Disagreement vs. Uncertainty: 
Investment Dynamics and Business Cycles

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Abstract

A firm’s uncertainty about its future profitability will depend on both the underlying volatility of its future productivity and the accuracy of the firm’s information about this productivity. I build a DSGE model of heterogeneous firms that disentangles the business cycle effects of time-varying fundamental volatility of productivity à la Bloom (2009) from those of time-varying uncertainty coming from variation in the precision of firms’ signals about the productivity. Firms rely on imperfect information to differentiate between aggregate and idiosyncratic productivity. More dispersed information increases the heterogeneity of firms’ expectations of productivity and leads to greater disagreement. This triggers larger capital misallocation and a drop in aggregate investment, which generates a recession followed by a sluggish recovery. Conversely, an increase in fundamental volatility induces a short-lived recession and an expansion in the medium run. The model jointly explains the pro-cyclicality of dispersion in firm investment rates and the differential impacts of volatility-based measures of uncertainty and forecast disagreement. A model with volatility shocks alone fails to explain the former and is silent on the latter.

JEL codes: D83, E22, E32, G14

Key Words: Uncertainty, Information Frictions, Firm Dynamics, Investment, Business Cycles

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1 Introduction

I develop a general equilibrium theory that finds that changes in the magnitude of disagreement about aggregate productivity among firms can be an important driver of business cycles. The theory incorporates two intrinsically different forces: shocks to the dispersion of idiosyncratic productivity, and shocks to the dispersion of heterogeneous information. The former, known as swings in real uncertainty, is argued to be a strong factor that triggered significant damages during the 08-09 recession (Bloom et al., 2014; Christiano et al., 2014). However, the business cycle impacts of time-varying information quality that shapes cross-sectional dispersion of firms’ beliefs are largely under-explored. More dispersed information causes more firms to act on imprecise signals, which prevent them from being well informed of the economic status quo and from forming good forecasts of future profitability. Larger information dispersion can increase the heterogeneity of firms’ beliefs about aggregate as well as firm-specific productivity, which results in greater informational disagreement, even if the distribution of fundamental productivity is unchanged. The model shows that larger informational disagreement makes more productive firms believe the unobserved good idiosyncratic productivity draws are not good enough to justify investment, which leads to greater capital mis-allocation and contractions.

How uncertainty affects business cycles has been intensively studied within the framework of micro-level capital and labor adjustment frictions in the form of irreversible investment and non-convex adjustment costs (Bernanke, 1983; Pindyck, 1991; Bloom, 2009). Uncertainty is conventionally measured by the second moment of a distribution of real economic fundamentals. For example, cross-sectional dispersion of firm-specific productivity captures the micro uncertainty, and time-series volatility of aggregate productivity measures the macro uncertainty. Shocks that increase fundamental uncertainty can raise the option value of adopting a “wait and see” policy towards investment or hiring. As more firms pause their actions, such real option effects can be sufficiently strong to generate recessions.

This paper addresses a key shortfall of models of real uncertainty shocks. When firms restart investing and hiring in better times, shrinkage of productivity dispersion tightens dispersion of firm-level investment rates as uncertainty is subdued. Therefore, models with shocks to fundamental uncertainty cannot simultaneously generate sizable business cycles and the procyclical dispersion of firm-level investment rates as found in the data (Bachmann and Bayer, 2014). This paper shows that effects of uncertainty shocks can survive a model with informational disagreement shocks, and more importantly, a new and quantitatively important channel brought about by the imperfect information can generate the procyclicality of investment rate dispersion.

In addition, evidence from Vector Autoregressions (VAR) suggests that the business cycle impacts of shocks to fundamental uncertainty and those of shocks that affect cross-sectional forecast
disagreement can be quite different. Bloom (2009) finds that major macro aggregates, including output and employment, drop and quickly rebound within the first two or three quarters in response to rises in conventional uncertainty measures. By contrast, Bachmann et al. (2013b) uncovers that, in response to rises in measures of firm-level business forecast disagreement, aggregate output can undergo a gradual decline and very slow recovery. Therefore, there are potentially two different channels through which fundamental uncertainty and forecast disagreement could affect business cycles. While the former is well studied through the models of uncertainty shocks, the mechanism for the latter shocks is unaccounted for in the literature.

This paper will show that a model of informational disagreement shocks can generate a sizable recession followed by a slow recovery through an investment channel, which is in line with the empirics. In Figure 1, I provide new VAR-based evidence on impulse responses of aggregate investment to shocks that affect cross-sectional dispersion of firm-level TFPs, a measure of uncertainty, and to shocks that shift the firm-level business forecast disagreement index, as in Bachmann et al. (2013b). Figure 1 shows that aggregate investment dynamics also experienced gradual decline and slow recovery. Appendix A shows that the results are robust across specifications of larger VAR system when both TFP dispersion and the disagreement index are present regardless of which one is ordered first. Specifically, consistent with the model prediction in Bloom (2009), aggregate investment drops in the first few quarters (despite wide standard error bands) and quickly rebounds towards an expansion region in response to a rise in firm-level TFPs dispersion. By contrast, it take two years for aggregate investment to reach its trough, and it recovers the loss at an extremely slow speed in response to shocks that shift cross-sectional disagreement across firms.

Regarding the model ingredients, this paper builds an imperfect information environment in which firms care about the difference between aggregate and idiosyncratic productivity. However, firms can only imperfectly disentangle the aggregate from idiosyncratic draws through noisy signals. How much they are “uncertain” about future profitability is driven by both factors: the dispersion of future idiosyncratic productivity, i.e. the fundamental uncertainty, and the imprecision of the information contained in their signals. This paper answers the following question: how can changes in the dispersion of pure noise, which drive cross-sectional disagreement about aggregate productivity, have real and sizable effects on business cycles? Importantly, without disregarding real uncertainty shocks, how can we disentangle effects of informational disagreement from those caused by fundamental uncertainty? Ultimately, the model economy helps evaluate which source of the exogenous disturbances, if both are present, is the more important driver of business cycle fluctuations.

The theoretical framework extends real business cycle general equilibrium models of heterogeneous firms with firm-level non-convex adjustment costs (Khan and Thomas, 2003, 2008) and

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1 Later sections will illustrate how these measures of uncertainty and disagreement are constructed.
those with uncertainty shocks (Bloom et al., 2014) to aid comparisons with existing work. It deviates from these benchmarks in the following ways: (1) subject to information frictions, firms cannot distinguish the aggregate from the idiosyncratic component despite their observations of the total TFP. (2) Idiosyncratic productivity is more persistent than the aggregate counterpart as suggested in the data (Cooper and Haltiwanger, 2006). Thus, firms extract separate beliefs about the levels of the two productivity components based on the observed total TFP and a noisy signal that indicates aggregate productivity. (3) Information precision of the signals is governed by an aggregate variable, the dispersion of firm-specific signals, or equivalently, the standard deviation of noises within the public signals. I show that modeling the imperfect signal either as public or private does not affect the results. It is the aggregate information precision that is time-varying and subject to exogenous disturbances. More dispersed information renders larger disagreement among firms. (4) Firm-level adjustment of capital incurs both convex and non-convex adjustment costs, though, for simplicity, labor hiring is free of adjustment costs.

The model economy is hit by two types of exogenous disturbances. The first is, second moment shocks to non-fundamentals, i.e., dispersion of noisy signals, along with those that affect

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A large number of very recent theoretical works study the effect of confidence, sentiment, exuberance, and news on business cycles. This literature mainly focuses on the first moment time variation of aggregate noise shocks (Lorenzoni, 2009; Angeletos and La’O, 2011; Schmitt-Grohé and Uribe, 2012; Blanchard et al., 2013; Benhabib...
fundamentals of real productivity: micro uncertainty shocks, which capture perturbations to the dispersion of idiosyncratic productivities across firms. The drawn distinction intends to emphasize that deterioration in information quality does not necessarily suggest the economy is undergoing larger variability in real productivity. The second type of exogenous disturbance is first moment shocks to the levels of aggregate productivity, idiosyncratic productivity, and the signal noise. All shocks are orthogonal.

In terms of results, this model predicts pro-cyclical investment rate dispersion. When fewer firms form beliefs with good precision, firms disagree more about future aggregate productivity. Very productive firms increasingly believe that the de facto good idiosyncratic productivity draws are not that good, and less productive firms further embrace a more optimistic view about the draw. This generates greater underinvestment and overinvestment, which leads to an increasing shrinkage in dispersion of firm-level investment rates. As there is more disagreement concentrated in bad times, as the data suggests (Bachmann et al., 2013b; Bloom, 2014), investment rate dispersion drops. During good times, firms then better invest and disinvest closer to the optimal with more precise information. As a result, dispersion of investment rates enlarges in good times. Crucially, this model prediction suggests that we can rely on dispersion in firm-level investment rates as the key data moment to identify whether firms become more uncertain for more variability in real productivity, or because they are more misinformed.

The reason more productive firms underestimate their idiosyncratic productivity is given by the presence of imperfect information for disentangling productivity components. For larger informational disagreement, firms increasingly mis-attribute more of the productivity variation due to the more persistent idiosyncratic productivity shocks to the less persistent aggregate counterpart and vice versa. Therefore, the magnitude of insufficient firm-level investment response to idiosyncratic productivity increases. This leads to greater capital mis-allocation and a drop in aggregate investment. In addition, firms assign smaller weights to new productivity draws when they know their information is getting less precise. Therefore, as firms carry the erroneous perceptions over time, capital mis-allocation can be persistent. Thus pure informational second moment shocks can generate a recession followed by a slow recovery even if fundamentals are not changed. Conversely, the real-option effect brought about by jumps in fundamental uncertainty is only short-lived. In the medium run, when firms are pushed out of the inaction band, the pent-up investment triggers a quick rebound. Therefore, this paper is the first to provide a theoretical explanation for why the impulse responses of macro aggregates to the changes in fundamental uncertainty and informational disagreement can differ.

Numerically, two thirds of the variation in aggregate investment that cannot be explained by aggregate TFP shocks is captured by time-variation in informational disagreement, whereas...
one third is attributable to changes in fundamental uncertainty. A strong implication of this paper is that pure noise dispersion can generate important business cycles even if, on average, the economy does not have aggregate noise. The policy implication suggests that more dispersed information can generate more heterogeneous mis-perceptions of aggregate and idiosyncratic productivity across firms, which renders government subsidy policy less effective.

The rest of the paper proceeds as follows: Section 2 summarizes the related literature. Section 3 illustrates a simple partial equilibrium model that compares and contrasts effects of fundamental uncertainty and informational disagreement. Section 4 gives the description of a full DSGE model. Section 5 discusses parameter values used to solve the full model. Section 6 presents the numerical results of the model. Section 7 concludes.

2 Related Literature

This paper is related to several strands of the literature. Firstly, this paper contributes to the stream of work that finds uncertainty and, more recently, stochastic volatility shocks can affect investment and hiring. This paper suggests that pure informational second moment shocks can affect business cycles similarly through the two well-documented channels by which real uncertainty drives business cycles: the convexity and the real-option mechanisms. Specifically, theories with convex capital adjustment cost or with convex marginal product of capital function in productivity predict that higher uncertainty increases investment given that expected marginal revenue (Oi, 1962; Hartman, 1972; Abel, 1983) of capital is higher. When micro-level non-convexity is also considered, the contractionary real-option effect dominates the expansionary convexity effect in the short run, though the convexity effect will kick in very quickly (Bloom, 2009; Bloom et al., 2014). This paper finds that larger informational disagreement can also raise the expected marginal product of capital while forcing some firms to pause investment until more precise information arrives.

Secondly, the recent literature finds that the sign and the quantitative importance of real uncertainty shocks for business cycles are sensitive to the model structure and parameterization. Counterfactual expansionary effects or moderate negative impacts on output or investment are found if the model lacks additional market frictions such as price rigidity (Bundick and Basu, 2014), credit market friction (Gilchrist et al., 2014), or search friction (Leduc and Liu, 2015). Bachmann and Bayer (2013) find that, when calibrating with German data, the role of the real-option effect, as a key channel through which uncertainty can trigger a recession, is very limited. Caldara et al. (2014) exhibit that VAR-based impulse responses of uncertainty shocks are sensitive to how we identify uncertainty shocks. My paper shows that the impacts of real uncertainty can survive in the presence of information frictions. However, to be able to generate procyclical investment rate
dispersion, it is crucial to disentangle shocks due to time-variation in information quality from real uncertainty shocks.

An emerging literature argues that various sorts of information frictions that affect precision of agents’ learning endogenously determine uncertainty and drive business cycles: changes in estimated tail risk for forecasting (Orlik and Veldkamp, 2014), time-variation in the cost of information acquisition (Benhabib et al., 2015a), and agents’ learning from the actions of others subject to information externalities (Fajgelbaum et al., 2015). Instead of modeling how exactly information precision is shifted by endogenous actions, this paper treats precision of signals firms receive as distinctive sources of exogenous perturbations. Importantly, the merit of this paper is that results do not rely on the assumption that worsened information precision is due to adverse first moment shocks. Rather, informational disagreement shocks can be the primitive shocks that drive the cycles.

Evidence from Eisfeldt and Rampini (2006) suggests that the cost of capital reallocation across firms must be countercyclical, given that capital reallocation is procyclical and the benefit of reallocation as measured by firm-level productivity dispersion is countercyclical. This paper rationalizes jumps in cross-sectional disagreement as the information cost that prevents more productive firms from accumulating capital in bad times. The most closely related paper that generates similar capital mis-allocation based on information frictions is David et al. (2014). However, their paper models information frictions such that firms cannot perfectly learn their firm-specific demands via private information. Instead, my paper builds information frictions on imperfect disentangling such that firms cannot distinguish aggregate from idiosyncratic productivity. I show that, as long as the additional information on which firms rely for disentangling purposes is imperfect, capital misallocation is a natural consequence regardless of whether information is public or firm-specific.

The idea that imperfect disentangling of two different types of shocks triggers partial adjustment in decision variables can be dated back to Lucas (1972). Recent papers apply the noisy and dispersed private information to study optimal monetary policy (Woodford, 2001; Adam, 2007; Lorenzoni, 2010) and business cycles (Lorenzoni, 2009; Blanchard et al., 2013). Differently, within a neoclassical framework, this paper studies the business cycle effects of second moment time variations in information quality rather than the impacts of the first moment shocks to aggregate noise.

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This stream of work addresses the concern in the literature on uncertainty shocks that the source of exogenous disturbances to uncertainty is unclear. Other endeavors to endogenize uncertainty shocks include Bachmann and Moscarini (2011) and Decker et al. (2014)
3 A Simple Model

I present a three-period partial equilibrium simple model to illustrate the key mechanisms at work in the full model. Specifically, firms care about differentiating between aggregate and firm-specific productivity as they are separately driven by different shocks. However, firms cannot perfectly disentangle what fraction of their observed total productivity should be attributed to aggregate component and what fraction to idiosyncratic counterpart. Therefore, firms rely on additional noisy information to form separate beliefs. I show that changes in precision of information, which shifts the extent of cross-sectional disagreement, affect how precise firms’ believes about their draws of productivity components are and how good the firms’ expectations about future marginal product of capital are.

Firstly, results suggest that firms’ investments in response to changes in either fundamental uncertainty or informational disagreement are affected by two offsetting forces: contractionary real-option effect and expansionary convexity effect. Secondly, I show that the capital misallocation due to rises in informational disagreement is the key mechanism that generates procyclical dispersion of firm-level investment rates while drives down aggregate investment. Importantly, this third channel helps identify shocks to fundamental uncertainty from shocks to precision of information that shifts cross-sectional disagreement.

3.1 Environment

The economy is populated by a unit measure of firms and each firm is indexed by \( i \). With the same initial capital stock \( k_0 > 0 \) across firms, firm \( i \) produces output for three periods. Firms’ profit function is given by AK technology for the first two periods

\[
y_{i,t} = A_{i,t}k_{i,t-1}
\]

where total factor productivity \( A_{i,t} \) has an aggregate component \( X_t \) and an idiosyncratic component \( Z_{i,t} \) such that \( A_{i,t} = X_tZ_{i,t} \). \( k_{i,t-1} \) is the predetermined capital stock; the only factor input for \( t = 1, 2 \). Output in period 3 (last period) has decreasing returns to scale (DRTS) in capital such that

\[
y_{i,3} = A_{i,3}^{1-\alpha}k_{i,2}^\alpha
\]

\( \alpha \in (0, 1) \) captures the magnitude of DRTS. The elasticity of output with respect to TFP is scaled down to \( 1 - \alpha \) simply for the purpose of deriving tractable analytical results.
Productivity and Uncertainty. Firm $i$ enters period 1 with observed TFP $A_{i,1}$, produces output $y_{i,1}$, makes investment decision $I_{i,1}$ but is uncertain about the to-be-realized $A_{i,2}$. Once $A_{i,2}$ is known to the firm at the beginning of period 2, uncertainty clears as firm knows perfectly that TFP will stay constant such that $A_{i,3} = A_{i,2}$. The firm then produces $y_{i,2}$ and decides on investment $I_{i,2}$ given the expected gain from producing in the last period. Uncertainty affects period 1’s investment decision only.

I use lower cases to denote productivity factors in natural log, which are assumed to follow AR(1) processes.

\[
\begin{align*}
x_1 &= \sigma_{v,0} \cdot v, & x_2 &= \rho_x x_1 + \sigma_v \cdot v_2 \\
z_{i,1} &= \sigma_{e,0} \cdot e_i, & z_{i,2} &= \rho_z z_{i,1} + \sigma_e \cdot e_{i,2}
\end{align*}
\tag{3a, 3b}
\]

Period 2 log productivity components are linked to their period 1 realizations through an auto-regressive system with persistence $\rho_j \in (0, 1)$ for $j \in \{x, z\}$. Evidence suggests idiosyncratic productivity is more persistent $\rho_z > \rho_x$ (Davis and Haltiwanger, 1992; Cooper and Haltiwanger, 2006), as is assumed throughout the model section of the paper.

In period 1, each log productivity component is written in products of the realized first moment shock innovations: $v$ and $e_i$ and the corresponding predetermined second moments: standard deviations $\sigma_{v,0}$ and $\sigma_{e,0}$ of innovations. These first-moment innovations are i.i.d. draws from $N(0, 1)$. We call $v$ and $e_i$ that affect period 1 TFP components respectively TFP shocks and firm-level TFP shocks, similarly for $v_2$ and $e_{i,2}$ with respect to period 2 TFP.

It exhibits that the second moments $\sigma_v$ and $\sigma_e$ scale the standard deviations of two future TFP shocks: $v_2$ and $e_{i,2}$. These second moments are treated as parameters here for simplicity. However, stochastic shocks to $\sigma_v$ and $\sigma_e$, as modeled in the full model, are known as uncertainty shocks à la Bloom (2009). Specifically, shocks to $\sigma_v$ that affect the volatility of future aggregate productivity are called the macro uncertainty shocks, whereas shocks to $\sigma_e$ that shape the dispersion of future idiosyncratic productivity across firms are known as the micro uncertainty shocks. Importantly, these second moments are predetermined variables. It implies that firms observe them and thus know how volatile aggregate productivity and how dispersed firm-specific productivity will be in the next period.

Noisy Information and Disagreement. Since aggregate and idiosyncratic components are governed by different dynamics, firms care about individually the aggregate and idiosyncratic productivity realized in period 1 in order to form the expectation about period 2 TFP $A_{i,2}$. Hence, investment $I_{i,1}$ in period 1 can be pinned down.

However, I assume firms do not separately observe $x_1$ and $z_{i,1}$ though they observe the total
sum of the log productivity factors $a_{i,1}$. In addition, firms have access to noisy public signals $s$ about aggregate productivity with an i.i.d. noise shock $\xi \sim N(0,1)$.

$$a_{i,1} = x_1 + z_{i,1}$$
$$s = x_1 + \sigma_\xi \cdot \xi$$ (4) (5)

Firms are assumed to know how imprecise the signal is by the standard deviation of noise shocks: $\sigma_\xi$. Knowing both $a_{i,1}$ and $s$ facilitates the firm to extract separate beliefs about the two productivity components $\mathbb{E}(x_1|s, a_{i,1}) = x_{i,1|1}$ and $\mathbb{E}(z_1|s, a_{i,1}) = z_{i,1|1}$.

I will delegate to later sections to show that as long as the signals are imperfect regardless of whether it’s public or firm-specific, the results are robust. Hence, forecast disagreement about $A_{i,2}$ arises given the heterogeneity in $x_{i,1|1}$ and the persistence of $\rho_x$. It is clear that if firms are perfectly informed of the two components as $\sigma_\xi \rightarrow 0$, firms would not have different beliefs about TFP since signal truly reveals the aggregate state $s = x_1$. Therefore, information precision $\sigma_\xi$ could shift the cross-sectional disagreement. When modeled as exogenous shocks in the full model, changes in $\sigma_\xi$ are called non-fundamental disagreement shocks. Calling it non-fundamental is because it measures the pure noisiness in the information, which differs from fundamental uncertainty that captures TFP volatility and firm-level TFP spread $\sigma_v$ and $\sigma_e$. Such distinction is to emphasize the idea that changes in measurement error in public signals may not necessarily imply or is affected by swings in fundamental uncertainty.

In summary, Figure 2 plots the time line along which firms make investment decisions. The reason why a third period of production is needed is because some firms may take delayed investment in period 2 after staying inaction in period 1. To justify period 2 investment, there should be expected gain from producing in period 3.

\footnote{Besides $\sigma_\xi \neq 0$, another condition for firms to differ in beliefs is that each firm should have its own idiosyncratic productivity draw. However, this requirement is trivial in the sense that shutting down productivity heterogeneity means shutting down cross-sectional dimension.}
Figure 2: Timeline Firm’s Decision

\[
\begin{array}{c|c|c}
 t = 1 & t = 2 & t = 3 \\
 \text{(Period of Uncertainty and Disagreement)} & A_{i,2} \text{ reveals} & \text{produces } y_{i,3} \\
 \text{observes TFP } A_{i,1} & \text{produces } y_{i,2} & \text{makes investment } I_{i,2} \\
 \text{produces } y_{i,1} & \text{knows } A_{i,3} = A_{i,2} & \\
 \text{receives signal } s & & \\
 \text{observes } \sigma_v, \sigma_e \text{ and } \sigma_{\xi} & & \\
 \text{form beliefs } x_{i,1|1} \text{ and } z_{i,1|1} & & \\
 \text{makes investment } I_{i,1} & & \\
\end{array}
\]

3.2 Productivity Beliefs, Disagreement, and Expectation

I examine how firms form separate productivity beliefs \( x_{i,1|1} \) and \( z_{i,1|1} \) and thus expected future productivity in this section as they are crucial for investment decisions. Greater \( \sigma_{\xi} \), lower the quality of information contained in a received signal. I will show \( \sigma_{\xi} \) not only affects firm’s beliefs and forecasts but also shifts the cross-sectional disagreement about future aggregate productivity.

**Productivity Beliefs.** Assume TFP shocks \( v \) and \( e_i \) and noise shocks \( \xi \) are orthogonal. Note that variability of period 1 TFP shocks \( \sigma_{v,0} \) and \( \sigma_{e,0} \) are predetermined. Applying Bayes’ Rule, we can characterize the firms’ posterior beliefs about aggregate productivity \( x_{1} \) and idiosyncratic productivity \( z_{i,1} \). The magnitude of information precision affects these perceptions.

**Lemma 1** With imperfect information, upon observing \( s \) and \( a_{i,1} \), firm \( i \)’s posterior expectations of \( x_{1} \) and \( z_{i} \) are given by

\[
\begin{bmatrix}
  x_{i,1|1} \\
  z_{i,1|1}
\end{bmatrix} = \kappa 
\begin{bmatrix}
  a_{i,1} \\
  s
\end{bmatrix}
\]

where \( \kappa = \begin{bmatrix}
  \kappa_{11} & \kappa_{12} \\
  \kappa_{21} & \kappa_{22}
\end{bmatrix} = \begin{bmatrix}
  b/(a+b+c) & c/(a+b+c) \\
  1-b/(a+b+c) & -c/(a+b+c)
\end{bmatrix} \]

\[
a = 1/\sigma_{v,0}^2, \quad b = 1/\sigma_{e,0}^2, \quad c = 1/\sigma_{\xi}^2
\]

**Proof.** See Appendix B \( \blacksquare \)

Lemma 1 suggests that with noisy signals, firms’ posterior beliefs about each productivity component are linear combinations of observables \( a_{i,1} \) and \( s \) weighted by precision parameters (inverse of variances) \( a, b, c \). The weights can be well summarized in a Kalman gain matrix \( \kappa \).
that elements sum up to 1. Changes in information quality (or, non-fundamental disagreement) clearly shifts individual beliefs. The following proposition sees that precision of information can shift cross-sectional forecast disagreement.

**Proposition 1** *Larger imprecision of signals, larger cross-sectional dispersion of beliefs about future aggregate productivity.*

**Proof.** By Lemma 1, the standard deviation of cross-sectional forecasts of $A_{i,2}$ in period 1 after applying law of large numbers $\int_0^1 z_{i,1} \, di = 0$ is given by $\sigma_{E(x_2)} = \rho_x \sigma_{x_{i,1}} = \frac{\rho_x \sqrt{b}}{a+b+c}$. Hence

$$\frac{\partial \sigma_{E(x_2)}}{\partial \sigma_\xi} > 0 \quad (7)$$

**Q.E.D.**

Then we compute firm $i$’s expected period 2 TFP $A_{i,2}$ exploiting the auto-regressive system (3a) and (3b). The following lemma summarizes the key result.

**Lemma 2** For $\rho_z > \rho_x$,

$$\mathbb{E}A_{i,2} = \exp[\mu_i + 0.5 \Sigma(\sigma_v, \sigma_e, \sigma_\xi)] \quad (8)$$

where

$$\mu_i = (\rho_x + M)x_1 + (\rho_x - N)z_{i,1} - P\xi \quad (9)$$

is the mean TFP in log. Terms $M$, $N$, $P$ and $Q$ are positive and functions of disagreement $\sigma_\xi$ in period 1 and predetermined $\sigma_v,0$ and $\sigma_e,0$. $M'(\sigma_\xi) > 0$, $N'(\sigma_\xi) > 0$, and $Q'(\sigma_\xi) > 0$. Forecast variance $\Sigma$ is a function of period 1 macro, micro uncertainty and disagreement.

$$M = a \cdot d, \quad N = b \cdot d, \quad P = \sqrt{c} \cdot d, \quad Q = (\rho_z - \rho_x)^2/(a + b + c)$$

$$d = (\rho_z - \rho_x)/(a + b + c) > 0, \quad \Sigma = \sigma_v^2 + \sigma_e^2 + Q$$

**Proof.** See Appendix C. **Q.E.D.**

$\mathbb{E}$ is firm $i$’s expectation operator conditional on its information set at the beginning of period 1. Lemma 2 shows that the expected next period TFP depends on period 1 realizations of TFP shocks $v$, firm-level TFP shocks $e_i$, noise shocks $\xi$, three second moment parameters of uncertainty and disagreement $\sigma_v$, $\sigma_e$ and $\sigma_\xi$ and predetermined uncertainty $\sigma_v,0$ and $\sigma_e,0$. Larger TFP and firm-level TFP shocks increase firm’s expected total TFP.

In addition, more persistent idiosyncratic shocks $\rho_z > \rho_x$, firms would mis-attribute part of the variation of productivity due to aggregate changes to the more persistent idiosyncratic shocks.
via a positive term $M$. Vice versa, firms mis-attribute part of variation due to the idiosyncratic changes in productivity to the less persistent aggregate shocks through a negative term $-N$. This mechanism of making perception errors is known as mis-attribution of signals.

Presence of noisy signals makes firms’ expected next period total productivity negatively affected by rising noise shocks $\xi$. This is because firms know that they will mis-attribute signals and the errors of having noises within signals is captured by $P > 0$. Apart from the fundamental uncertainty terms $\sigma_v^2$ and $\sigma_\xi^2$, $Q$ captures the additional forecast variance brought by non-fundamental disagreement in period 1.

If firms have perfect information (zero noise variation for $s = x_1$) across firms as $\sigma_\xi \to 0$, the mis-attribution of signals effect is completely gone as $M$, $N$, $P$ and $Q$ all collapse to zero. This mis-attribution mechanism is pivotal as it differentially enhances and mutes the marginal impact of aggregate productivity shocks $v$ and idiosyncratic shocks $e_i$ on firms’ expected total productivity relative to the perfect information benchmark $^5$. In addition, by $\frac{\partial M}{\partial \sigma_\xi} > 0$ and $\frac{\partial N}{\partial \sigma_\xi} > 0$, we see that the larger disagreement among firms when firms act upon more imprecise signals to extract separate beliefs, the magnitudes of both enhancing and dampening effects will increase.

### 3.3 Effects of Disagreement vs. Uncertainty

In this section, I show that imperfect information brings forth a distinctive effect of capital misallocation due to disagreement. Rise in disagreement reduces aggregate investment and affects the dispersion of firm-level investment rates. In addition, changes in both fundamental uncertainty and non-fundamental disagreement can trigger the expansionary convexity effect and contractionary real-option effect. Mis-allocation effect and convexity effect shift aggregate investment on the intensive margin whereas real-option effect is through the extensive margin.

#### 3.3.1 Firm-level Investment

I study the effects of uncertainty and non-fundamental disagreement ($\sigma_v$, $\sigma_e$ and $\sigma_\xi$) on investment in period 1 when firms are subject to uncertainty and imperfect information.

**Non-convex and Convex Capital Adjustment Costs.** Use $\delta$ to denote the capital depreciation rate in period 1. Investment in period 1 is given by $I_{i,1} = k_{i,1} - (1 - \delta)k_0$. For simplicity,

\footnote{Note that if $\rho_x = \rho_z$, there is no mis-attribution effect. With no difference in persistence between aggregate and idiosyncratic productivity shocks, firms do not need to differentiate them at all because firms’ investment decisions in response to these level shocks will be no different. However, if $\rho_x > \rho_z$, though contradictory to the empirical evidence, mis-attribution mechanism is still there but the directions of enhancing and dampening effects will be reversed. The bottom line is that firms would always want to mis-attribute the changes due to less persistent shocks to more persistent changes. (See Appendix L for details about other scenarios.)}
I assume after period 2 and 3 productions, capital stock $k_{i,1}$ and $k_{i,2}$ will be fully depreciated. Therefore, period 2 investment is easily characterized as $I_{i,2} = k_{i,2} \geq 0$. Firms would only incur non-negative investment in period 2. By contrast, firms may invest ($I_{i,1} > 0$), disinvest ($I_{i,1} < 0$) and take no investment action ($I_{i,1} = 0$) in period 1.

I assume that if investment is non-zero $I_{i,t} \neq 0$ in period 1 or 2, firm has to pay a non-convex fixed cost $c_k$ per unit of existing capital stock. In addition, following Lee and Shin (2000), I assume the fixed cost is avoidable in period 2 if a firm already paid the cost for non-zero investment in period 1. This assumption of cost avoidance is for the purpose of maintaining tractability of results only, the full model will be in line with standard fixed cost assumption as in Gilchrist et al. (2014). Investment in period 1 is also subject to a quadratic convex adjustment cost of capital $\frac{1}{2}I_{i,1}^2$.

**Firms’ Problem** Firm $i$ maximizes the sum of three expected dividends with no inter-temporal discounting.

$$\Pi = \max_{I_{i,1}, I_{i,2}} A_{i,1}k_0 - I_{i,1} - \frac{1}{2}I_{i,1}^2 - c_k[\mathbb{I}_k k_0 + (1 - \mathbb{1}_c)k_{i,1}] + \mathbb{E}[A_{i,2}k_{i,1} - k_{i,2} + A_{i,3}^{1-\alpha}k_{i,2}]$$

(10)

where

$$\mathbb{1}_c = \begin{cases} 1, & \text{if } I_{i,1} \neq 0 \\ 0, & \text{if } I_{i,2} \neq 0 \end{cases}, \quad A_{i,3} = A_{i,2} = e^{x_2 + x_{i,2}}, \quad I_{i,1} = k_{i,1} - (1 - \delta)k_0$$

Solving the problem backwards, conditional on realized $A_{i,2}$, if firms take positive investment in period 2, then investment for period 3 production is given by

$$k_{i,2}^* = \alpha \frac{1}{1-\alpha} A_{i,2} > 0$$

(11)

Equation (11) says that higher realized period 2 productivity induces larger capital demand in order to reap higher profit from period 3 production.

**Firms with non-zero investment in period 1.** If a firm has paid a fixed cost for non-zero investment in period 1, period 2 and period 3 total profit that is free of fixed cost conditional on realized $A_{i,2}$ is given by

$$\Pi_{i,2}(I_{i,1} \neq 0; A_{i,2}) = A_{i,2}(k_{i,1} + \psi)$$

where

$$\psi = \alpha \frac{1}{1-\alpha} - \alpha \frac{1}{1-\alpha} > 0$$

This shows if firms adjust capital in period 1, it is always optimal to invest positive $k_{i,2}^*$ to reap revenue of period 3 production rather than have zero investment with a smaller profit $A_{i,2}k_{i,1}$. The
marginal gain from investing in period 2 as captured by $\psi$ appears not dependent on investment actions in period 1. The optimal investment in period 1 is given by

$$I^*_i,1 = \mathbb{E}A_{i,2} - 1$$

(12)

Investment in period 1 positively responds to expected productivity in period 2, the forecast. It must be the case that $\mathbb{E}A_{i,2} > 1$ for positive investment ($I^*_i,1 > 0$) and $\mathbb{E}A_{i,2} < 1$ for negative investment ($I^*_i,1 < 0$). Then we can express the expected total profit for firms taking non-zero action (Adj) of investment or disinvestment in period 1 in the following

$$\Pi^{Adj} = A_{i,1}k_0 + \mathbb{E}A_{i,2}I^*_i,1 + \mathbb{E}A_{i,2}(1 - \delta)k_0 + \mathbb{E}A_{i,2}\psi - I^*_i,1 - \frac{1}{2}I^*_i,1 - c_kk_0$$

(13)

The first four terms in Equation (13) consist of the revenue to firms with non-zero action in period 1: period 1 output, expected gain from taking additional non-zero investment, expected gain from production using the net depreciation capital stock, and the expected return from taking positive investment in period 2. The other terms capture the cost expenditure including investment goods input, quadratic and fixed costs of capital adjustment.

**Firms taking no investment action in period 1.** Firms will take positive investment in period 2 for $\psi > 0$ even if they do not adjust capital stock in period 1. However, those firms that enter period 2 with existing capital stock $(1 - \delta)k_0$ have to pay a fixed cost to have optimal investment as given by Equation (11). The period 2 and period 3 total profit conditional on taking no investment in period 1 is

$$\Pi_{i,2}(I^*_i,1 = 0; A_{i,2}) = \max\{A_{i,2}((1 - \delta)k_0 + \psi) - c_k(1 - \delta)k_0, A_{i,2}(1 - \delta)k_0\}$$

Expected total profit for these firms with no action (Non-Adj) is thus given by

$$\Pi^{Non-Adj} = A_{i,1}k_0 + \mathbb{E}A_{i,2}(1 - \delta)k_0 + \mathbb{E}[\max\{A_{i,2}\psi - c_k(1 - \delta)k_0, 0\}]$$

(14)

For firms taking no action in period 1, they also receive output from period 1 and the expected return from net depreciation capital. However, they also retain an option value from waiting in period 1 as captured by the third term in Equation (14). This option is said to be “in the money” when the realized period 2 productivity $A_{i,2}$ is greater than a fixed cost that a firm has to pay for such a delayed capital adjustment.

**Option of Waiting.** The option value is defined as the expected value of payoffs from all
scenarios. The value cannot be negative because if the realized period 2 productivity is small enough, the firm can walk away from this option. Therefore, the higher the option value, the more likely a firm would wait a period without taking any investment or disinvestment actions. Exploiting the property of max function and truncated log-normality, we can re-express the option value as below:

\[
V_{\text{option}} = \int_{A}^{\infty} [\psi A_{i,2} - c_k(1 - \delta)k_0]d\hat{F}(A_{i,2}) = [1 - \hat{F}(A)]\psi\phi E A_{i,2} - c_k(1 - \delta)k_0 \geq 0 \quad (15)
\]

where

\[
A = \frac{c_k(1 - \delta)k_0}{\psi}, \quad \phi = \frac{\Phi(\sqrt{\Sigma} - \log A - \mu_i \sqrt{\Sigma})}{\Phi(-\log A - \mu_i \sqrt{\Sigma})} > 1.
\]

\(\Phi\) is the CDF of standard normal distribution and \(\hat{F}(A_{i,2})\) is the posterior cumulative distribution about \(A_{i,2}\) which follows log-normal distribution \(\sim \ln N(\mu_i, \Sigma)\) with mean and variance as in Equation (9).

**Profit Maximization** Then we can recast firm’s problem using Equations (13) and (14)

\[
\Pi = \max_{I_{i,1}} \{\Pi^{Adj}, \Pi^{Non-Adj}\} \quad (16)
\]

We then define a \(\Psi\) function that captures the gain from taking non-zero investment action in period 1 relative to waiting.

\[
\Psi = \Pi^{Adj} - \Pi^{Non-Adj} = \frac{(E A_{i,2})^2}{2} - \zeta \cdot E A_{i,2} + \gamma \quad (17)
\]

where \(\gamma = \frac{1}{2} - c_k k_0[1 - (1 - \delta)(1 - \hat{F}(A))].\) I further assume \(\zeta = 1 - \psi + \psi \cdot \phi[1 - \hat{F}(A)] > 0\) and \(\gamma > 0\) to examine equilibrium of interest. We plot this difference function in Figure 3 in which it traces out a parabolic and symmetric function about the expected value of period 2 total productivity. To simplify notations, use \(E_i\) to denote firm \(i\)’s forecast of \(A_{i,2}\). Two roots \(E^I_i\) and \(E^D_i\) of function \(\Psi\) make firms are indifferent between taking non-zero investment and waiting in period 1 \((\Pi^{Adj} = \Pi^{Non-Adj})\). Note that the two trigger points are firm-specific because each firm has its own posterior belief of the distribution of future \(A_{i,2}\). Therefore, we have the following lemma and we can characterize the optimal policy function for investment.

**Lemma 3 (Investment/Disinvestment Thresholds)** For \(\zeta > 0\) and \(\gamma > 0\), there exist thresholds about expected next period TFP factor \(E^I\) and \(E^D\) such that \(\Pi^{Adj} = \Pi^{Non-Adj}\) where \(E^I = \zeta + \sqrt{\zeta^2 - 2\gamma} > 0\) and \(E^D = \zeta - \sqrt{\zeta^2 - 2\gamma} > 0\), which are functions of aggregate variables \(\sigma_v, \sigma_z,\) and \(\sigma_\xi\) and depend on posterior beliefs about distribution of future \(A_{i,2}\). Loss from waiting increases in \(E_i\) when \(E_i > \zeta\) and decreases in \(E_i\) when \(E_i \in (0, \zeta)\).
Proposition 2 Firm $i$ invests if $E_i > E_i^I$, disinvests if $E_i < E_i^D$ and takes no action if $E \in [E_i^D, E_i^I]$.

Proposition 2 suggests that firm $i$ would take positive (negative) investment in period 1 only if its expected value of next period total productivity is sufficiently higher (lower) than some threshold. This suggests that with mediocre future forecast, the option value of waiting outweighs all the other gains from taking investment or disinvestment right away.

3.3.2 Effects of Fundamental Uncertainty and Non-fundamental Disagreement

Through three propositions in the following, I compare and contrast between effects of macro and micro uncertainty and those of disagreement upon firm-level and aggregate investment.

Capital Mis-allocation Effect. This is the effect that forces firms’ forecasts to deviate from the truth, which pushes quantity of investment and disinvestment away from optimal. The effect arises from information frictions and is determined by the magnitude of non-fundamental disagreement. We consider a quasi-elasticity of investment with respect to the three second moments for simplicity. Define $i_{i,1} = \log(1 + I_{i,1})$,

Lemma 4 In an economy with imperfect disentangling, firm-level investment and disinvestment would over-react to TFP shocks and under-react to firm-level TFP shocks relative to a perfect information scenario conditional on firm’s taking non-zero investment.
Proof. Evaluate the partial derivatives of \( i_{i,1} \) with respect to the TFP \( x_1 \) and firm-level TFP \( z_{i,1} \). We have \( \phi_x > 0 \) and \( \phi_z > 0 \) in the following:

\[
\phi_x = \frac{\partial i_{i,1}}{\partial x_1} = \rho_x + M > \rho_x = \phi_x^* \tag{18a}
\]

\[
\phi_z = \frac{\partial i_{i,1}}{\partial z_{i,1}} = \rho_z - N < \rho_z = \phi_z^* \tag{18b}
\]

The inequality conditions are given by \( M > 0 \) and \( N > 0 \).

- \( \phi_x^* \) and \( \phi_z^* \) are the corresponding partial derivatives in case of perfect information when \( \sigma_\xi \to 0 \).

Regarding disinvestment, the overreaction and underreaction still go through as \( |\phi_x| > |\phi_x^*| \) and \( |\phi_z| > |\phi_z^*| \). ■ Q.E.D.

Lemma 4 is a direct result of firms’ mis-attributing signals rooted in the information frictions. Firms are unable to perfectly identify productivity components and thus mis-attribute variation of productivity due to changes in less persistent aggregate TFP shocks \( v \) to changes in more persistent idiosyncratic shocks \( e_i \) and vice versa. Presence of imperfect information creates a gap between firm’s belief and the realized but unobserved draw, which leads to the over-reaction and under-reaction of firm-level investment.

Importantly, this gap of misperception is governed by how much firms disagree with each other or how much firms are mis-led by the noisy signals. In addition, these gaps of misperception are the magnitudes of amplified and dampened investment. Given \( M \) and \( N \) both increase in \( \sigma_\xi \), we have the following result:

**Proposition 3** Larger disagreement increases (1) the amplification of firm-level investment to TFP shocks \( v \) and (2) further dampens effect of its response to idiosyncratic shocks \( e_i \), conditional on firm’s taking non-zero action.

Proof. Evaluate the cross-partial derivative of \( i_{i,1} \) with respect to both TFPs and disagreement:

\[
\frac{\partial^2 i_{i,1}}{\partial x_1 \partial \sigma_\xi} = \frac{\partial M}{\partial \sigma_\xi} > 0 \tag{19a}
\]

\[
\frac{\partial^2 i_{i,1}}{\partial z_{i,1} \partial \sigma_\xi} = -\frac{\partial N}{\partial \sigma_\xi} < 0 \tag{19b}
\]

■ Q.E.D.

According to Equations (19a) and (19b), amplified and dampened effects of firm-level investment both increase in disagreement. This is because rising disagreement enhances the firm-level extent of signal misattribution for all firms. A key proposition regarding the capital misallocation is stated below:
Proposition 4  When aggregate TFP shocks are at \( v = 0 \), more productive firms increasingly cut investment and less productive firms increasingly increase investment in response to greater disagreement among firms.

Given imperfect disentangling, firms with good realized draws of firm-level TFP shocks would not believe the draws are that good enough to justify investment. Therefore, Proposition 4 says more productive firms are not investing enough while less productive firms are investing too much relative to the truth which are unobserved to all firms. Capital is hence mis-allocated. Critically, when the magnitude of non-fundamental disagreement rises due to more imprecise information, the extent of capital mis-allocation rises. We call this the capital mis-loc-allocation effect.

Convexity Effect. The following proposition considers another effect which applies to both fundamental uncertainty and disagreement conditional on firms’ taking investment or disinvestment. We consider the case when all first moment shocks: TFP shocks \( v \), firm-level TFP shocks \( e_i \), and the noise shocks \( \xi \) are at zero in order to isolate the impact of second moments.

Proposition 5  When first moment shocks are at zeros, conditional on taking non-zero investment action, firm-level investment increases in both macro and micro uncertainty and disagreement.

Proof. By Equations (8) and (12), with shocks \( v = 0 \), \( e_i = 0 \) and \( \xi = 0 \), we have

\[
\phi_{\sigma_v} = \frac{\partial i_{i,1}}{\partial \sigma_v} > 0 , \quad \phi_{\sigma_e} = \frac{\partial i_{i,1}}{\partial \sigma_e} > 0 , \quad \phi_{\sigma_\xi} = \frac{\partial i_{i,1}}{\partial \sigma_\xi} > 0
\]

(20)

Similarly, for disinvestment, higher second moments all reduce firm-level disinvestment. Q.E.D.

By Equation (12), the amount of firm-level investment is positively responding to firm’s expected marginal product of capital, i.e. the expected productivity in period 2. When the marginal production function \( A_{i,2} = e^{x_2 + z_{i,2}} \) is convex in log productivity component, higher variance increases the expected value \( E A_{i,2} \) due to Jensen’s Inequality. What’s new in this noisy information environment is that non-fundamental disagreement brings about a second source of forecast variance \( Q \) that is added to the fundamental uncertainty \( \sigma_v \) and \( \sigma_e \). As a result, investment or disinvestment would respond to disagreement similarly as if it responds to changes in fundamental uncertainty.

Real-Option Effect. We consider the second shared effect of uncertainty and disagreement with respect to their impacts on firm’s hazard of capital adjustment.

\[ ^6 \text{In the perfect information case, the forecast variance is just given by } \sigma_v^2 + \sigma_e^2 . \]
Proposition 6 For $A \geq e^{\sqrt{\Sigma}}$, a firm with all first moment shocks at zero in face of larger uncertainty and greater disagreement sees greater gain from waiting and taking no investment action.

Proof. See proof in Appendix D

This proposition says rise in either uncertainty or non-fundamental disagreement could enlarge a firm’s inaction region such that for $j \in \{x, z, \xi\}$

\[
\frac{\partial E^I_i}{\partial \sigma_j} > 0, \quad \frac{\partial E^D_i}{\partial \sigma_j} < 0
\] (21)

Therefore, as the probability of taking positive (negative) investment when $E_i > E^I_i$ ($E_i < E^I_i$) both decreases, a firm becomes more likely to be pushed into the enlarged inaction region of waiting. The reason is that larger uncertainty or disagreement increases the option value of waiting.

The following section is then devoted to check the macro implications of all these effects on aggregate investment. I show that the misallocation effect arising from jump in disagreement can generate recession.

3.3.3 Aggregate Investment

Here we examine the aggregate economy when aggregate TFP shocks $v = 0$ and aggregate noise shocks $\xi = 0$ whereas firms differ in firm-specific TFP shocks as if we are considering the ergodic distribution of firms in a dynamic setting at steady state. As firm $i$ would incur positive investment $E_i - 1 > 0$ if its forecast $E_i > E^I_i$ or disinvestment $E_i - 1 < 0$ if $E_i < E^D_i$. I assume a single crossing property in the following to exclude the possibility for multiple equilibria in order to ensure that conditional upon taking non-zero action, further favorable or unfavorable firm-level productivity draws should not push the firm back to the inaction region.

Assumption 1 At steady state with $v = 0$ and $\xi = 0$, conditional on taking positive (negative) investment, increase of the option value of waiting for a firm in response to larger (lower) firm-level productivity, is bounded such that $e^{\mu_i + \Sigma} |e^{\mu_i + \Sigma} - \zeta(e_i)| \geq |\zeta'(\mu_i) e^{(\mu_i + \Sigma)} - \gamma'(\mu_i)|$ where $\mu_i = (\rho - N) e_i$.

The assumption above says no firm with a better forecast of the future relative to an investing firms would want to take inaction and the reverse is true for disinvesting firms with even worse expected future productivity. This global continuity assumption helps us to use idiosyncratic productivity shocks $e_i$ to describe the entire firm distribution and to examine an aggregate “inaction band” in the following lemmas.
Lemma 5  For $v = 0$ and $\xi = 0$, firms would invest if firm-specific productivity draw $e_i$ is greater than $e^I$ and disinvest if $e_i < e^D$ where $e^I$ uniquely solves $E_i(e^I) = E^I_i$ and $e^D$ uniquely solves $E_i(e^D) = E^D_i$.

Proof. See proof in Appendix E □

Lemma 6 If \( \frac{\partial E^I_i}{\partial \sigma_j} > E'_i(\sigma_j) \) and \( \frac{\partial E^D_i}{\partial \sigma_j} < E'_i(\sigma_j) \), higher uncertainty or disagreement expands the aggregate inaction band such that \( \frac{\partial e^I}{\partial \sigma_j} > 0 \) and \( \frac{\partial e^D}{\partial \sigma_j} < 0 \).

Proof. See proof in Appendix F □

Lemma 6 holds when the magnitude of firm-level real option effect dominates the convexity effect, the inaction band of the aggregate economy expands in face of higher uncertainty and disagreement. This implies that more firms would do nothing for higher uncertainty or disagreement because they see larger gains from waiting relative to increased expected marginal return of capital.

Then we can characterize the aggregate investment $I$ by integrating the capital increments across firms who are investing and the firms who are disinvesting.

\[
I = \int_{e^I}^{\infty} (E_i - 1) d\Phi(e) - \int_{-\infty}^{e^D} (1 - E_i) d\Phi(e)
\]  

Proposition 7 In an economy with more investing firms relative to disinvesting firms $1 - \Phi(e^I) > \Phi(e^D)$, in response to larger disagreement, aggregate investment responds (1) negatively to the capital mis-allocation effect; (2) positively to convexity effect; (3) negatively to the real-option effect that reduces the number of firms who are investing.

Proof. We take the partial derivatives of $I$ with respect to disagreement

\[
\frac{\partial I}{\partial \sigma_{\xi}} = \int_{e^I}^{\infty} E_i[-\mathbf{N}_{\sigma_{\xi}} e_i + \Sigma'(\sigma_{\xi})]d\Phi(e) - (e^{(\rho - N)e^I + \Sigma} - 1) \frac{\partial e^I}{\partial \sigma_{\xi}} + (e^{(\rho - N)e^D + \Sigma} - 1) \frac{\partial e^D}{\partial \sigma_{\xi}} + \int_{-\infty}^{e^D} E_i[-\mathbf{N}_{\sigma_{\xi}} e_i + \Sigma'(\sigma_{\xi})]d\Phi(e)
\]

Rearrange this equation, we have

\[
\frac{\partial I}{\partial \sigma_{\xi}} = \left\{ \begin{array}{ll}
\int_{e^I}^{\infty} E_i \Sigma'(\sigma_{\xi}) d\Phi(e) + \int_{-\infty}^{e^D} E_i \Sigma'(\sigma_{\xi}) d\Phi(e) & \text{intensive margin: convexity effect} > 0 \\
(e^{(\rho - N)e^I + \Sigma} - 1) \frac{\partial e^I}{\partial \sigma_{\xi}} & \text{extensive margin: fewer disinvesting firms} > 0 \\
-\left( e^{(\rho - N)e^I + \Sigma} - 1 \right) \frac{\partial e^I}{\partial \sigma_{\xi}} & \text{extensive margin: fewer investing firms} < 0 \\
-\int_{e^I}^{\infty} E_i \mathbf{N}_{\sigma_{\xi}} e_i d\Phi(e) - \int_{-\infty}^{e^D} E_i \mathbf{N}_{\sigma_{\xi}} e_i d\Phi(e) & \text{intensive margin: capital mis-allocation effect}
\end{array} \right.
\]  

(23)
Regarding the sign of the mis-allocation effect, we consider the case when $1 - \Phi(e^I) > \Phi(e^D)$. If $e^D < e^I < 0$, we have

$$- \left[ \int_{-\infty}^{\infty} E_i N_{\sigma e} e_i d\Phi(e) + \int_{-\infty}^{e^D} E_i N_{\sigma e} e(i) d\Phi(e) \right]$$

$$= - N_{\sigma e} \left[ \int_0^\infty e^{-2N_{\sigma e}e e_i} e_i d\Phi(e) - \int_{e^D}^{e^I} E_i e_i d\Phi(e) \right] < 0$$

In case of $e^D < 0 < e^I$, we can again rearrange the integral to be

$$\int_{e^I}^{\infty} (E_i - 1) N_{\sigma e} e_i d\Phi(e) + \int_{-\infty}^{e^D} (E_i - 1) N_{\sigma e} e(i) d\Phi(e) - \int_{e^D}^{e^I} N_{\sigma e} e(i) d\Phi(e) > 0$$

Because investment (disinvestment) is positive (negative), the first two terms in the bracket are positive. The last term (with the minus sign) will be positive due to the fact that there are more investing firms. □ Q.E.D.

Absent the channel of capital misallocation, in response to higher fundamental uncertainty, aggregate investment is subject to two offsetting forces: expansionary convexity effect and contractionary real-option effect. To be consistent with VAR-based impulse response evidence, it shows that in order for aggregate investment to drop in case of higher disagreement, combined effects of capital mis-allocation and real-option should dominate convexity effect in the short run. For a slow recovery after the adverse shock, the capital misallocation effect should be persistent enough to offset the effect when more firms are pushed out of inaction region and restart investing. I delegate the full model to quantitatively assess their joint impacts on aggregate investment. In addition, I delegate Appendix J to show that building upon firm-specific signals, these effects on aggregate investment are robust.

3.3.4 Dispersion of Investment Rates: Shocks Identification

The simple model predicts that if uncertainty jumps during recessions, then more firms pause investing and hiring. Therefore, dispersion of investment rates could shrink. However, as uncertainty subdues in good times, conditional on more firms restarting to invest and hire, the the dispersion of investment rates should be driven by the shrunk dispersion of productivities.

By contrast, if larger disagreement is seen in bad times, firms can be pushed into inaction region to wait for clearer information. In addition, greater disagreement generates capital mis-allocation, which reinforces the shrinkage in dispersion of investment rates. As it’s turning to good times with more precise information, firms then would better invest and disinvest closer to the optimal quantity. As a result, dispersion of investment rates can be pro-cyclical. Therefore,
dispersion of investment rates naturally serves as critical data moment that helps identify shocks that shift productivity dispersion from shocks that hit non-fundamental information precision.

Note that results suggest that macro and micro uncertainty affect firm-level and aggregate investment exactly via the same channels: convexity effect and real-option effect. These two types of shocks are less distinguishable. I thus delegate the full model to compare and contrast only between disagreement shocks and the micro uncertainty shocks that shifts cross-sectional productivity dispersion. I will show that investment rate dispersion is a sufficient statistic for isolating the non-fundamental second moment shocks.

4 The Full Model

A dynamic stochastic general equilibrium framework is built to quantitatively evaluate the impacts of uncertainty and disagreement on investment and the business cycles. As suggested by David et al. (2014), firms primarily learn from firm-specific information. I thus consider firm-specific signals instead of public signals about aggregate productivity in the full model. The model economy is hit by exogenous shocks to dispersion of information or signals (non-fundamental or informational disagreement shocks) and shocks to dispersion of idiosyncratic productivity (real uncertainty shocks). Heterogeneity on the production side is thus driven by persistent differences in firm-specific productivity and different draws of signals about aggregate productivity each period. Firms make investment decisions subject to a fixed cost and a quadratic capital adjustment cost.

The model differs from the neoclassical model with non-convex adjustment costs (Khan and Thomas, 2008) in that firms infer the unobserved productivity components with imperfect signals. The precision of signals is determined by the dispersion of firm-specific signals. This setup thus augments the framework with conventional uncertainty shocks (Bloom, 2009) by having shocks to dispersion of non-fundamental noises.

4.1 Firms

4.1.1 Technology

There are a large number of production units in the model economy. In period $t$, firm $i$ produces output $y_{i,t}$ using predetermined capital stock $k_{i,t-1}$ and labor $n_{i,t}$ via a Cobb-Douglas Decreasing Returns to Scale (DRS) production technology:

$$y_{i,t} = e^{x_{1}+z_{i,t}}k_{i,t}^{\alpha_{k}}n_{i,t}^{\alpha_{n}}.$$  

I do not differentiate the terms "firm", "establishment" or "plant"
The log stochastic productivity has a common component $x_t$ (aggregate productivity) and a firm-specific component $z_{i,t}$ (idiosyncratic productivity). $\alpha_k$ and $\alpha_n$ respectively refers to share of capital and labor in production. $\alpha_k + \alpha_n \in (0, 1)$ captures the degree of decreasing returns to scale for all firms. Firm $i$ has an infinite horizon.

4.1.2 Imperfect Information: Recursive Signal Extraction

At the beginning of period $t$, firm $i$ does not separately observe the realized components of productivity. Rather, it solves a signal extraction problem at the beginning of each period upon observing the productivity sum $a_{i,t}$ and the signal $s_{i,t}$. These exogenous processes are defined in the following.

**Exogenous Processes.** The aggregate and idiosyncratic productivity components $x_t$ and $z_{i,t}$ of firm $i$’s total productivity, $a_{i,t} = x_t + z_{i,t}$, are assumed to follow stationary AR(1) processes:

\[
\begin{align*}
    x_t &= \rho_x x_{t-1} + \sigma_v v_t \quad \text{(25a)} \\
    z_{i,t} &= \rho_z z_{i,t-1} + \sigma_{e,t-1} e_{i,t} \quad \text{(25b)}
\end{align*}
\]

$\rho_j \in (0, 1)$ with $j \in \{x, z\}$ are persistence parameters with $\rho_z > \rho_x$. Innovations $v_t \sim \mathcal{N}(0, 1)$ and $e_{i,t} \sim \mathcal{N}(0, 1)$ are identically and independently distributed over time and across firms. These are first moment shocks that affect productivity levels. $\sigma_{e,t}$ are time-varying standard deviations that scale the dispersion of next period $t+1$ idiosyncratic productivity shocks, which are realized at the beginning of period $t$. The dynamics of productivity uncertainty is given by:

\[
\log(\sigma_{e,t}) = (1 - \rho_{\sigma_e}) \log(\bar{\sigma}_z) + \rho_{\sigma_e} \log(\sigma_{e,t-1}) + \eta_{\sigma_e} \epsilon_{\sigma_e,t}. \quad \text{(26)}
\]

The standard normal innovations $\epsilon_{\sigma_e}$ known as **micro uncertainty shocks** affect the dispersion of firms’ cross-sectional idiosyncratic productivity. $\bar{\sigma}_z$ is the unconditional mean of uncertainty.

The signal that contains information regarding the aggregate productivity is contaminated by the idiosyncratic noise shocks $\xi_{i,t}$, which are an i.i.d. draw from $\mathcal{N}(0, 1)$ over time and across firms such that

\[
    s_{i,t} = x_t + \sigma_{\xi,t} \xi_{i,t}. \quad \text{(27)}
\]

The common parameter $\sigma_{\xi,t}$ captures the spread of heterogeneous information quality or “noisiness” across firms at the beginning of period $t$, a non-fundamental shifter of cross-sectional disagreement. I similarly assume that $\sigma_{\xi,t}$ in log follows a stationary AR(1) process with unconditional
\[
\log(\sigma_{\xi,t}) = (1 - \rho_{\sigma_{\xi}}) \log(\bar{\sigma}_{\xi}) + \rho_{\sigma_{\xi}} \log(\sigma_{\xi,t-1}) + \eta_{\sigma_{\xi}} \varepsilon_{\sigma_{\xi},t}
\]  

(28)

The innovation term \(\varepsilon_{\sigma_{\xi},t} \sim N(0,1)\) denotes shocks to the dispersion of firm-specific noises. Similarly to modeling imperfect information as public signal, larger \(\sigma_{\xi,t}\) makes more firms acting on imprecise firm-specific signals, which similarly measures the aggregate information precision. Thus, I call \(\varepsilon_{\sigma_{\xi},t}\) shocks to information precision that affect disagreement, the non-fundamental disagreement shocks. \(\rho_{\sigma_{\xi}} \in (0,1)\) is the persistence parameter and \(\eta_{\sigma_{\xi}}\) is the standard deviation of the innovation to disagreement.

Swings in this noise spread can be interpreted as firms exogenously hold more or less dispersed beliefs about the latent unknown, which arises from the sources other than the true dynamics of economic fundamentals.\(^8\) I assume uncertainty shocks and disagreement shocks \(\{\varepsilon_{\sigma_{e}}, \varepsilon_{\sigma_{\xi}}\}\) are mutually orthogonal for the sake of identifying their individual contributions to investment dynamics and business cycles. Note that with this assumptions, firms disagree due to dispersed information does not necessarily suggest the real economy is undergoing changes in uncertainty.

**Information Set.** The only information learned by the firm in order to infer the productivity components is the received noisy signal about aggregate productivity and the observable productivity sum. I assume firms do not learn from the other firms’ information sets and thus do not act upon other firms’ capital and labor decisions.\(^9\) In addition, I assume that firms would never know what the history of true realizations of aggregate TFPs is. It means that all the past other aggregate state variables do not reveal the true realizations.

As firm \(i\) enters period \(t\), it carries a few state variables that characterize the imperfect information environment: (1) period \(t-1\) posterior beliefs about the then productivity components \(x_{i,t-1|t-1}\) and \(z_{i,t-1|t-1}\); (2) posterior variance and co-variance matrix, or forecast variance about \(x_{t-1}\) and \(z_{i,t-1}\) in period \(t-1\), \(\hat{\Sigma}_{t-1|t-1}\). This variance matrix is an aggregate variable as all firms are subject to the same second moment dynamics that affect the precision of forecasting; (3) \(t-1\) uncertainty realizations that govern how dispersed the period \(t\)’s idiosyncratic productivity \(\sigma_{e,t-1}\).

As period \(t\) unfolds, firm \(i\) knows (1) how disagreed they are among themselves about aggregate productivity \(x_{t}\) due to imprecise information \(\sigma_{\xi,t}\); (2) how uncertain the \(z_{i,t+1}\) productivity will

---

\(^8\)The driver of distributional changes in firms’ beliefs about fundamentals may be due to a sunspot variable that “conveys no information about technology, preference or endowments and does not directly enter the equilibrium conditions” (Woodford, 1990). It is possible that changes in belief dispersion are in fact endogenous and driven by optimal information updating by firms. While this is an interesting idea, it is beyond the scope of this paper.

\(^9\)This model thus abstracts from the complication that firms need to care about what other firms think about what others think, i.e. higher order beliefs. Implicitly, firms are assumed not to communicate with each other and do not coordinate to reach a consensus or maintain a given noise dispersion.
be: $\sigma_{e,t}$; observes (3) the realized total productivity sum $a_{i,t}$, and (4) receives the signal $s_{i,t}$.

**Signal Extraction.** Firm $i$’s profit maximization problem consists of solving a signal extraction problem and making optimal labor and investment decisions. Bayesian firms use a recursive Kalman Filtering way to optimally update prior beliefs to form posterior estimates of productivity components. Acting upon the posterior beliefs about current productivity $x_{i,t|t}$ and $z_{i,t|t}$, firms compute future expectation of marginal product of capital in order to pin down the investment decision given they know the shock persistence.

How firms form new beliefs are derived in Appendix G. Important to note that the uncertainty realizations about current period productivity $\sigma_{e,t-1}$, along with the disagreement $\sigma_{\xi,t}$ affect the precision of forming current beliefs about period $t$. However, precision of future expectation for period $t+1$ profitability will be combining the current belief imprecision with the additional variance given by newly realized uncertainty $\sigma_{e,t}$.

4.1.3 Capital Adjustment: Non-convex and Convex Costs

I assume firms have to pay a fixed cost $c_k > 0$ per unit of their existing capital stock $k_{i,t-1}$ as long as it decides to invest or disinvest ($I_{i,t} \neq 0$) each period. In addition, capital adjustment incurs a quadratic adjustment cost given by

$$\frac{\theta}{2} \frac{I_{i,t}}{k_{i,t-1}}^2 k_{i,t-1}$$

where $\theta > 0$ indexes the level of the cost.

Empirical evidence lends support to modeling adjustment costs (Cooper and Haltiwanger, 2006). In addition, non-convex fixed cost is necessary to generate a region of inaction in which firms do not take investment and disinvestment actions in equilibrium (Bloom, 2009). Hence, it is pivotal for uncertainty and disagreement to affect aggregate investment through the real-option effect channel. Quadratic adjustment cost is to attenuate the excessive responses of investments to productivity shocks.

4.1.4 Profit Maximization

Each firm can be denoted by its predetermined stock of capital $k_{i,t-1}$ and its previous period posterior estimates of productivity components $m_{i,t-1|t-1} = \{x_{i,t-1|t-1}, z_{i,t-1|t-1}\}$. Then we can fully describe the distribution of firms over the Borel algebra $S$ for the space $S = R^+ \times R^2$ on which the probability measure $\mu_{t-1}$ is defined. $\mu_{t-1}$ denotes the firm distribution in the end of period $t-1$ (beginning of period $t$) and is varying over time. The capital stock $k_{i,t}$ at any point
of time is non-negative.

The aggregate state of the economy at the beginning of period \( t \) is described by \( \Omega_t = \{x_t, \sigma_{e,t}, \sigma_{\xi,t}, \sigma_{e,t-1}, \mu_{t-1} \} \) as disagreement, and uncertainty are time-varying. The reason why \( \sigma_{e,t-1} \) enters the aggregate state vector is because it matters for firms to form current period posterior beliefs about productivity components. Law of large numbers average out idiosyncratic productivity shocks and noise shocks and they do no enter as aggregate state variables. I assume a mapping \( \Gamma \) of \( \Omega_t \) as some aggregate laws of motion, which moves the firm distribution over time such that \( \mu_t = \Gamma(\Omega_t) \).

**Labor Demand.** Given existing capital stock \( k_{i,t-1} \), aggregate law of motion \( \Gamma(\Omega_t) \), and wage \( w_t \), along with a stochastic discount factor \( \beta Q_{t+1|t} \) in numeraire of consumption goods, firm \( i \) maximizes expected profit among options of being inaction and taking non-zero investment subject to the adjustment costs.

Firm \( i \)'s labor demand can be separately determined apart from the dynamic programming problem by solving a static optimization problem each period:

\[
\max_{n_{i,t}} e^{x_t + z_{i,t}} k_{i,t-1}^{\alpha_n} n_{i,t}^{\alpha_n} - w_t n_{i,t}
\]

to have the optimal labor demand

\[
n_{i,t} = \left[ \frac{\alpha_n e^{x_t + z_{i,t}} k_{i,t-1}^{\alpha_n}}{w_t} \right]^{\frac{1}{1-\alpha_n}}
\]

Net the wage bill payment, firm \( i \)'s operating profit is \((1 - \alpha_n) y_{i,t}\) where

\[
y_{i,t} = \left[ \frac{\alpha_n}{w_t} \right]^{\frac{1}{1-\alpha_n}} \exp \left( x_t + z_{i,t} \right) k_{i,t-1}^{\alpha_n}(1-\alpha_n)
\]

**Profit Maximization.** Then firm’s dynamic optimization problem is then defined below:

\[
V(k_{i,t-1}, m_{i,t-1}|t-1; \Omega_t) = \max_{k_{i,t}} \{ V^{Adj}, V^{Non-Adj} \}
\]

where

\[
V^I(k_{i,t-1}, m_{i,t-1}|t-1; \Omega_t) = \max_{k_{i,t}} (1 - \alpha_n) y_{i,t} - I_{i,t} - c_k k_{i,t-1} - \frac{\theta}{2} \left[ \frac{I_{i,t}}{k_{i,t-1}} \right]^2 k_{i,t-1} + \beta E Q_{t+1|t} V(k_{i,t}, m_{i,t}|t; \Omega_{t+1})
\]

\[
V^{Non-Adj}(k_{i,t-1}, m_{i,t-1}|t-1; \Omega_t) = \max_{k_{i,t}} (1 - \alpha_n) y_{i,t} + \beta E Q_{t+1|t} V((1 - \delta)k_{i,t-1}, m_{i,t}|t; \Omega_{t+1})
\]

\( y_t \) by Equation (30) and \( I_{i,t} = k_{i,t} - (1 - \delta)k_{i,t-1} \).
4.2 Households

I assume there is a representative household who has quasi-linear utility in labor hours and owns all the firms. It solves the following lifetime utility maximization problem:

\[
W(\Omega_t) = \max_{\{c_t, n_t^h\}_{t=0}^{\infty}} \log(c_t) + \psi(1 - n_t^h) + \beta \mathbb{E}_t W(\Omega_{t+1})
\] (34)

subject to

\[
c_t = w_t n_t^h + \int_S \Pi_{i,t} \mu(d[k_{i,t-1}, x_{i,t-1}| t-1, z_{i,t-1}| t-1]).
\] (35)

\(\psi\) is the marginal disutility of labor. Take wage \(w_t\) as given, household chooses consumption \(c_t\) and total labor hours \(n_t^h\), which are to be allocated among firms. The household does not save but receives the profits of all the firms each period \(^{10}\). Optimization yields the following first order conditions:

\[
w_t = \psi c_t \tag{36}
\]

\[
\Lambda_t = \frac{1}{c_t} \tag{37}
\]

\(\Lambda_t\) is the Lagrangian multiplier associated with the budget constraint and has the interpretation of the marginal utility of consumption. In general equilibrium, this term enters firm \(i\)’s stochastic discount factor.

4.3 Recursive Competitive Equilibrium

Use prime and subscript -1 to respectively denote future and predetermined variables, a recursive competitive equilibrium is defined as collection of functions \(\{V, W, N, K, \lambda, \Lambda, C, w, N^h, \Gamma, \Xi\}\) such that

1. Given predetermined capital stock \(k_{-1}\), observables \(a\), and signal \(s\), wage \(w\), and stochastic discount factor \(\beta Q\), the firm’s value function \(V\), the policy function of optimal capital stock demand \(K\), and labor demand policy \(N\) satisfy the firm’s recursive problem (31).

2. Given wage \(w\), the welfare function \(W\) satisfies household’s utility maximization problem (34). The marginal value of consumption \(\Lambda\) and the policy function for consumption \(C\) satisfy (37) and (36).

\(^{10}\) Household can save through buying shares of firms. Abstract from saving, the results about identifying responses of aggregate investment dynamics to different types of second moment shocks are not changed.
3. The labor market clears with wage $w$. The labor supply $N^h$ and demands satisfy

$$N^h = \int_S N(k_{-1}, m_{-1|t-1}, \mu_{-1}; x, z, \xi, \sigma_e, \sigma_\xi) \mu(d[k_{-1}, m_{-1|t-1}])$$

4. The goods market clears:

$$C = \int_S \left\{ [\alpha_n]^{\frac{\alpha_n}{1-\alpha_n}} \exp \left[ \frac{a}{1-\alpha_n} k_1^{\frac{\alpha_n}{1-\alpha_n}} - \frac{\theta}{2} \right] \right\} \mu(d[k_{-1}, m_{-1|t-1}])$$

where $K = K(k_{-1}, m_{-1|t-1}, \mu_{-1}; x, z, \xi, \sigma_e, \sigma_\xi)$ and $\mathbb{I}_P$ and $\mathbb{I}_{c_k}$ some indicator functions such that

$$\mathbb{I}_P = \left\{ \begin{array}{ll} 1 : K \neq (1-\delta)k_{-1} \\ 0 : K = (1-\delta)k_{-1} \end{array} \right\}, \quad \mathbb{I}_{c_k} = \left\{ \begin{array}{ll} 1 : K \neq (1-\delta)k_{-1} \\ 0 : K = (1-\delta)k_{-1} \end{array} \right\}$$

5. Stochastic discount factor is given by:

$$\Xi = \frac{C}{C'} = \frac{w}{w'}$$

6. The aggregate law of motion defines the dynamics of the probability measure $\mu_t$ of firms over space $S$: $\mu = \Gamma(x, \sigma_e, \sigma_\xi, \sigma_e_{-1}, \mu_{-1})$

7. The state variables of posterior beliefs $m$ satisfy the recursive time-varying Kalman Filter conditions given by Equations (41a), (41b) and (41c). In addition, the productivity level and second moment stochastic processes are given by Equations (25a)-(28).

### 4.4 Approximate Aggregation

Firm $i$ enters period $t$ carrying the key state variable, $\mu(k_{i,t-1}, m_{i,t-1|t-1})$, end of period $t-1$ joint distribution of firm-level capital stock and beliefs about aggregate TFP and firm-level TFP across firms. Firm’s investment decision in period $t$ and newly formed posterior beliefs, which move firms in the distribution over time, depend on a range of aggregate state variables and $\mu_{t-1}$, i.e. defined in $\Omega_t$ vector. Hence the aggregate law of motion $\mu_t = \Gamma(\Omega_t)$ cannot be written in closed form.

Following literature on the general equilibrium of heterogeneous agents, I assume firms is bounded rational in that they only use the a finite number of distributional moments to infer the state variable of the joint distribution and then the market-clearing wage (Krusell et al., 1998). Specifically, I use the cross-sectional means of capital stocks $\bar{k}_{t-1}$ and of posterior beliefs about the
log aggregate productivity $\bar{x}_{t-1|t-1}$ to describe $\mu_{t-1}$, which turns out to be sufficient to describe the joint distribution. I delegate later chapters to demonstrate the validity of this approximate aggregation approach. Market-clearing wage $w_t$ taken as given for firms’ investment decisions is also assumed to be function of these two means.

Firms thus take the following log-linear perceived laws of motion to infer the distribution dynamics and equilibrium wage:

$$\begin{bmatrix}
\log(\bar{k}_t) \\
\bar{x}_{t|t} \\
\log(w_t)
\end{bmatrix} = \Gamma_0 + \Gamma_1 \begin{bmatrix}
\log(\bar{k}_{t-1}) \\
\bar{x}_{t-1|t-1} \\
\log(\sigma_{\xi,t}) \\
\log(\sigma_{e,t}) \\
\log(\sigma_{e,t-1})
\end{bmatrix} + \Gamma_2 \begin{bmatrix}
\log(\sigma_{\xi,t}) \\
\log(\sigma_{e,t}) \\
\log(\sigma_{e,t-1})
\end{bmatrix}$$ (38)

Where $\Gamma_0, \Gamma_1$ and $\Gamma_2$ are conformable vectors or matrices of coefficients. It should be noted that the de facto aggregate TFP $x_t$ is not included as aggregate state variable because firms never truly act upon the realized TFP per the information frictions. In addition, a number of second moment state variables are to augment the laws of motion in order to capture the aggregate impacts of uncertainty and disagreement shocks. The lagged uncertainty enter the laws of motion simply because it directly affects the newly formed posterior beliefs about current period TFP components.

4.5 Sketch of Model Solution

The equilibrium definition requires that with the acceptable error tolerance, these approximate law of motions should “rationalize” the actual equilibrium dynamics. Precisely, the actual market-clearing wage series, aggregate capital stock and mean forecast of TFP series move closely enough as if their dynamics are following these laws. Therefore, these equations will be determined in equilibrium.

Taking conjectured policy function of a firm’s optimal investment as solved for a given parameter conjecture of aggregate laws of motion, I simulate the economy for a fixed number periods after burning 500 initial periods of data. Simulation is done following Young (2010), which moves firm density across grid nodes over time. Then a non-linear solver is used to clear the labor market to obtain a time series of equilibrium real wage. Then I re-estimate the equation systems (38) using OLS based on the simulated data and update the $\Gamma_j, j \in \{0, 1, 2\}$ coefficients. The equilibrium is solved by looping over policy function and updating coefficients until full convergence is reached.

To save computational complexity, the exogenous processes: aggregate productivity, idiosyncratic productivity, noise, uncertainty, and disagreement are discretized into two-state Markov

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11By law of large numbers, mean of posterior beliefs about idiosyncratic productivity should be intrinsically zero, which does not affect the aggregate dynamics.
chain processes. The exact solution algorithm is detailed in Appendix M.

5 Parameter Values

The full set of parameter values that feeds into the full quantitative model is pinned down through estimations as well as calibrations.

5.1 Measurement and Estimation

First, to well approximate the micro uncertainty process in the model, I use the cross-sectional standard deviation of log firm-level TFP data as computed by Imrohoroglu and Tüzel (2014), which builds up a panel of TFP data ranging from 1963 to 2013 based on Compustat Database. Appendix H shows how these firm-level TFPs are estimated. Regarding the firm-level belief data, given its limited availability, I follow Bachmann et al. (2013b) to construct a business forecast disagreement series based on Philadelphia Fed’s Business Outlook Survey, which surveys 100-125 manufacturers in the Third Federal Reserve District about their beliefs in the direction of change of their business operations every month since May 1968. The numbers of firms each month who report increase, decrease or no change in their beliefs are recorded.\footnote{From May 1968 up to March 2015, there are 97 firms that responded to the surveys every month on average}

A disagreement index can be constructed to capture the cross-sectional belief dispersion based on the numbers of firms that responded to a question probing their views about their own business conditions in six months relative to the survey date.\footnote{The question is framed as “General Business Conditions: What is your evaluation of the level of general business activity six months from now versus [CURRENT MONTH]: decrease, no change, increase”. The word “general” can be misleading. However, Trebing (1998) finds that answers to this question are highly correlated with answers to questions that specifically ask firms’ beliefs about their firm-specific factors, e.g. shipment and orders. Hence, both Trebing (1998) and Bachmann et al. (2013b) suggest this question well measures firms’ own beliefs about their idiosyncratic business activity} The disagreement measure is defined as

\[
DIS_t = \sqrt{F_t^+ + F_t^- - (F_t^+ - F_t^-)^2}
\]  

The fractions of responding firms with beliefs of increase and decrease in its own business are denoted by \(F_t^+\) and \(F_t^-\) respectively. The closer of this index is to 1, the greater magnitude of cross-sectional disagreement about their own future profitability is among firms. However, it should be noted that this disagreement index does not perfectly correspond to the dispersion of firms’ beliefs about idiosyncratic TFP as in the model.

The parameters of persistence and innovation S.D.s for fundamental uncertainty and informational disagreement processes are obtained by estimating a large VAR system with both uncer-
tainty and disagreement measures included along with major macro aggregates, which is in line with Leduc and Liu (2015). I estimate a Structural VAR system with Cholesky ordering restriction that augments the ordering structure as specified in Bloom (2009). Specifically, in addition to the stock market valuation, federal funds rate, average hourly earnings in manufacturing sector, consumer price index, weekly average hours in manufacturing, employment in manufacturing, industrial production in manufacturing, I include the real gross private domestic investment, disagreement index, and uncertainty measure are in the estimation. The estimation takes quarterly disagreement index series and within-year linearly interpolated standard deviations of firm-level TFPs from 1969 to 2013 and both are in logs. The benchmark ordering restriction puts uncertainty measure first which is followed by the disagreement index. This is to isolate the non-fundamental sources of second moment shocks that shift cross-sectional disagreement but do not immediately affect the actual dispersion of real idiosyncratic productivity. I also show that the ordering between uncertainty and disagreement does not affect the estimation results and the estimates are robust across VAR specifications in Appendix A.\footnote{Other estimated impulse responses to disagreement shocks and uncertainty shocks can be also found in Appendix A.}

The VAR-based impulse responses of uncertainty and disagreement to their own innovations are plotted in Figure 4. Figure 4 shows that after reaching the peak of jumps in one year, it takes four years for uncertainty to fully decay. On average, uncertainty falls about 25% of its peak within a year. Then by the AR(1) formulation, the quarterly uncertainty shock persistence is implied by $\rho_{\sigma_e} = (1 - 0.25)^{1/4} = 0.93$. By contrast, within a year, the disagreement index drops about 85.7% off its peak, the quarterly persistence is thus given by $\rho_{\sigma_\xi} = (1 - 0.857)^{1/4} = 0.615$. On impact, the percent jump directly translates into the standard deviation numbers that capture changes in innovations to uncertainty or disagreement. Hence, the estimation shows that uncertainty shocks are more persistent than that of informational shocks to disagreement. However, shocks to uncertainty has one tenth of its innovation size relative to that of disagreement shocks.

Parameter values for the aggregate productivity and idiosyncratic productivity processes are directly borrowed from Cooper and Haltiwanger (2006), which estimates the persistence and standard deviation using constructed plant capital series based on data on retirements and investment constructed from the Longitudinal Research Database (LRD). Following Edmond and Veldkamp (2009), I convert their annual persistence numbers to quarterly counterparts using $\rho_{\text{quarter}} = \rho_{\text{annual}}^{1/4}$. Then I convert S.D. of their aggregate productivity innovations by $\sigma_{\text{quarter}} = \sigma_{\text{annual}}/(1 + \rho_{\text{quarter}} + \rho_{\text{quarter}}^2 + \rho_{\text{quarter}}^3)$. Hence, $\rho_x = 0.93 < \rho_z = 0.97$, $\sigma_v = 0.014$ and the unconditional dispersion of log idiosyncratic productivity $\bar{\sigma}_e = 0.15$. I note that the sample mean of cross-sectional annual S.D. of log firm-level TFPs based on Compustat data is 0.42, which falls in the range of 0.3 as in Gilchrist et al. (2014) and 0.64 as in Cooper and Haltiwanger (2006).
Figure 4: VAR-based IRFs: Bloom (2009) with Disagreement, Uncertainty and Investment

Notes: The figure plots the impulse response of cross-sectional firm-level disagreement about future profitability to a 1% increase in innovation in Disagreement Index as in Bachmann et al. (2013b) and the impulse response of uncertainty measure, firm-level TFPs dispersion to a 1% increase in innovation in cross-sectional standard deviation of firm-level TFPs as in Imrohoroglu and Tüzel (2014). They are obtained from estimation of a ten-variable VAR system with the following Cholesky recursive ordering: log(S&P500 stock market index), log(TFP dispersion), log(disagreement index), Federal Funds Rate, log(average hourly earnings in manufacturing), log(consumer price index), weekly average hours in manufacturing, log(manufacturing employment), log(real gross private domestic investment), and log(industrial production: manufacturing). The frequency of data is quarterly, and the VARs are estimated with four lags. The sample covers 1969Q1 to 2013Q4. Red dashed lines define \( \pm 2 \) S.E. confidence bands.

Therefore, the magnitudes of firm-level productivity dispersion \( \bar{\sigma}_e \) are close enough regardless of whether it’s based on Compustat or LRD sample, despite I set it to be the number that is more consistent with Cooper and Haltiwanger (2006)’s other estimates on productivity levels. For the sample average of business outlook disagreement index is around 0.68, given that this index does not perfectly correspond to any disagreement or dispersion measure in the full model, I thus simply set the unconditional noise dispersion \( \bar{\sigma}_\xi = \sqrt{0.68^2 - 0.15^2} = 0.66 \), assuming that the dispersion of real productivity and the dispersion of noises are additively shaping the business outlook disagreement index. In Table 1, I summarize the parameter values as estimated based on the VAR system.

5.2 Calibration

Some parameter values are standard as in the literature. One period corresponds to a quarter. I take numbers on the capital and labor shares in production technology from Khan and Thomas (2008) such that \( \alpha_k = 0.256 \) and \( \alpha_n = 0.64 \) so that capital cost share is about one third. The
Table 1: Parameter Values: Fundamental Uncertainty and Informational Disagreement

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_{\sigma_e}$</td>
<td>0.930</td>
<td>Persistence of Micro Uncertainty</td>
</tr>
<tr>
<td>$\rho_{\sigma_\xi}$</td>
<td>0.615</td>
<td>Persistence of Informational Disagreement</td>
</tr>
<tr>
<td>$\bar{\sigma}_z$</td>
<td>0.150</td>
<td>Unconditional Mean of Uncertainty</td>
</tr>
<tr>
<td>$\bar{\sigma}_\xi$</td>
<td>0.660</td>
<td>Unconditional Mean of Informational Disagreement</td>
</tr>
<tr>
<td>$\eta_{\sigma_e}$</td>
<td>0.007</td>
<td>Standard Deviation of Uncertainty Shocks</td>
</tr>
<tr>
<td>$\eta_{\sigma_\xi}$</td>
<td>0.070</td>
<td>Standard Deviation of Informational Disagreement Shocks</td>
</tr>
</tbody>
</table>

**Notes:** Parameters are based on 10 variable Structural VAR estimations. See details in the text.

subjective discount factor $\beta = 0.99$ implies an annualized real interest rate of 4% at the steady state.

A stationary distribution of firms is computed by solving the model with all aggregate shocks turned off. I discretize idiosyncratic productivity and the firm-specific noise into three states. Wage is pinned down by having the wage taken by the firms close enough to the market clearing wage. Equilibrium is solved based on the exact firm mass distribution over the grids. The steady state firm distribution in capital stock helps calibrate parameters on capital depreciation, adjustment costs, and marginal disutility of labor ($\delta, \theta, c_k, \psi$). They are jointly pinned down by matching the model-implied moments to the data moments of the cross-sectional distribution of plant-level annual investment rates as in Cooper and Haltiwanger (2006).\(^{15}\) Table 2 lists the moment targets and reports the calibrated parameters. The process of calibrating the model is supposed to align the model so as to generate the right fraction of firms who incur large investment spikes as defined to have annual investment rate greater than 20%, the lumpy investment, and the fraction of firms with disinvestment, i.e. annualized investment rate smaller than -1%.

Table 2: Calibrated Parameters and Data Targets

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Parameter</th>
<th>Target</th>
<th>Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>0.0656</td>
<td>Quadratic Cost</td>
<td>Fraction investment spikes</td>
<td>18.6 %</td>
</tr>
<tr>
<td>$c_k$</td>
<td>0.0270</td>
<td>Fixed Cost</td>
<td>Fraction of disinvestment</td>
<td>10.4 %</td>
</tr>
<tr>
<td>$\psi$</td>
<td>3.3606</td>
<td>Marginal Disutility of Labor</td>
<td>Average Labor Hours</td>
<td>0.33</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.0237</td>
<td>Quarterly Capital Depreciation Rate</td>
<td>Average Investment Rate</td>
<td>12.2 %</td>
</tr>
</tbody>
</table>

The calibrated adjustment costs, both convex and non-convex, are small in size, which are in line with estimates based on Simulated Method of Moment estimation of a partial equilibrium model as in Cooper and Haltiwanger (2006). The calibrated quarterly capital depreciation rate is

\(^{15}\) The moment targets are estimates of annual rates. I convert my model-predicted quarterly investment rates to get annual equivalents according to $\frac{k_{t+1}}{k_{t+1}} - (1 - \delta)^t$. 

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2.37 %, which is close enough to 2.5 % as usually assumed in the literature, for example, Gilchrist et al. (2014) and Bloom (2014). The marginal dis-utility of labor falls in the range of $2 \sim 4$ as commonly documented in the literature, e.g. Hansen (1985).

6 Quantitative Results

6.1 Steady State

The model generates a number of steady state moments of firm-level investment rate distribution, which closely matches the key moment targets implied by micro-level data according to Table 3. A few trade-offs of targets exist in the calibration process, which affect the model-implied firm-level investment rate distribution. Note that the actual band of inaction region is not targeted, the presence of fixed cost still generates 2.4 % of firms that do not take investment or disinvestment action. Larger fixed cost parameter $c_k$ can increase this range of inaction band. However, as it is modeled as a symmetric fixed cost that has to be paid for capital adjustment from both investing and disinvesting firms, higher of this can reduce the cross-sectional average of investment rates. In addition, to generate larger mass of investment spikes, a smaller convex adjustment cost $\theta$ can simultaneously increase the number of firms who would incur negative investments.

Table 3: Calibrated Model: Steady State Moments

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data Source</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Labor Hours</td>
<td>Standard</td>
<td>0.33</td>
<td>0.322</td>
</tr>
<tr>
<td>Fraction of disinvesting firms</td>
<td>LRD</td>
<td>10.4 %</td>
<td>12.5 %</td>
</tr>
<tr>
<td>Fraction of investment spikes</td>
<td>LRD</td>
<td>18.6 %</td>
<td>15 %</td>
</tr>
<tr>
<td>Average Investment Rate</td>
<td>LRD</td>
<td>12.2 %</td>
<td>10.2 %</td>
</tr>
<tr>
<td>Fraction of investing firms</td>
<td>LRD</td>
<td>81.6 %</td>
<td>85.1 %</td>
</tr>
</tbody>
</table>

Notes: The moments of the steady state cross-sectional firm-level investment rates are computed based on a continuous simulation of firm’s grid-based investment policy function after 200 quarters are burned for a panel of 10,000 firms.

Despite all these trade-offs, the model is able to generate the good amount of investing firms at steady state, which is about seven to eight times the mass of disinvesting firms which is consistent with the data. Importantly, this guarantees that in response to rises in informational disagreement, drops in investments from more productive firms net the increased investments from less productive firms is going to reduce the aggregate investment.
6.2 Impacts on Aggregate Investment: Uncertainty and Disagreement Shocks

In order to examine the net effects of disagreement shocks and uncertainty shocks on aggregate investment, I compute Impulse Response Functions of aggregate investment rate respectively to one standard deviation increase in micro uncertainty and dispersion of information. Simulations of economies are done 10000 times. One standard deviation increase in innovation is imposed in period 101 with first 100 quarters burned, the economy is evolving stochastically thereafter. Investments are aggregated and average out across simulations.

Figure 5: Model-based IRFs: Shocks to uncertainty or disagreement in period 2

Figure 5 plots the impulse responses of quarterly aggregate investment rate given one S.D. jump in two different second moments in period 2. Aggregate investment drops immediately in response to uncertainty jump. Convexity effect brought by fundamental uncertainty jump quickly kicks in after one quarter which pushes firms outside the inaction region. Conditional on adjusting capital, firms restart investing and capital goes to very productive firms. Therefore, the effects of uncertainty shocks are robust even in this framework of imperfect information. The “drop and quick rebound” finding is in line with the literature of uncertainty shocks (Bloom, 2009).

However, for more dispersed information that triggers non-fundamental disagreement, extensive margin real-option effect is limited so that there are fewer firms pause and wait in response to the shock. Conditional on the intensive margin of taking investment and disinvestment, convexity effect dominates for a quarter and firms see increase in the expected product of capital. However, after two quarters, when firms never observing real fundamental improvement in productivity and thus productive firms start disinvesting because they become even more reluctant to believe their
productivity draws are good enough to justify further investment. This capital mis-allocation effect starts driving down aggregate investment and creates a recession.

Importantly, we see capital mis-allocation effect can be persistent so that the drop in aggregate investment cannot fully recover its loss after three years. The reason is that imperfect information builds in a belief inertia as we look into how firms’ dynamically extract beliefs over time following a Kalman Filtering way. Larger non-fundamental dispersion of noises force firms to increasingly under-weigh new observation of total productivity $a_{i,t}$ by reducing the Kalman gain from learning. Therefore, productive firms’ negative beliefs carry over time. This persistence continues forcing firms to cut investment. Therefore, we see a gradual decline in investment and then it takes quite a while for firms to realize there was no fundamental shocks initially. Hence, capital mis-allocation effect is crucial to generate this gradual decline and sluggish recovery.

In addition, this figure plots the aggregate investment rate response to 1 standard deviation shocks to both second moment measures. Such impulse response to both shocks help understand in reality when both shocks can be present such that belief heterogeneity can be affected. It shows that the real option effect of real uncertainty shocks dominate in the very short run whereas the mis-allocation effect of informational noisiness shocks kicks in that prolongs the recovery. It is clear to see that the sluggish recovery due to imperfect learning is crucial to propagate the impacts of real uncertainty shocks.

### 6.3 Identification of Real vs. Noise-dispersion Shocks: Cyclicality of Investment Rate Dispersion

We then ask the question how we can identify shocks that shifts fundamental uncertainty and non-fundamental disagreement. It can be shown that the firm-level investment rate dispersion can be the identifier that tells apart the two types of shocks.

Figure 6 plots the dispersion of investment rate in response to two types of shocks imposed in period 2. It shows that real-option effect that is common to both shocks shrinks investment rate dispersion by pushing more firms into inaction region. Then, conditional on firms restart investing and disinvesting, firms are building up capital due to the expansionary convexity effect. However, given the fact that real-option effect due to non-fundamental disagreement shocks is weaker as compared to effect due to uncertainty shocks, convexity effects can even expand the dispersion of investment rates in case of non-fundamental second moment shocks in quarter 2.

However, then intensive margin effect of capital mis-allocation of disagreement shocks starts kicking in and blurs investing and disinvesting firms view about how productive they are. This creates capital mis-allocation which brings more firms to build up capital amount similar to each other. Conversely, convexity effect of uncertainty continues generating pent-up demands for invest-
ment which increases the heterogeneity in firm-level investment rates. It exhibits that for about three to four years going forward, more dispersed real firm-level productivity due to jump in micro uncertainty drives more dispersed investment rates relative to the case when uncertainty is lower. By contrast, the persistent dampening effects of investment’s response to true productivity draws largely shrinks the investment rate dispersion.

Clearly, as argued in Bachmann and Bayer (2014) that the fact that investment rate dispersion is pro-cyclical limits the macro implications of models of uncertainty shocks. As we see so far, a model with both disagreement shocks and uncertainty shocks can resuscitate the business cycle effects of uncertainty shocks. I’ll show in the following that this framework helps generate pro-cyclicality of investment rate dispersion.

Firstly, I turn off aggregate TFP shocks and simulate the economies assuming second moment shocks are the primitive shocks. Table 4 below checks the correlation between dispersion of investment rates as measured by cross-sectional standard deviation and fundamental uncertainty, along with how this dispersion is correlated with non-fundamental disagreement.

<table>
<thead>
<tr>
<th></th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>+1</th>
<th>+2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Disagreement</td>
<td>-0.186</td>
<td>-0.238</td>
<td>-0.283</td>
<td>-0.372</td>
<td>-0.329</td>
</tr>
<tr>
<td>Uncertainty</td>
<td>0.170</td>
<td>0.224</td>
<td>0.281</td>
<td>0.148</td>
<td>0.434</td>
</tr>
</tbody>
</table>

Notes: Quarterly correlation coefficients are averaged across 500 economies simulations. ”-“: S.D. of Investment Rates Lags and ”+“: S.D. of Investment Rates Leads.
of pure noises and investment rate dispersion while more dispersed real productivity enhances the investment rate dispersion. Importantly, this table suggests that in a model we can rely on investment rate dispersion to simply tell if the second moment shocks that are real or not. Note that we do not need aggregate first moment shocks to generate this clear identification scheme.

Next, we proceed by considering correlations between second moment shocks with TFP shocks. As suggested by Bloom (2014) and by evidence shown before, uncertainty shocks and non-fundamental disagreement shocks are counter-cyclical. Therefore, I impose negative correlations between TFP shocks and the two second moment shocks respectively. Then I simulate the economy and check if investment rate dispersion is truly pro-cyclical, that is, if there is positive correlation between the dispersion and TFP shocks. Table 5 compares the cyclicity of investment rate dispersion when considering pairwise negative correlations between TFP shocks and second moment shocks.

<table>
<thead>
<tr>
<th>Imposed TFP Correlation w/</th>
<th>Uncertainty</th>
<th>Disagreement</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1</td>
<td>-0.056</td>
<td>0.088</td>
</tr>
<tr>
<td>-0.3</td>
<td>-0.092</td>
<td>0.111</td>
</tr>
<tr>
<td>-0.5</td>
<td>-0.087</td>
<td>0.133</td>
</tr>
<tr>
<td>-0.7</td>
<td>-0.107</td>
<td>0.155</td>
</tr>
<tr>
<td>-0.9</td>
<td>-0.094</td>
<td>0.191</td>
</tr>
</tbody>
</table>

Notes: Correlation coefficients are averaged across 500 economies simulations. Simulated 500 periods with 100 periods burned. Correlation between TFP and S.D. of firm-level investment rates.

It shows that regardless of how much the non-fundamental shocks to disagreement are negatively correlated with TFP shocks, investment rate dispersion is procyclical. On the contrary, given the fact that uncertainty is counter-cyclical, investment rate dispersion is consistently counter-cyclical, which is at odds with the data. Therefore, we conclude that separating second moment shocks that are pure noise-driven from fundamental uncertainty shocks are crucial so as to better align the model with the data. Note that the key channel that non-fundamental shocks to disagreement brings about is the capital-misallocation effect. Hence this effect is crucial for its strong impacts on driving the investment rate dispersion through which aggregate investment dynamics is affected.
6.4 Historical Variance Decomposition: Relative Magnitude of Second Moment Shocks

I show that a model that isolates fundamental uncertainty shocks from non-fundamental disagreement shocks can generate (1) different business cycle effects upon aggregate investment and investment rate dispersion. It predicts (2) investment rate dispersion can be used to disentangle them because disagreement shocks bring about a key channel that leads to procyclical investment rate dispersion. In order to assess the relative magnitude of effects due to these two different types of shocks on macro aggregates. Table 6 below reports the historical variance decomposition with respect to three aggregate shocks based on the simulated data.

**Table 6: In-sample Variance Decomposition: Aggregate Shocks**

<table>
<thead>
<tr>
<th>Macro Series</th>
<th>TFP</th>
<th>Uncertainty</th>
<th>Disagreement</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>77.96</td>
<td>12.7</td>
<td>9.32</td>
</tr>
<tr>
<td>Investment</td>
<td>79.55</td>
<td>8.8</td>
<td>11.67</td>
</tr>
<tr>
<td>Labor</td>
<td>72.5</td>
<td>3.7</td>
<td>23.76</td>
</tr>
<tr>
<td>Consumption</td>
<td>58.0</td>
<td>37.1</td>
<td>4.91</td>
</tr>
</tbody>
</table>

Notes: Numbers In Percent. Aggregate series are HP-filtered with $\lambda = 1600$. Quarterly simulation of 1500 periods. Averaged across 500 economies.

It shows that second moment shocks on average explain about 20 to 30% changes in macro aggregates. However, with respect to aggregate investment, shocks to non-fundamental disagreement explain 33% more of variation in investment relative to the variation that can be explained by uncertainty shocks. Non-fundamental disagreement shocks can be of a more crucial driver of aggregate investment dynamics relative to uncertainty shocks if both are indeed important business cycle drivers. In addition, the reason why aggregate labor series is crucially driven by disagreement shocks is because disagreement shocks are more important in terms of shaping capital distribution and the effects are persistent. Labor demand which is set based on cross-sectional distribution of capital is critically affected by disagreement shocks.

7 Conclusion

I study a general equilibrium model of heterogeneous firms that face shocks to aggregate productivity and more persistent firm-specific productivity in an environment with imperfect information. Firms care about each productivity component but can only imperfectly disentangle them from the total TFP through imperfect signals regardless of whether the signals are public or private. I disentangle two distinctive sources of business cycles: uncertainty shocks as in
Bloom (2009) and shocks to the precision of signals through which firms learn about aggregate productivity.

Information imprecision driven by pure noise dispersion can shift the cross-sectional dispersion in firms’ beliefs about aggregate and idiosyncratic productivity over time. I refer to the exogenous informational shocks that affect disagreement among firms as “informational disagreement shocks”. When the level of pure disagreement rises, firms become increasingly confused. Investment decisions continue under-reacting to idiosyncratic productivity draws. When more productive firms perceive that good idiosyncratic productivity draws are not good enough to justify investments, this leads to underinvestment. Aggravated capital mis-allocation can have real and sizable impacts on business cycles by driving down aggregate investment, which is simply due to the presence of non-fundamental sources of informational disturbances, even if there are no adverse TFP shocks nor shocks that affect dispersion of real firm-level productivity.

This paper makes three main contributions. (1) It generates a pro-cyclical dispersion of investment rates, a key empirical regularity that, according to Bachmann and Bayer (2014), models of uncertainty shocks cannot explain well. (2) Through an aggregate investment channel, the model explains why we see macro aggregates undergo a quick rebound after a rise in volatility-based measures of uncertainty but a gradual decline and slow recovery in response to larger forecast disagreement in the data. (3) The model also suggests that we can use dispersion in firm-level investment rates as a key identifier to determine when the economy is hit by real uncertainty shocks and when the economy is simply driven by informational disagreement shocks. Nonetheless, this paper models uncertainty and disagreement variations exogenously. Endogenizing the process of information acquisition to justify how these second moment variations can be time-varying in different ways can be an interesting extension. In addition, constructing different empirical measures of real uncertainty and informational disagreement based on investment rate dispersion will be very useful for better examining their individual impacts in other contexts on labor market dynamics, corporate borrowing, and taxation policy, among other domains.
A Appendix

A VAR-Based Impulse Responses

I present additional VAR-based impulse response evidence in this section. It exhibits that regardless of ordering between disagreement index or cross-sectional dispersion of firm-level TFPs, or in a bigger VAR system, aggregate investment along with other macro aggregate quantity variables see a quick rebound in face of fundamental uncertainty jump as measured by more dispersed TFPs. On the contrary, in response to larger cross-sectional firm-level forecast disagreement, very slow recovery is detected.

Figure 7 shows that in a three-variable system, assuming disagreement does not contemporaneously respond to shocks to uncertainty by being ordered first. Quick rebound in case of uncertainty shocks and slow recovery when there is larger disagreement are still there.

Figure 7: VAR-based IRFs: Aggregate Investment Responses

Notes: This figure plots impulse responses of U.S. Real Gross Private Domestic Investment to 1 % increase in Disagreement Index as in Bachmann et al. (2013b) and 1 % increase in cross-sectional S.D. of firm-level TFPs as in Imrohoroglu and Tüzeli (2014) (all in log levels), obtained from estimation of a three-variable system of VAR. Top panel and bottom panel differs in the Cholesky recursive ordering: Disagreement, TFP dispersion, and real investment (Top) and TFP dispersion, Disagreement, and real investment (Bottom). The frequency of data is quarterly and the VARs are estimated with 4 lags. The sample covers 1969Q1 to 2013Q4. Red dashed lines define ± 2 S.E. confidence bands.

Figure 8 plots the impulse responses of manufacturing output and employment based on estimation of a VAR system that augments the eight-variable system in Bloom (2009) with the
disagreement index, which is placed after uncertainty measure. It turns out the ordering between uncertainty and disagreement does not alter the results either. The second difference is that Bloom’s baseline uncertainty measure is based on stock market volatility. In addition, this system is abstract from aggregate investment.

In particular, recessions are triggered in face of larger disagreement, which are followed by slow recovery and the losses in output cannot be fully recovered after five years. However, the mean employment and output drop and quickly rebound within two years despite the standard errors associated with uncertainty-related impulse response are much larger.

![Emp To Firm-level Disagreement](image1)

![Emp To TFP Dispersion](image2)

![MP To Firm-level Disagreement](image3)

![MP To TFP Dispersion](image4)

**Figure 8: VAR-based IRFs: Bloom (2009) with Both Disagreement and Uncertainty**

**Notes:** This figure plots impulse responses of U.S. manufacturing employment (first row) and manufacturing production (second row) to 1% increase in Disagreement Index as in Bachmann et al. (2013b) and 1% increase in cross-sectional S.D. of firm-level TFPs as in Imrohoroglu and Tüzel (2014), obtained from estimation of a nine-variable system of VAR with the following Cholesky recursive ordering: log(S&P500 stock market index), log(TFP dispersion), log(disagreement index), Federal Funds Rate, log(average hourly earnings in manufacturing), log(consumer price index), weekly average hours in manufacturing, log(manufacturing employment), and log(industrial production: manufacturing) in line with Bloom (2009). The frequency of data is quarterly and the VARs are estimated with 4 lags. The sample covers 1969Q1 to 2013Q4. Red dashed lines define ±2 S.E. confidence bands.

Figure 9 further augments the nine-variable VAR system with real gross private domestic investment which is ordered after employment and before industrial production. Compared to Figure 8, higher disagreement consistently generates sluggish recovery after a big drop whereas uncertainty triggers quick rebound in the medium run. More importantly, aggregate investment
can be pushed into expansion in the medium run in face of uncertainty jump, which is consistent with bi-variate or tri-variate VAR estimations. Conversely, aggregate investment gradually declines and slowly catches up without fully recovering itself even after five years when firms are more disagreed among themselves about their future profitability.

Figure 9: VAR-based IRFs: Bloom (2009) with Disagreement, Uncertainty and Investment

Notes: This figure plots impulse responses of U.S. real private domestic investment (first row), manufacturing employment (second row), and manufacturing production (third row) to 1% increase in Disagreement Index as in Bachmann et al. (2013b) and 1% increase in cross-sectional S.D. of firm-level TFPs as in Imrohoroglu and T¨ uzel (2014), obtained from estimation of a ten-variable system of VAR with the following Cholesky recursive ordering: log(S&P500 stock market index), log(TFP dispersion), log(disagreement index), Federal Funds Rate, log(average hourly earnings in manufacturing), log(consumer price index), weekly average hours in manufacturing, log(manufacturing employment), log(real gross private domestic investment), and log(industrial production: manufacturing). The frequency of data is quarterly and the VARs are estimated with 4 lags. The sample covers 1969Q1 to 2013Q4. Red dashed lines define ± 2 S.E. confidence bands

B Proof of Lemma 1

Proof. Define vector $X_i$ as firm $i$’s information set known at the onset of period 1:

$$X_i = \begin{bmatrix} s \\ a_i \\ \end{bmatrix} = \begin{bmatrix} x_1 + \sigma \xi \\ x_1 + z_{i,1} \end{bmatrix}.$$
$x_{i,1|1}$ and $z_{i,1|1}$ are linear projections of $x_1, z_i$ on $X_i$ such that

$$x_{i,1|1} = \mu_x + \Sigma_{xX} \Sigma_{XX}^{-1} (X - \mu_X)$$
$$z_{i,1|1} = \mu_z + \Sigma_{zX} \Sigma_{XX}^{-1} (X - \mu_X)$$

where $\mu_x, \mu_z, \mu_X$ are prior means. Given the zero mean and orthogonality properties, variance co-variance matrix $\Sigma_{XX}$ is defined as below. The firm index and period 1 index in the expectation operator are suppressed:

$$\Sigma_{XX} = \mathbb{E}(XX') = \begin{bmatrix} \sigma^2_{v,0} + \sigma^2_{\xi} & \sigma^2_{v,0} \\ \sigma^2_{v,0} & \sigma^2_{v,0} + \sigma^2_{e,0} \end{bmatrix}$$

By the matrix inverse property,

$$\Sigma_{XX}^{-1} = \frac{1}{\sigma^2_{e,0} \sigma^2_{e,0} + \sigma^2_{e,0} \sigma^2_{\xi} + \sigma^2_{v,0} \sigma^2_{\xi}} \cdot \begin{bmatrix} \sigma^2_{v,0} + \sigma^2_{e,0} & - \sigma^2_{v,0} \\ - \sigma^2_{v,0} & \sigma^2_{v,0} + \sigma^2_{e,0} \end{bmatrix}$$

Similarly, $\Sigma_{zx} = [\sigma^2_{v,0}, \sigma^2_{v,0}]$ and $\Sigma_{xz} = [0, \sigma^2_{e,0}]$. Therefore, it yields

$$x_{i,1|1} = \frac{\sigma^2_{v,0} \sigma^2_{\xi} \cdot a_i + \sigma^2_{v,0} \sigma^2_{e,0} \cdot s}{\sigma^2_{v,0} \sigma^2_{e,0} + \sigma^2_{e,0} \sigma^2_{\xi} + \sigma^2_{v,0} \sigma^2_{\xi}}$$
$$z_{i,1|1} = \frac{\sigma^2_{e,0} (\sigma^2_{v,0} + \sigma^2_{\xi}) \cdot a_i - \sigma^2_{v,0} \sigma^2_{e,0} \cdot s}{\sigma^2_{v,0} \sigma^2_{e,0} + \sigma^2_{e,0} \sigma^2_{\xi} + \sigma^2_{v,0} \sigma^2_{\xi}}$$

Redefine $a = 1/\sigma^2_{v,0}$, $b = 1/\sigma^2_{e,0}$ and $c = 1/\sigma^2_{\xi}$, we get the formulations of $x_{i,1|1}$ and $z_{i,1|1}$ in Lemma 1 $Q.E.D.$

C Proof of Lemma 2

Proof. By Lemma 1, $x_{i,1|1}$ and $z_{i,1|1}$ are linear combinations of standard normal variables and thus follow normal distribution. Hence, expectation of the exponential of these variables are log-normal. We have

$$\mathbb{E}e^{a_{i,2}} = \exp [\mathbb{E}(a_{i,2}) + 0.5 \Sigma] = \exp [\rho_x x_{i,1|1} + \rho_z z_{i,1|1} + 0.5 \Sigma]$$
where $\Sigma$ is the forecast variance co-variance term of $A_{i,2}$ conditional on receiving the noisy signal, which measures the precision of expectation. Hence we have

$$\Sigma = \mathbb{E}(x_2 - \rho_x x_{i,1}|1)^2 + \mathbb{E}(z_{i,2} - \rho_z z_{i,1}|1)^2 + 2\mathbb{E}(x_2 - \rho_x x_{i,1}|1)(z_{i,2} - \rho_z z_{i,1}|1)$$

$$= \mathbb{E}(\rho_x x_1 + \sigma_v v_2 - \rho_x x_{i,1}|1)^2 + \mathbb{E}(\rho_z z_{i,1} + \sigma_e e_{i,2} - \rho_z z_{i,1}|1)^2$$

$$+ 2\mathbb{E}(\rho_x x_1 + \sigma_v v_2 - \rho_x x_{i,1}|1)(\rho_z z_{i,1} + \sigma_e e_{i,2} - \rho_z z_{i,1}|1)$$

By Lemma 1, substituting out $x_{i,1}|1$ and $z_{i,1}|1$, we then have

$$\Sigma = \sigma_v^2 + \sigma_e^2 + \frac{(\rho_z - \rho_x)^2}{a + b + c}$$

Therefore, it yields

$$\mathbb{E}e^{a_i,2} = \exp[(\rho_x + M)x + (\rho_z - N)z_{i,1} - P\xi + 0.5(\sigma_v^2 + \sigma_e^2 + Q)]$$

Where $Q = \frac{(\rho_z - \rho_x)^2}{a + b + c}$ and

$$M = \frac{(\rho_z - \rho_x)a}{a + b + c} > 0, \ N = \frac{(\rho_z - \rho_x)b}{a + b + c} > 0, \ P = \frac{(\rho_z - \rho_x)\sqrt{c}}{a + b + c} > 0$$

It shows that $M'(\sigma_\xi) > 0, N'(\sigma_\xi) > 0$ and thus $Q'(\sigma_\xi) > 0$. We see for perfect information case $\sigma_\xi \to 0$ when $x_1$ and $z_{i,1}$ are separately observed, $M, N, P$ and $Q$ terms go to zero. The expectation term is standard given by

$$\mathbb{E}e^{a_i,2} = \exp[\rho_x x_1 + \rho_z z_{i,1} + 0.5(\sigma_v^2 + \sigma_e^2)]$$

$\blacksquare$ Q.E.D.

### D Proof of Propositions 6

**Proof.** $E_i^I$ and $E_i^D$ are two roots of the gain from taking non-zero investment $\Psi$ function. Within these two bounds, firms would not take any investment action for $\Pi_{Non-Adj} > \Pi^{Adj}$. By the implicit function theorem:

$$\frac{dE_i}{d\sigma_j} = -\frac{\partial \Psi/\partial \sigma_j}{\partial \Psi/\partial E_i} = \frac{[c_k(1 - \delta)k_0 - \psi \cdot \phi E_i] \frac{d\hat{F}(\lambda)}{d\sigma_j} + \psi E_i[1 - \hat{F}(\lambda)] \frac{d\phi}{d\sigma_j}}{E_i - \zeta}$$

It shows that for range $E_i^I \in (\zeta, \infty)$, denominator is positive whereas range $E_i^D \in (0, \zeta)$ leads to negative denominator where $\zeta = 1 - \psi(1 - \phi)$. 

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By Equation (15), the numerator of the equation above is exactly given by the partial derivative of the option value with respect to the second moment parameter:

\[
\frac{\partial V_{\text{option}}}{\partial \sigma_j} = \int_\mathcal{A} \left[ \frac{1}{\sqrt{2\pi} A_{i,2}} \int_\mathcal{A} \left[ \frac{-0.5}{2\Sigma} e^{-\frac{(\log A_{i,2} - \mu_i)^2}{2\Sigma}} \right] dA_{i,2} \right]
\]

where \( \int_\mathcal{A} = \psi A_{i,2} - c_k (1 - \delta) k_0 > 0 \), \( \Sigma' (\sigma_j) > 0 \) and \( A_{i,2} \geq A > 0 \). By the assumption of \( A \geq e^{\sqrt{\Sigma}} > 1 \), when all first moment shocks are at zeros: \( v = 0 \), \( e_i = 0 \), and \( \xi_i = 0 \) that makes \( \mu_i = 0 \), we have the following

\[
\frac{(\log A_{i,2})^2}{\Sigma} \geq 1
\]

for all \( A_{i,2} \geq A \). Thus the numerator is positive.

The results above are largely due to the fact that a mean-preserving spread increase of a convex function (max function) such as uncertainty or disagreement, increases the expected value of future TFP thus the option value of waiting. Therefore larger fundamental uncertainty and non-fundamental disagreement enlarges firm’s inaction band by having \( E_i' (\sigma_j) > 0 \) and \( E_i'^D (\sigma_j) < 0 \).

Therefore, firms see greater gain from waiting and pausing actions in case of larger uncertainty or in more disagreed environment. This is true regardless whether or not this is for macro or micro uncertainty.

\[ Q.E.D. \]

\[ \textbf{E} \quad \text{Proof of Lemma 5} \]

**Proof.** We consider the partial derivative of gain from taking non-zero action (Ψ) with respect to idiosyncratic TFP shocks \( e_i \):

\[
\frac{\partial \Psi}{\partial e_i} = \frac{\partial}{\partial e_i} \left[ \frac{e^{2(\mu_i + \Sigma)}}{2} - \zeta(e_i) e^{(\mu_i + \Sigma)} + \gamma(e_i) \right] = \mu'_i(e_i) [e^{2(\mu_i + \Sigma)} - \zeta(e_i) e^{(\mu_i + \Sigma)} - \zeta' (\mu_i) e^{(\mu_i + \Sigma)} + \gamma' (\mu_i)]
\]

\( \mu'_i(e_i) = (\rho_z - \mathbf{N}) \sigma_{e,0} = \frac{\rho_a \rho \sigma_{e} + \rho \rho_b}{a + b} > 0 \). For \( e^{\mu_i + \Sigma} > E_i' > \zeta(e_i) \), by Assumption (1), We have \( \frac{\partial \Psi}{\partial e_i} > 0 \) while \( \frac{\partial \Psi}{\partial e_i} < 0 \) for \( e^{\mu_i + \Sigma} < E_i'^D < \zeta(e_i) \). For \( e' \) solves \( E_i' = e^{\mu_i + \Sigma} \) such that \( \Pi^{Adj} = \Pi^{Non - Adj} \), firms would invest if \( e_i > e' \). Similarly, firms would disinvest if \( e_i < e^{D} \) where \( e' \) and \( e^{D} \) are common to all firms for a given \( \xi_i \). \[ Q.E.D. \]
Proof of Lemma 6

Proof. By implicit function theorem, we have

\[
\frac{\partial e^I}{\partial \sigma_j} = -\frac{\partial (E_i(e_i) - E^I_i)}{\partial \sigma_j} / \frac{\partial e_i}{\partial e_i}, \\
\frac{\partial e^D}{\partial \sigma_j} = -\frac{\partial (E_i(e_i) - E^D_i)}{\partial \sigma_j} / \frac{\partial e_i}{\partial e_i}
\]

By Assumption (1),

\[
e^{\mu_i+\Sigma} - \frac{|\zeta'(\mu_i)e^{\mu_i+\Sigma} - \gamma'(\mu_i)|}{|e^{\mu_i+\Sigma} - \zeta(e_i)|} \geq 0
\]

Hence, \( \frac{\partial (E_i(e_i) - E^D_i)}{\partial e_i} > 0 \) and \( \frac{\partial (E_i(e_i) - E^I_i)}{\partial e_i} > 0 \). Therefore, for \( \frac{\partial e^I_i}{\partial \sigma_j} > E'_i(\sigma_j) \) and \( \frac{\partial e^D_i}{\partial \sigma_j} < E'_i(\sigma_j) \),

\[
\frac{\partial e^I}{\partial \sigma_j} > 0, \quad \frac{\partial e^D}{\partial \sigma_j} < 0
\]

The inaction band expands for a given level of \( \xi_t \). \( \blacksquare \) Q.E.D.

Time-varying Recursive Kalman Filtering

Firm's inference problem has a state space representation:

\[
m_{i,t} = Fm_{i,t-1} + \zeta_{i,t} \quad \text{(State Equation)} \\
n_{i,t} = Hm_{i,t} + u_{i,t} \quad \text{(Measurements Equation)}
\]

where

\[
m_{i,t} = \begin{bmatrix} x_t \\ z_{i,t} \end{bmatrix}, \quad F = \begin{bmatrix} \rho_x & 0 \\ 0 & \rho_z \end{bmatrix}, \quad \zeta_{i,t} = \begin{bmatrix} \sigma_{v,t} \\ \sigma_{e,t-1}e_{i,t} \end{bmatrix}, \quad \zeta_{i,t} \sim \mathcal{N}(0, \chi_t)
\]

\[
n_{i,t} = \begin{bmatrix} a_t \\ s_{i,t} \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad u_{i,t} = \begin{bmatrix} 0 \\ \sigma_{\xi,t}\xi_{i,t} \end{bmatrix}, \quad u_{i,t} \sim \mathcal{N}(0, R_t)
\]

\[
\chi_t = \begin{bmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_e^2 \end{bmatrix}, \quad R_t = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{\xi,t}^2 \end{bmatrix}
\]

Firm uses a recursive linear projection algorithm each period to form new posterior estimates object \( \hat{m}_{i,t|t} = \mathbb{E}(m_t|a_t, s_{i,t}) \). The posterior expectations are functions of period \( t - 1 \) posterior estimates \( \hat{m}_{i,t-1|t-1} \), period \( t - 1 \) posterior variance covariance matrix (matrix of imprecision)
\[ \hat{\Sigma}_{t-1|t-1} = \mathbb{E}[(m_{i,t-1} - \hat{m}_{i,t-1|t-1})(m_{i,t-1} - \hat{m}_{i,t-1|t-1})'] \], newly observed magnitude of disagreement \( \sigma_{z,t} \) and predetermined uncertainty \( \sigma_{e,t-1} \), as well as the observable object \( n_{i,t} \). The projection rules are stated below:

\[ \hat{m}_{i,t|t} = (I - \kappa_t H)\hat{m}_{i,t-1|t-1} + \kappa_t \tilde{\nu}_t \]  
where \[ \kappa_t = \left( F\hat{\Sigma}_{t-1|t-1}F' + \chi_t \right) \left( \hat{\Sigma}_{t|t-1} + \chi_t \right)^{-1} \]  
\[ \hat{\Sigma}_{t|t} = (I - \kappa_t H)(F\hat{\Sigma}_{t-1|t-1}F' + \chi_t) \]

\( \tilde{\nu}_t \) is a 2 by 2 identity matrix and a prime denotes the matrix transpose. Equations (41b) and (41c) show that all firms attach the same weights to previous period posterior estimates and the new observables. Specifically, \( \kappa_t \) known as the Kalman gain is the optimal weights for \( n_{i,t} \). \( \kappa_t \) resembles the weights we see in Lemma 1. Equation (41c) is the discrete time Riccati equation that updates posterior variance covariance matrix and the thus Kalman gain each period.

By Equation (41b), rising disagreement for higher noise dispersion passes larger \( R_t \) term into a higher discount of the Kalman gain whereas larger uncertainty terms increases the \( \kappa_t \) term. This suggests that noise dispersion makes firm underweigh the new observations of productivity draws rather than overweigh as uncertainty shocks do. It also implies noise dispersion shocks can create some expectation inertia such that firms still weigh more on previous period estimates.

Specifically, to derive Equations (41a)(41b)(41c), suppress the firm \( i \) index, the updating is through a linear projection rule as derived below following Hamilton (1994):

\[ \hat{m}_{t|t} = \hat{m}_{t|t-1} + \kappa_t \tilde{\nu}_t \]  
\[ \hat{\Sigma}_{t|t} = (I - \kappa_t H)\hat{\Sigma}_{t|t-1} \]

where

\[ \hat{m}_{t|t-1} = F\hat{m}_{t-1|t-1} \]  
\[ \hat{\Sigma}_{t|t-1} = F\hat{\Sigma}_{t-1|t-1}F' + \chi_t \]  
\[ \hat{\Sigma}_{t|t} = (I - \kappa_t H)(F\hat{\Sigma}_{t-1|t-1}F' + \chi_t) \]

\( \tilde{\nu}_t = n_t - H\hat{m}_{t|t-1} \)

\( \kappa_t = \hat{\Sigma}_{t|t-1}H'(H\hat{\Sigma}_{t|t-1}H' + R_t)^{-1} \)

H Data on Firm-level TFPs

Data that is used to approximate the dispersion of firm-level productivity is from Imrohoroglu and Tüzel (2014) which computes firm-level TFPs for a unbalanced panel of Compustat firms.
dating from 1969 to 2013. The firm level TFPs are estimated using the following specification and assuming log Cobb-Douglas production function: $y_{i,t} = \beta_0 + \beta_k k_{i,t} + \beta_h h_{i,t} + TFP_{i}^{Firm} + \eta_{i,t}$ with industry-specific time dummies that rule out effects of industry and aggregate TFP in a given year. Estimated firm level TFPs are the difference between actual and predicted production such that $TFP_{i}^{Firm} = y_{i,t} - \hat{\beta}_0 - \hat{\beta}_k k_{i,t} - \hat{\beta}_h h_{i,t}$.

I take the cross-sectional standard deviation of log TFPs to measure the dispersion of $z_{i,t}$. I also use the cross-sectional log range of log TFPs such that $\ln(TFP_{i}^{Firm, max} - TFP_{i}^{Firm, min})$ for robustness checks. Estimates are not sensitive to which measure used for estimation. With estimates using annual data, I then infer the quarterly persistence and innovation S.D.s.

I. Goodness of Fit Checks

Following Krusell et al. (1998), the accuracy of the approximated aggregate laws of motion is measured based on the two goodness of fit statistics, i.e. adjusted $R^2$ and the standard error of residuals from regressions per the system (38). Wouter Den Haan (2010) also proposes a number of robustness checks. Aggregate time series are simulated given the optimal investment policy functions. The table below reports the statistics.

| State Var   | log($\bar{k}_t$) | log($w_t$) | $\bar{x}_{t|t}$ |
|-------------|-----------------|-----------|-----------------|
| constant    | 0.51904         | -0.67227  | 0.00411         |
| log($\bar{k}_{t-1}$) | 0.88565         | 0.95341   | -0.00047        |
| log($\bar{x}_{t-1|t-1}$) | 0.33            | 1.7086    | 0.98236         |
| $\sigma_{\xi,t}$ | -0.02425       | 0.3058    | 0.00019         |
| $\sigma_{e,t}$ | -0.0976         | 2.1719    | 0.00067         |
| $\sigma_{e,t-1}$ | 0.02589         | -0.27906  | -0.00019        |
| Adj.$R^2$   | 0.9974          | 0.85      | 0.9704          |

**Notes:** Regressions are done using OLS in line with Equations (38).

The aggregate laws of motion are estimated and converged to a tolerance range of smaller than 1 and the policy function is solved with convergence tolerance of 0.1. Estimations of coefficients are sensitive to tolerance levels. For more refined tolerance, the convergence may not be achievable. This is largely due to the curse of dimensionality in that the grids are loosely dispersed. More more refined grids, a very slow speed of convergence will result. The aggregate capital stock equation
is solved with very high precision. In order to improve the accuracy of other equations, I tried to augment with interactions and other higher order moments. However, adding more moments creates explosiveness in the sense that the path-searching is drifting away from the equilibrium. Therefore, I maintained with a smaller number of state variables. It suggests high non-linearity in the model is presence. Estimated coefficients for the persistence of aggregate capital stock has comparable magnitudes and signs as in Khan and Thomas (2008).

J Robustness Checks: Public Signals vs. Private Signals

I check if or not when signals are defined as firm-specific signals, the key results would be altered. The answer is no. Following Lemma 1, by derivation of Appendix B, we can redefine information vector $X_i$ at the onset of period 1 by labeling signal as firm-specific $s_i$ attached with a firm-specific noise.

$$X_i = \begin{bmatrix} s_i \\ a_i \end{bmatrix} = \begin{bmatrix} x_1 + \sigma \xi_i \\ x_1 + z_{i,1} \end{bmatrix}.$$  

Note that the aggregate parameter that governs the dispersion of information still measures the aggregate information precision $\sigma \xi$: more dispersed information, more fraction of firms are acting upon less precise information with noises of larger magnitude $|\xi_i|$. As a result, firms still follow the same rule of extracting beliefs about current period aggregate and idiosyncratic productivity as given by Lemma 1.

The only difference resulting from firm-specific noises is that by derivation of Appendix C, firm $i$’s expectation of future productivity will now depend on firm-specific noise $\xi_i$.

$$\mathbb{E} e^{a_{i,2}} = \exp \left[ (\rho_x + M)x + (\rho_z - N)z_{i,1} - P\xi_i + 0.5(\sigma_v^2 + \sigma_e^2 + Q) \right]$$

Therefore, firm’s over-reaction of investment to aggregate TFP shocks and under-reaction of investment to idiosyncratic TFP shocks are still there. Change in non-fundamental disagreement parameter $\sigma \xi$ would change the extent of capital mis-allocation. On average, when first moment shocks are at zero, both jumps in uncertainty and disagreement can increase firm’s option value of waiting and also increase expected value of marginal product of capital by increasing forecast variance term $\Sigma = \sigma_v^2 + \sigma_e^2 + Q$.

For aggregate implication, the following shows that we can still define the aggregate cutoff points in terms of real firm-specific productivity conditional on a distribution of firm-specific
noises. The impact of disagreement would not be changed at all.

$$\frac{\partial I}{\partial \sigma_\xi} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_i \Sigma'(\sigma_\xi) d\Phi(e) d\Phi(\xi) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_i \Sigma'(\sigma_\xi) d\Phi(e) d\Phi(\xi)$$

\begin{align*}
\text{intensive margin : convexity effect} &> 0 \\
+ \int_{-\infty}^{\infty} (e^{(\muz-N)eD-P\xi_t+\Sigma} - 1) \frac{\partial eD}{\partial \sigma_j} d\Phi(\xi) - \int_{-\infty}^{\infty} (e^{(\rhoz-N)eD-P\xi_t+\Sigma} - 1) \frac{\partial eD}{\partial \sigma_j} d\Phi(\xi) \\
\text{extensive margin : fewer disinvesting firms} &> 0 \\
- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_i N_{\sigma_\xi} e_i d\Phi(e) d\Phi(\xi) - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_i N_{\sigma_\xi} e_i d\Phi(e) d\Phi(\xi) \\
\text{extensive margin : fewer investing firms} &< 0 \\
- \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_i N_{\sigma_\xi} e_i d\Phi(e) d\Phi(\xi) - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_i N_{\sigma_\xi} e_i d\Phi(e) d\Phi(\xi)
\end{align*}

\begin{align*}
\text{intensive margin : capital mis-allocation} &< 0 \\
\end{align*}

The reason why macro implications are not affected is that firm-specific noises on average will equal to zero, and the noises are i.i.d. over time. Therefore, aggregation of firm-level investment decisions from intensive margin washes out the effect of firm-specific noise when log expectation is linear in firm-specific noises. Therefore, convexity effect and misallocation effect go through for aggregate. Regarding the real-option effect, firm-specific noise does not affect the cutoff points of investing mass and dis-investing mass conditional on distribution of firm-specific noises. It is that uncertainty and disagreement who will affect forecast variance alter the relative mass since firms see greater gain from waiting on average.

In sum, modeling signals as private or public do not affect the main results of this paper. The full model solution that builds on firm-specific information further confirms this robustness.

**K Robustness Checks: Noisy Signal About the Idiosyncratic Productivity**

I examine the case when firm $i$ receives a firm-specific signal about its idiosyncratic productivity rather than the aggregate plus a firm-specific noise. The information set at the onset of period 1 is reformulated below:

$$X_i = \begin{bmatrix} s_i \\ a_i \end{bmatrix} = \begin{bmatrix} z_{i,1} + \sigma_\xi \xi_i \\ x_1 + z_{i,1} \end{bmatrix}.$$  

By the matrix inverse property,

$$\Sigma_{XX}^{-1} = \frac{1}{\sigma_{v,0}^2 + \sigma_{e,0}^2 + \sigma_{\xi}^2 + \sigma_{v,0}^2 \sigma_{\xi}^2} \cdot \begin{bmatrix} \sigma_{v,0}^2 + \sigma_{e,0}^2 & \sigma_{e,0}^2 \\ \sigma_{e,0}^2 & \sigma_{e,0}^2 \end{bmatrix}$$
Since $\Sigma_{x_tX} = [0, \sigma_{v,0}^2]$ and $\Sigma_{z_tX} = [\sigma_{e,0}^2, \sigma_{e,0}^2]$, it yields

$$x_{i,1|1} = \frac{\sigma_{e,0}^2 (\sigma_{e,0}^2 + \sigma_{x,0}^2) \cdot a_i - \sigma_{e,0} \cdot s_i}{\sigma_{e,0}^2 \cdot \sigma_{x,0}^2 + \sigma_{e,0}^2 \cdot \sigma_{x,0}^2}$$

$$z_{i,1|1} = \frac{\sigma_{e,0}^2 \cdot a_i + \sigma_{e,0}^2 \cdot s_i}{\sigma_{e,0}^2 \cdot \sigma_{x,0}^2 + \sigma_{e,0}^2 \cdot \sigma_{x,0}^2}$$

Reevaluate Equation (8), using notations of inverse of variances, we have

$$\mathbb{E}(e^{x_t + z_t}) = \left[ \rho_x + \frac{(\rho_z - \rho_x) a}{a + b + c} \right] x_1 + \left[ \rho_z - \frac{(\rho_z - \rho_x) b}{a + b + c} \right] z_{i,1} + \frac{(\rho_z - \rho_x) \sqrt{c}}{a + b + c} \xi_i + 0.5(\sigma_v^2 + \sigma_e^2 + Q)$$

We see that firms’ investment would exactly under-react to idiosyncratic productivity and over-react to aggregate productivity in line with what we had in Appendix B.

In summary, as long as the idiosyncratic productivity component is more persistent than the aggregate component, amplification and dampening effects are still present. The reason is that firms still use imperfect information to disentangle the two components even if the information is about the idiosyncratic productivity.

## L Robustness Checks: Permanent and Transitory Component + Firm-specific Noisy Signal

Use notations $x$ and $z$ to denote two different but unobserved aggregate productivity components that enter production function. Absent the idiosyncratic shocks, a firm observes total productivity sum $a = x + z$ but relies on noisy signal $s_i = x + \sigma_x \xi_i$ to disentangle them. Without loss of generality, the signal can be defined about $z$ as well. We can think of one is permanent and one is transitory component. Then the firm solves a signal extraction problem in line with Appendix B. Firm $i$ has separate beliefs as below

$$x_{i,1|1} = \frac{\sigma_{e,0}^2 \sigma_{x,0}^2 \cdot a_i + \sigma_{e,0}^2 \cdot s_i}{\sigma_{x,0}^2 \cdot \sigma_{t,0}^2 + \sigma_{e,0}^2 \cdot \sigma_{x,0}^2}$$

$$z_{i,1|1} = \frac{\sigma_{e,0}^2 (\sigma_{e,0}^2 + \sigma_{x,0}^2) \cdot a_i - \sigma_{e,0}^2 \cdot s_i}{\sigma_{e,0}^2 \cdot \sigma_{x,0}^2 + \sigma_{e,0}^2 \cdot \sigma_{x,0}^2}$$

Using notations of inverse of variances, we have

$$\mathbb{E}(e^{x_t + z_t}) = \left[ \rho_x + \frac{(\rho_z - \rho_x) a}{a + b + c} \right] x_1 + \left[ \rho_z - \frac{(\rho_z - \rho_x) b}{a + b + c} \right] z_{i,1} + \frac{(\rho_z - \rho_x) \sqrt{c}}{a + b + c} \xi_i + 0.5(\sigma_v^2 + \sigma_e^2 + Q)$$
Clearly, in line with what we had in Appendix B. Specifically, to take an extreme example, if \( x \) is the permanent component with \( \rho_x = 1 \) whereas \( z \) is transitory with \( \rho_z = 0 \). We have the following such that

\[
\mathbb{E}(e^{x_2 + z_2}) = (1 - \frac{a}{a + b + c})x + \frac{b}{a + b + c}z + \frac{\sqrt{c}}{a + b + c}\xi_i + 0.5(\sigma_e^2 + \sigma^2 + Q)
\]

In summary, depending on the relative persistence of aggregate components, firm’s investment would always over-react to the one with smaller persistence and under-react to the one with larger persistence.

M Iterative Steps for Equilibrium Solution

The following are the general steps used to solve for the recursive competitive equilibrium. I borrow ingredients from the Approximate Aggregation procedure in Krusell et al. (1998).

a. Assume that the aggregate capital stock \( \bar{k}_{t-1} \) and the mean level of TFP forecasts \( \bar{x}_{t-1|t-1} \) at the beginning of period \( t \) are sufficient to summarize the beginning-of-period distribution of firms, \( \mu_{t-1} \), we come up with Equation system (38). Taking parameter conjectures \( \Gamma_j, j \in \{0, 1, 2\} \), solve for the individual firm’s policy functions using Value Function Iteration that accounts for micro-level nonconvexity.

b. Using the capital stock decision rule, simulate actual firm distribution following Young (2010) for \( T \) periods and compute the aggregate capital stock and posterior mean belief about TFP time series by taking the cross-sectional average. Each period in this model corresponds to a quarter. I set \( T = 5000 \) periods with the first 1000 periods burned. Using the same innovations that generate the panel, the goods market clearing condition generates a time series of the equilibrium real wage. Do OLS regressions on the burned-in sample using actual series and obtain new estimates of parameters \( \Gamma_j', j \in \{0, 1, 2\} \).

c. Evaluate if \( \max(|\Gamma_j' - \Gamma_j|) < \epsilon \), a tolerance range. If true, stop. Otherwise, update conjectures via \( \Gamma_j = \lambda \Gamma_j + (1 - \lambda)\Gamma_j \), where \( \lambda \) is the convergence speed control on how much weights are assigned to the previous parameter conjectures.

d. With updated parameters \( \Gamma_j \), solve the individual policy functions again until the vector of \( \Gamma_j \) converges. Each converged parameter vector is associated with a particular set of exogenous innovations on which aggregation is obtained. Repeat the convergence for \( T_{mc} \) times and obtain the average \( \frac{1}{T_{mc}} \sum_{t=1}^{T_{mc}} \Gamma_j \).
References


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