from late 1960s to mid 1980s, which are typically above the unconditional mean and thus volatility is higher when the level is higher. For the 1984-2012 sample we find $\phi_2 < 0$ and $\gamma = 0$. In this period the extreme observations are equally likely to be positive or negative and thus $\gamma = 0$ is reasonable. $\phi_2 < 0$ implies that interest rate fall faster than they rise. This seems to be consistent with Federal Reserve's operating procedures in the post-1983 sample and it can have two separate explanations. First, the Federal Reserve may have an asymmetric policy rule, in which reactions to deviations from targets may depend on the sign of the deviation. This can happen, for example, if the Federal Reserve is risk averse and wants to avoid recessions: when GDP growth falls, the central bank is willing to cut the policy rate quickly, but when GDP growth starts to improve, it is reluctant to increase the policy rate immediately. Second, the variables that the Federal Reserve track may have asymmetries themselves. Given our finding that $\phi_2 < 0$ for GDP in this sample, the second explanation is certainly reasonable. There is some evidence about the first explanation as well. For example Dolado, Maria-Dolores, and Ruge-Murcia (2004) and Cukierman and Muscatelli (2008) estimate a non-linear Taylor rule using GMM and find that U.S. monetary policy is better characterized by a nonlinear policy rule after 1983, especially with respect to the reaction to output gap deviations.

To sum up, the estimation of QAR(1,1) models provides evidence of interesting and substantial nonlinearities in the U.S. macroeconomic time series. For the two samples that start in 1960 and extend beyond the 1990s the nonlinearities are reflected in the run-up in inflation in the 1970s, with spill-overs to nominal wage growth and the federal funds rate. In the shorter post-1983 samples, there are two important nonlinearities: the asymmetries in GDP growth, which is particularly pronounced if the 2008-09 recession is included in the sample, and the federal funds rate. In the remainder of this paper we examine whether a DSGE model with asymmetric adjustment costs for prices and wage can possibly generate the nonlinearities documented in this section.

5 A DSGE Model with Asymmetric Price and Wage Adjustment Costs

By now there exists a large empirical literature on the estimation of New Keynesian DSGE models, including small-scale models such as the one studied in Lubik and Schorfheide (2004) and Rabanal and Rubio-Ramírez (2005), as well as variants of the Smets and Wouters (2007) model. It turns out that in the absence of zero-lower-bound constraints on nominal interest rates, high degrees of risk aversion, large shocks, or exogenous nonlinearities such as stochastic volatility, these models do not generate strong nonlinearities, in the sense that first-order and higher-order perturbation approximations deliver very similar decision rules. In order to generate stronger nonlinearities that can be captured in higher-order perturbation approximations, we consider a model with potentially asymmetric price and wage adjustment costs that builds on Kim and Ruge-Murcia (2009).

The model economy consists of final goods producing firms, a continuum of intermediate goods producing firms, a representative household, and a monetary as well as a fiscal authority. The model replaces Rotemberg-style quadratic adjustment cost functions by linex adjustment cost functions, which can capture downward (as well as upward) nominal price and wage rigidities. Our model abstracts from capital accumulation. In the subsequent empirical analysis we examine whether the asymmetric adjustment costs can generate the observed nonlinearities in inflation and wage growth and whether the effects of asymmetric adjustment costs translate into nonlinearities in GDP growth and the federal funds rate. In a nutshell, asymmetric price adjustments should lead to asymmetric quantity adjustments. To the extent that the central bank sets interest rates in response to inflation and output movements, nonlinearities in the target variables may translate into nonlinearities in the interest rate itself.

Final Good Production. The perfectly competitive, final-goods-producing firms combine a continuum of intermediate goods indexed by $j \in [0,1]$ using the technology

$$Y_t = \left(\int_0^1 Y_t(j)^{1-\lambda_{p,t}} dj \right)^{\frac{1}{1-\lambda_{p,t}}}. \tag{22}$$

Here $1/\lambda_{p,t} > 1$ represents the elasticity of demand for each intermediate good. The firm takes input prices $P_t(j)$ and output prices P_t as given. Profit maximization and free entry imply that the demand for intermediate goods is

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-1/\lambda_{p,t}} Y_t. \tag{23}$$

The relationship between intermediate goods prices and the price of the final good is

$$P_t = \left(\int_0^1 P_t(j)^{\frac{\lambda_{p,t}-1}{\lambda_{p,t}}} dj \right)^{\frac{\lambda_{p,t}}{\lambda_{p,t}-1}}. \tag{24}$$

Intermediate Goods. Intermediate good j is produced by a monopolist who has access to the following production technology:

$$Y_t(j) = A_t H_t(j), \tag{25}$$

where A_t is an exogenous productivity process that is common to all firms. Intermediate-goods producers buy labor services $H_t(j)$ at a nominal price of W_t. Moreover, they face nominal rigidities in terms of price adjustment costs. These adjustment costs, expressed as a fraction of the firm's revenues, are defined by the linex function

$$\Phi_p(x) = \varphi_p \left(\frac{\exp\left(-\psi_p\left(x-\pi\right)\right) + \psi_p\left(x-\pi\right) - 1}{\psi_p^2} \right), \tag{26}$$

where we let $x = P_t(j)/P_{t-1}(j)$ and π is the steady state inflation rate associated with the final good. The parameter φ_p governs the overall degree of price stickiness and ψ_p controls the asymmetry of the adjustment costs. Taking as given nominal wages, the price of the final good, the demand schedule for intermediate products and technological constraints, firm j chooses its labor inputs $H_t(j)$ and the price $P_t(j)$ to maximize the present value of future profits

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty} \beta^s Q_{t+s|t} \left(\frac{P_{t+s}(j)}{P_{t+s}} \left(1 - \Phi_p\left(\frac{P_{t+s}(j)}{P_{t+s-1}(j)} \right) \right) Y_{t+s}(j) - \frac{1}{P_{t+s}} W_{t+s} H_{t+s}(j) \right) \right]. \tag{27}$$

Here, $Q_{t+s|t}$ is the time t value of a unit of the consumption good in period $t+s$ to the household, which is treated as exogenous by the firm.

Labor Packers. Labor services used by intermediate good producers are supplied by perfectly competitive labor packers. The labor packers aggregate the imperfectly substitutable labor services of households according to the technology:

$$H_t = \left(\int_0^1 H_t(k)^{1-\lambda_w} dk \right)^{\frac{1}{1-\lambda_w}}. \tag{28}$$

The labor packers choose their demand for each type of labor in order to maximize profits, taking as given input prices $W_t(k)$ and output prices W_t. Optimal labor demand is then given by:

$$H_t(k) = \left(\frac{W_t(k)}{W_t} \right)^{-\frac{1}{\lambda_w}} H_t. \tag{29}$$

Perfect competition implies that labor cost W_t and nominal wages paid to workers are related as follows:

$$W_t = \left(\int_0^1 W_t(k)^{\frac{\lambda_w - 1}{\lambda_w}} dk \right)^{\frac{\lambda_w}{\lambda_w - 1}}. \tag{30}$$

Households. Each household consists of a continuum of family members indexed by k. The family members provide perfect insurance to each other which equates their marginal utility in each state of the world. A household member of type k derives utility from consumption $C_t(k)$ relative to a habit stock. We assume that the habit stock is given by the level of technology A_t. This assumption ensures that the economy evolves along a balanced growth path even if the utility function is additively separable in consumption, leisure, and real money balances (omitted in the subsequent formula). The household member derives disutility from hours worked $H_t(k)$ and maximizes

$$\mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s \left(\frac{(C_{t+s}(k)/A_{t+s})^{1-\tau} - 1}{1-\tau} - \chi_H \frac{H_{t+s}^{1+1/\nu}(k)}{1+1/\nu} \right) \right], \tag{31}$$

where β is the discount factor, $1/\tau$ is the intertemporal elasticity of substitution, χ_H is a scale factor that determines the steady state hours worked. Moreover ν is the Frisch labor supply elasticity.

The household is a monopolist in the supply of labor services. As a monopolist, he chooses the nominal wage and labor taking as given the demand from the labor packer. We assume that labor market frictions induce a cost in the adjustment of nominal wages. Adjustment costs are payed as a fraction of labor income and they have the same linex structure assumed for prices

$$\Phi_w(x) = \varphi_w \left(\frac{\exp\left(-\psi_w \left(x - \gamma\pi\right)\right) + \psi_w \left(x - \gamma\pi\right) - 1}{\psi_w^2} \right), \tag{32}$$

where $x = W_t(k)/W_{t-1}(k)$ and $\gamma\pi$ is the growth rate of nominal wages where γ is the average growth rate of technology as we define below. Beside his labor choices, the household member faces a standard consumption/saving trade-off. He has access to a domestic bond market where nominal government bonds $B_t(k)$ are traded that pay (gross) interest R_t. Furthermore, he receives aggregate residual real profits $D_t(k)$ from the firms and has to pay lump-sum taxes T_t. Thus, the household's budget constraint is of the form

$$\begin{aligned} & P_t C_t(k) + B_t(k) + T_t \\ = \;& W_t(k) H_t(k) \left(1 - \Phi_w \left(\frac{W_t(k)}{W_{t-1}(k)} \right) \right) + R_{t-1} B_{t-1}(k) + P_t D_t(k) + P_t SC_t, \end{aligned}$$

where $SC_t(k)$ is the net cash inflow that household k receives from trading a full set of state-contingent securities. We denote the the Lagrange multiplier associated with the budget constraint by λ_t. The usual transversality condition on asset accumulation applies, which rules out Ponzi schemes.

Monetary and Fiscal Policy. Monetary policy is described by an interest rate feedback rule of the form

$$R_t = R_t^{*\,1-\rho_R} R_{t-1}^{\rho_R} e^{\epsilon_{R,t}}, \tag{33}$$

where $\epsilon_{R,t}$ is a monetary policy shock and R_t^* is the (nominal) target rate. Our specification of R_t^* implies that the central bank reacts to inflation and deviations of output growth from its equilibrium steady state γ:

$$R_t^* = r\pi^* \left(\frac{\pi_t}{\pi^*} \right)^{\psi_1} \left(\frac{Y_t}{\gamma Y_{t-1}} \right)^{\psi_2}. \tag{34}$$

Here r is the steady state real interest rate, π_t is the gross inflation rate defined as $\pi_t = P_t/P_{t-1}$, and π^* is the target inflation rate, which in equilibrium coincides with the steady state inflation rate. The fiscal authority consumes a fraction ζ_t of aggregate output Y_t, where $\zeta_t \in [0, 1]$ follows an exogenous process. The government levies a lump-sum tax (subsidy) to finance any shortfalls in government revenues (or to rebate any surplus).

The model economy is perturbed by four exogenous processes. Aggregate productivity evolves according to

$$\ln A_t = \ln \gamma + \ln A_{t-1} + \ln z_t, \quad \text{where} \quad \ln z_t = \rho_z \ln z_{t-1} + \epsilon_{z,t}. \tag{35}$$

Thus, on average technology grows at the rate γ and z_t captures exogenous fluctuations of the technology growth rate. Define $g_t = 1/(1-\zeta_t)$. We assume that

$$\ln g_t = (1-\rho_g)\ln g + \rho_g \ln g_{t-1} + \epsilon_{g,t}. \tag{36}$$

The inverse demand elasticity for intermediate goods evolves according to a first order autoregressive processes in logs:

$$\ln \lambda_{p,t} = (1-\rho_p)\ln \lambda_p + \rho_p \ln \lambda_{p,t-1} + \epsilon_{p,t}. \tag{37}$$

Finally, the monetary policy shock $\epsilon_{R,t}$ is assumed to be serially uncorrelated. The four innovations are independent of each other at all leads and lags and are normally distributed with means zero and standard deviations σ_z, σ_g, σ_p, and σ_R, respectively.

6 Estimation and Evaluation of DSGE Model

The estimation and evaluation of the DSGE model proceeds in three steps. In Section 6.1 the DSGE model is estimated for two samples: 1960-2007 and 1984-2007, using the same series that were studied in Section 4. In Section 6.2 we use posterior predictive checks that are based on posterior mean estimates of QAR(1,1) parameters to assess whether the nonlinearities captured in the second-order approximated DSGE model are commensurate with the nonlinearities captured by the QAR(1,1) model. Finally, we assess the effect of adjustment cost asymmetries on the model's ability to generate nonlinear inflation and wage growth dynamics in Section 6.3.

6.1 DSGE Model Estimation on U.S. Data

We now estimate the DSGE model based on the same data that was used to estimate the QAR(1,1) models in Section 4. The marginal prior distributions for the DSGE model parameters are summarized in columns 2 to 4 of Table 3. We use pre-sample evidence to quantify *a priori* beliefs about the average growth rate of the economy, as well as average inflation and real interest rates. We use the same priors for both samples. The prior mean for τ implies a risk aversion coefficient of 2. Our prior for the Frisch labor supply elasticity covers some of the low values estimated in the microeconometrics literature as well as a value of 2 advocated in the real-business-cycle literature based on steady-state considerations. The prior for the price-adjustment-cost parameter φ_p is specified indirectly through a prior for the slope $\kappa(\varphi_p)$ of the New Keynesian Phillips curve. This prior encompasses values that imply an essentially flat as well as a fairly steep Phillips curve. The prior for the wage rigidity is directly specified on φ_w and spans values in the range from 0 to 30. The priors for the asymmetry parameters ψ_p and ψ_w are centered at zero (symmetric adjustment costs) and have a large variance, meaning that the asymmetries could potentially be strong. We do not restrict the signs of ψ_p and ψ_w, i.e., allowing *a priori* for both downward and upward price and wage rigidities. The priors for the monetary policy rule coefficients are centered at 1.5 (reaction to inflation), 0.2 (output growth), and 0.5 (interest rate smoothing). Finally, we use priors for the parameters associated with the exogenous shock processes then generate *a priori* reasonable magnitudes for the persistence and volatility of the observables.

The DSGE model presented in Section 5 is solved using a second-order approximation, which leads to a nonlinear state-space representation. As in Fernández-Villaverde and Rubio-Ramírez (2007), we use bootstrap particle filter to evaluate the likelihood function of the DSGE model. To facilitate the likelihood evaluation with the particle filter, the measurement equation contains mean-zero *iid* Gaussian measurement errors. The measurement error variances are set equal to 10% of the sample variances of GDP growth, inflation, interest rates, and nominal wage growth. Posterior inference is implemented with a single-block RWMH algorithm, described in detail in An and Schorfheide (2007) and Herbst and Schorfheide (2015). Theoretical convergence properties of so-called particle MCMC approaches are established in Andrieu, Doucet, and Holenstein (2010).

Posterior summary statistics for the DSGE model parameters are reported in Table 3. The most interesting and important estimates are the ones of the asymmetry parameters in

Table 3: Posterior Estimates for DSGE Model Parameters

		Prior		Posterior			
				1960:Q1-2007:Q4		1984:Q1-2007:Q4	
Parameter	Distribution	Para (1)	Para (2)	Mean	90% Interval	Mean	90% Interval
$400\left(\frac{1}{\beta}-1\right)$	Gamma	2.00	1.00	0.47	[0.08, 1.04]	1.88	[0.47, 3.01]
π^A	Gamma	3.00	1.00	3.19	[2.57, 3.84]	3.34	[2.44, 4.32]
γ^A	Gamma	2.00	1.50	2.04	[1.57, 2.77]	1.98	[1.59, 2.36]
τ	Gamma	2.00	1.00	4.83	[2.75, 7.28]	4.10	[2.35, 6.06]
ν	Gamma	0.50	1.00	0.37	[0.21, 0.52]	0.10	[0.05, 0.17]
$\kappa(\varphi_p)$	Gamma	0.30	0.20	0.02	[0.01, 0.04]	0.21	[0.12, 0.35]
φ_w	Gamma	15.0	7.50	18.7	[8.47, 38.1]	11.7	[5.34, 20.2]
ψ_w	Uniform	-200	200	67.4	[33.2, 99.5]	59.4	[21.7, 90.9]
ψ_p	Uniform	-300	300	150	[130, 175]	165	[130, 192]
ψ_1	Gamma	1.50	0.50	1.77	[1.51, 2.12]	2.57	[1.93, 3.26]
ψ_2	Gamma	0.20	0.10	1.41	[0.97, 1.85]	0.79	[0.42, 1.18]
ρ_r	Beta	0.50	0.20	0.81	[0.23, 0.72]	0.73	[0.64, 0.80]
ρ_g	Beta	0.80	0.10	0.95	[0.92, 0.98]	0.96	[0.94, 0.98]
ρ_z	Beta	0.20	0.10	0.48	[0.23, 0.72]	0.07	[0.01, 0.20]
ρ_p	Beta	0.60	0.20	0.89	[0.86, 0.94]	0.90	[0.76, 0.98]
$100\sigma_r$	InvGamma	0.20	2.00	0.17	[0.14, 0.21]	0.17	[0.12, 0.23]
$100\sigma_g$	InvGamma	0.75	2.00	0.88	[0.58, 1.29]	0.83	[0.49, 1.30]
$100\sigma_z$	Beta	0.75	2.00	0.44	[0.31, 0.62]	0.47	[0.38, 0.56]
$100\sigma_p$	Beta	0.75	2.00	2.62	[0.46, 7.23]	6.54	[4.56, 9.37]

Notes: $1/g$ is fixed at 0.85. Para (1) and Para (2) list the means and the standard deviations for Beta, Gamma, and Normal distributions; the upper and lower bound of the support for the Uniform distribution; and s and ν for the Inverse Gamma distribution, where $p_{\mathcal{IG}}(\sigma|\nu, s) \propto \sigma^{-\nu-1}e^{-\nu s^2/2\sigma^2}$. The effective prior is truncated at the boundary of the determinacy region. As 90% credible interval we are reporting the 5th and 95th percentile of the posterior distribution.

the price and wage adjustment cost function. The wage and price rigidity estimates differ substantially across subsamples. For instance, the estimated slope of the New Keynesian Phillips curve is 0.02 for the 1960-2007 sample, whereas it increases to 0.2 for the post-1983 sample. Likewise, the estimated wage rigidity is larger over the long sample. The positive estimates of ψ_p and ψ_w imply that it is more expensive to lower prices and wages than to raise them and that the asymmetry in prices is more pronounced than in wages. The asymmetry of the adjustment costs is more pronounced for prices ($\hat{\psi}_p$ equals 150 and 165, respectively) than for wages ($\hat{\psi}_w$ equals 67 and 59, respectively).

Compared to the estimates reported by Kim and Ruge-Murcia (2009) and Abbritti and Fahr (2013) who report estimates of $\hat{\psi}_w = 3,844$ and $\hat{\psi}_w = 24,700$, respectively, our estimates of the ψ_w's are considerably smaller.[9] In our experience such large values of ψ_w lead to a clear deterioration of the model's ability to track U.S. data. Moreover, the second-order solution of the DSGE model relies on a third-order approximation of the linex cost function which becomes very inaccurate for large values of ψ. In particular, we found that for values of ψ_w above 500 the the adjustment costs for large positive wage changes (that lie in the support of the ergodic distribution) would become negative due to the polynomial approximation of the linex function.

We estimate the risk-aversion parameter τ to be fairly large, around 4, and the Frisch labor supply elasticity to be fairly low, ranging from 0.1 to 0.4. The estimates of ν are in line with those reported in Ríos-Rull, Schorfheide, Fuentes-Albero, Kryshko, and Santaeulalia-Llopis (2012). The policy rule coefficient estimates are similar to the ones reported elsewhere in the DSGE model literature. The coefficient ψ_1 on inflation is larger for the post-1983 sample, which is consistent with the view that after the Volcker disinflation the Federal Reserve Bank has responded more aggressively to inflation movements. The government spending shock, which should be viewed as a generic demand shock, is the most persistent among the serially correlated exogenous shocks: ρ_g is approximately 0.95. The estimated autocorrelation ρ_z of technology growth shock, which generates most of the serial correlation in output growth rates, drops from 0.48 for the long sample to 0.07 for the post-1983 sample.

6.2 Posterior Predictive Checks

We proceed by examining whether QAR(1,1) parameter estimates obtained from data that are simulated from the estimated DSGE model are similar to the estimates reported in Section 4 computed from actual data. This comparison is formalized through a posterior

[9] Kim and Ruge-Murcia (2009) estimated their DSGE model Simulated Method of Moments (SMM). While they also used consumption and hours worked data in their estimation, the SMM objective function only includes second moments. The authors find that the covariance of consumption and hours worked, respectively, with wage growth plays a crucial role for their estimation. Abbritti and Fahr (2013) use a calibration approach to parameterize their model. Given their preferred calibration of the exogenous technology, discount-factor, and monetary-policy shocks, they find that a very large value of ψ_w is needed to match the volatility and skewness of wage growth observed in the data.

predictive check. The role of posterior predictive checks in Bayesian analysis is discussed in the textbooks by Lancaster (2004) and Geweke (2005) and reviewed in the context of the evaluation of DSGE models in Del Negro and Schorfheide (2011). The posterior predictive checks is implemented with the following algorithm.

Posterior Predictive Check. Let $\theta^{(i)}$ denote the i'th draw from the posterior distribution of the DSGE model parameter θ.

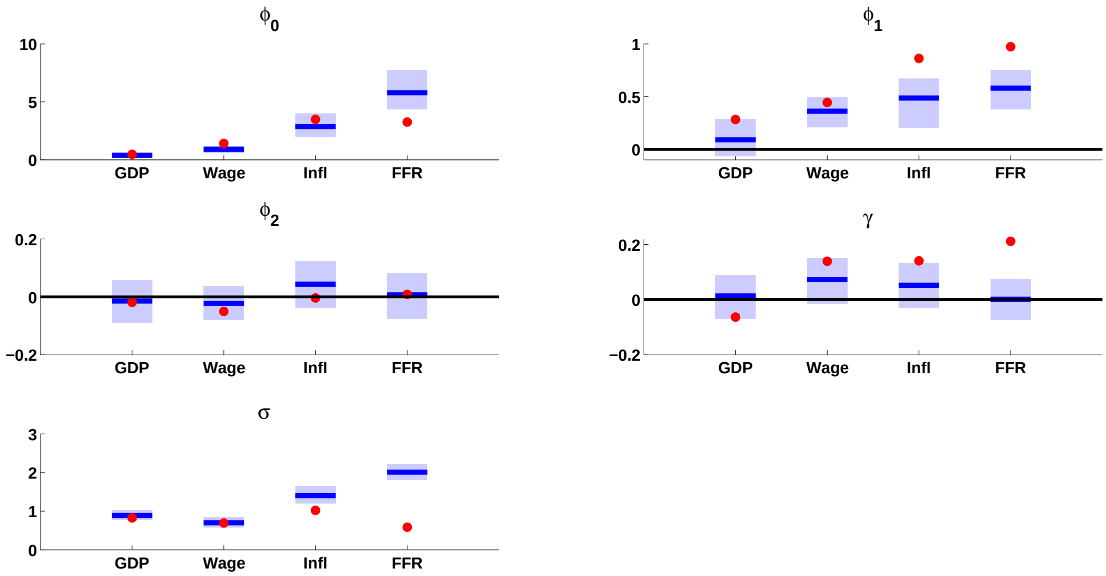
1. For $i = 1$ to n:
   (i) Conditional on $\theta^{(i)}$ simulate a pre-sample of length T_0 and an estimation sample of size T from the DSGE model. The second-order approximated DSGE model is simulated using the pruning algorithm described in Kim, Kim, Schaumburg, and Sims (2008). A Gaussian *iid* measurement error is added to the simulated data. The measurement error variance is identical to the one imposed during the estimation of the DSGE model. Denote the simulated data by $Y^{(i)}_{-T_0+1:T}$.
   (ii) Based on the simulated trajectory $Y^{(i)}_{-T_0+1:T}$ estimate the QAR(1,1) model as described in Section 4.1. The prior for the QAR(1,1) parameters is elicited from the presample $Y^{(i)}_{-T_0+1:0}$ and the posterior is based on $Y^{(i)}_{1:T}$. Denote the posterior median estimates of the QAR parameters by $\mathcal{S}\big(Y^{(i)}_{-T_0+1:T}\big)$.

2. The empirical distribution of $\big\{\mathcal{S}\big(Y^{(i)}_{-T_0+1:T}\big)\big\}_{i=1}^{n}$ approximates the posterior predictive distribution of $\mathcal{S}|Y_{-T_0+1:T}$. Examine how far the actual value $\mathcal{S}(Y_{1:T})$, computed from U.S. data, lies in the tail of its predictive distribution. $\square$

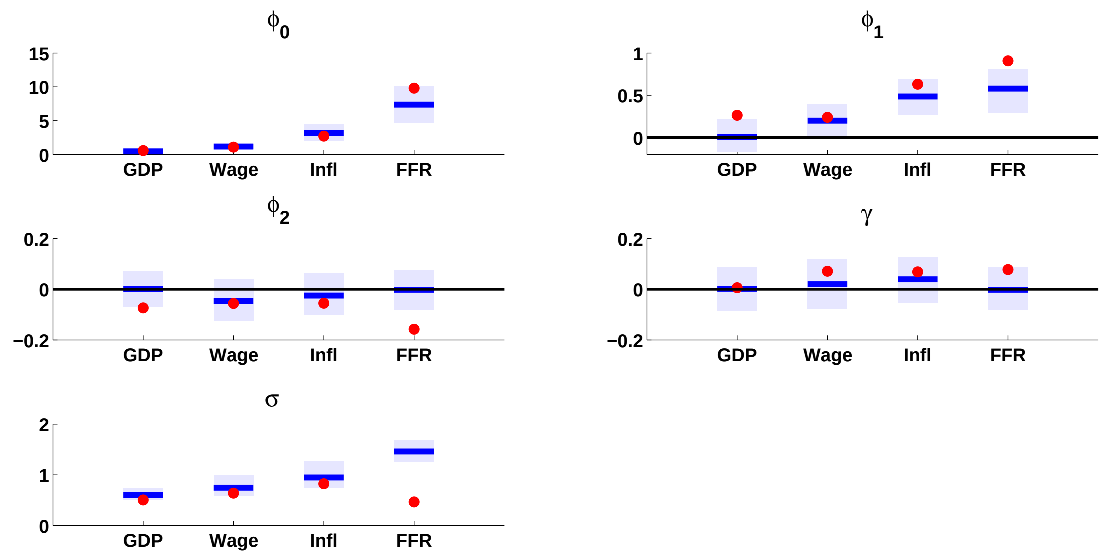
The predictive check is carried out for each QAR(1,1) parameter estimate separately. The results are summarized in Figure 5. The top panel corresponds to the 1960-2007 sample, whereas the bottom panel contains the results from the 1984-2007 sample. The dots signify the posterior median estimates obtained from U.S. data and correspond to the horizontal bars in Figure 1. The rectangles delimit the 90% credible intervals associated with the posterior predictive distributions and the solid horizontal bars indicate the medians of the predictive distributions. The length of the credible intervals reflects both parameter uncertainty, i.e., the fact that each trajectory $Y^{(i)}_{-T_0+1:T}$ is generated from a different parameter draw $\theta^{(i)}$, and

Figure 5: Predictive Checks Based on QAR(1,1)Estimates

1960-2007 Sample



1984-2007 Sample



Notes: Dots correspond to posterior median estimates from U.S. data. Solid horizontal lines indicate medians of posterior predictive distributions for parameter estimates and the boxes indicate 90% credible associated with the posterior predictive distributions.

sampling uncertainty, meaning that if one were to hold the parameters θ fixed, the variability in the simulated finite-sample trajectories generates variability in posterior mean estimates. Because the posterior variance of the DSGE model parameters is fairly small, these intervals mostly capture sampling variability. Accordingly, they tend to be larger in the bottom panel (short sample) than in the top panel (long sample).

By and large, the QAR parameter estimates for output growth, wage growth, and inflation from model-generated data are very similar to the ones obtained from actual data – in the sense that most of the actual estimates do not fall far in the tails of their respective posterior predictive distributions. Only interest rates exhibit large discrepancies between actual and model-implied estimates of the QAR(1,1) parameters.

Overall, the estimated DSGE model does not generate very strong nonlinearities. Posterior predictive distributions for $\hat{\phi}_2$ and $\hat{\gamma}$ typically cover both positive and negative values. The only exceptions are the predictive distributions for wage growth and inflation $\hat{\gamma}$ conditional on the 1960-2007 sample, which imply that $\hat{\gamma}$ is positive. Recall from Table 3 that for this sample we estimate sizeable adjustment costs ($\hat{\kappa} = 0.02$ and $\hat{\varphi}_w = 18.7$). Moreover, the asymmetry parameter estimates are substantially larger than zero: $\hat{\psi}_p = 150$ and $\hat{\psi}_w = 67.4$. The model-implied positive estimates of γ imply that high inflation and wage-growth rates are associated which high levels of volatility, which describes the experience of the U.S. economy in the 1970s and early 1980s. However, the nonlinear inflation and wage dynamics do not generate any spillovers to nonlinearities in GDP growth or the interest rate. Figure 5 indicates that the predictive distribution for the corresponding $\hat{\phi}_2$ and $\hat{\gamma}$ are centered at zero. For the 1984-2007 sample the overall magnitude of the estimated adjustment costs are smaller, which flattens the adjustment cost functions, makes the asymmetries less important for equilibrium dynamics, and shifts the predictive distribution for the inflation and wage growth $\hat{\gamma}$'s toward zero.

There are two types of nonlinearities present in the data that the estimated DSGE model does not predict. First, for the short sample $\hat{\phi}_2$ for GDP growth is negative, because the post-1983 sample exhibits pronounced drop in output growth during the recessions but does not feature positive growth rates of similar magnitudes in early parts of expansions. Second, the interest rate exhibits strong nonlinearities in the data, i.e., a large positive $\hat{\gamma}$ in the 1960-2007 sample and a large negative $\hat{\phi}_2$ in the 1984-2007 sample, that the DSGE model is unable to reproduce.

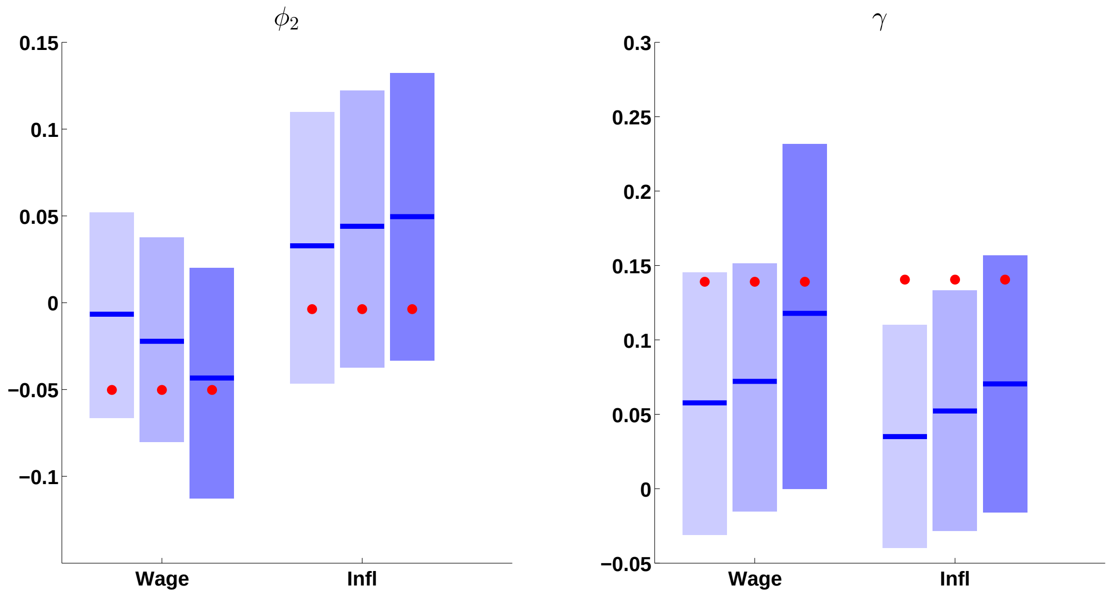
To sum up, of the nonlinearities we identified in Section 4, the only ones the DSGE model seems to be able to deliver are the conditional heteroskedasticity in inflation and nominal wage growth. It is able to do so relying on the asymmetric adjustment costs which penalize downward adjustments more than upward adjustments. However, while ex-ante reasonable, these asymmetries in prices do not spill over to quantities. Moreover, since the interest-rate feedback rule in the model does not feature any asymmetries, which would result from the central bank having an asymmetric loss function, and since there are no asymmetries in GDP growth in the model, the policy instrument does not display the asymmetry we identified in the data.

6.3 The Role of Asymmetric Adjustment Costs

To further study the role of asymmetric adjustment costs in generating nonlinear wage and inflation dynamics we repeat the predictive checks based on $\hat{\phi}_2$ and $\hat{\gamma}$ for alternative choices of ψ_p and ψ_w. We focus on the 1960-2007 sample because the nonlinearities are more pronounced than in the post-1983 sample. For each draw $\theta^{(i)}$ from the posterior distribution of the DSGE model parameters, we replace $\psi_p^{(i)}$ and $\psi_w^{(i)}$ by alternative values $\bar{\psi}_p$ and $\bar{\psi}_w$. In particular, we consider an elimination of the asymmetries, i.e., $\bar{\psi}_p = \bar{\psi}_w = 0$ and an increase to $\bar{\psi}_p = \bar{\psi}_w = 300$. The results are plotted in Figure 6. A decrease of the asymmetry in the adjustment costs moves the predictive distributions of $\hat{\phi}_2$ and $\hat{\gamma}$ toward zero, whereas an increase shifts them further away from zero. Relative to the overall width of the predictive intervals the location shifts are fairly small. This highlights that a precise measurement of nonlinearities is very difficult using quarterly observations.

For nominal wage growth the increase in the asymmetry parameters essentially eliminates the gap between the median of the posterior predictive distributions for $\hat{\phi}_2$ and $\hat{\gamma}$ and the estimates obtained from actual data, which are -0.05 and 0.14, respectively. For inflation the median of the predictive distributions for $\hat{\phi}_2$ and $\hat{\gamma}$ shift slightly upward, toward 0.05 and 0.06, respectively. This implies that the actual value of $\hat{\phi}_2$ lies further in the tail of the predictive distribution if ψ_w is increased, whereas the actual value of $\hat{\gamma}$ is less far in the tails. While an increase of ψ_w improves the outcome of the predictive check constructed from

Figure 6: Effect of Adjustment Costs on Nonlinearities



Notes: 1960-2007 sample. Box plots of posterior predictive distribution for ϕ_2 and γ estimates for different parameter values of the adjustment cost functions. No Asymmetric Costs is $\psi_p = \psi_w = 0$ (light blue); High Asymmetric Costs is $\psi_p = \psi_w = 300$ (dark blue). Large Dots correspond to posterior median estimates based on U.S. data.

the QAR parameter estimates for nominal wage growth, judging from the overall posterior distribution, the increased asymmetries lead to a deterioration of fit in other dimensions of the model, which is why the posterior estimates for ψ_p and ψ_w are only about 150 and 68, respectively.

7 Conclusion

Over the past decade, the DSGE model literature has shifted its focus toward models with explicit nonlinearities, including stochastic volatility, the effective lower bound on nominal interest rates, and occasionally-binding financial constraints. While there has been a lot of progress with respect to model solution and estimation, the evaluation of DSGE model nonlinearities is lagging behind and more methodological and empirical work is needed: Do the nonlinearities built into the model match the nonlinearities in the data? Do nonlinearities in one block of the model correctly propagate to other parts of the model? Are the DSGE model nonlinearities strong enough to be measurable in samples of a realistic size? Does the introduction of nonlinearities improve the model fit during "crisis" periods while at the same time deteriorate its fit during "normal" periods? How can one distinguish nonlinear dynamics from structural change?

While this paper has by no means provided answers to all of the above questions, it has developed and applied novel tools for the assessment of DSGE model nonlinearities and made an important step toward tackling these challenges. Building on the specification of generalized autoregressive models, bilinear models, and LARCH models, we have used a perturbation approximation of a nonlinear difference equation to obtain a new class of nonlinear time series models that can be used to assess nonlinear DSGE models. We use these univariate QAR(1,1) models to identify nonlinearities in the U.S. data and to construct predictive checks to assess a DSGE model's ability to capture nonlinearities that are present in the data. The QAR(1,1) estimates obtained from U.S. data highlight nonlinearities in output growth, inflation, nominal wage growth, and interest rate dynamics. Output growth displays sharp declines and slow recoveries in the post-1983 sample. Inflation and nominal wage growth both display conditional heteroskedasticity in the 1960-2007 sample. Finally, downward adjustments in the federal funds rate seem to be easier than upwards adjustments in the post-1983 sample.

Among the nonlinearities identified through the estimation of the QAR models, the only ones that our estimated DSGE model seems to be able to capture, are the conditional heteroskedasticity in inflation and nominal wage growth. The model does so by relying on the asymmetric adjustment costs which penalize downward adjustments more than upward adjustments. The model is not able to generate the apparent nonlinearities in output growth and the federal funds rate. The tools developed in this paper can be used to identify nonlinearities in any time series and doing this for other key series such as labor market variables in the U.S., and for key variables in other countries will be a useful exercise. The predictive checks simply require a simulation from the model and can be applied to any model, whether or not it is estimated, and should be a part of the toolbox for researchers working with DSGE models. Finally, we leave multivariate extensions of the QAR model, where the main challenge is to cope with the dimensionality of the model, to future research.

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