Abstract

I explore the role of financial intermediaries in supplying liquidity over the business cycle. I consider a model similar to Holmstrom and Tirole (1998), in which firms hold assets to shift current funds to future periods when they may be borrowing constrained. Financial intermediaries insure against loss of credit access by issuing credit lines, but intermediaries are also subject to agency costs whose severity depends on their asset holdings. This creates a linkage between bank balance sheets and the supply of liquidity, in which a fall in bank assets raises the liquidity premium and reduces investment. This provides a novel channel through which a financial crisis may affect the real economy. I analyze optimal policy assuming the government can issue bonds that commit future tax revenue. I find that when there is a positive liquidity premium it will be optimal for the government to issue a positive quantity of bonds. I find that the optimal supply of public liquidity is decreasing in bank assets. This implies that optimal policy in the wake of a financial crisis is to increase the supply of government bonds due to their liquidity properties.

Keywords: Liquidity, Financial intermediation, Financial crisis.

JEL Classification: E43, E44.
1 Introduction

An important function of the financial system is liquidity provision, i.e. the issuance of securities that serve as stores of value. Liquidity shortages played a prominent role in the 2008 financial crisis, yet most models of the financial sector ignore its role in liquidity provision. This is particularly relevant to the analysis of policy, since government liabilities also serve as stores of value. This paper analyzes the role of financial intermediaries and the government in liquidity provision. I investigate two questions: How do bank balance sheets affect the supply of liquidity, interest rates, and investment? What is the role of government liabilities in liquidity provision?

I use the term liquidity to refer to assets held by firms to transfer funds forward in time. Firms hold such assets to meet financing needs should they become borrowing constrained. If firms anticipate the possibility of shocks that reduce their future ability to borrow, they will borrow funds now to purchase assets that can be liquidated in the event of a shock. If the current borrowing rate is the same as the return on these assets, firms will be able to secure financing to purchase an efficient quantity of such assets. However, if liquid assets are in short supply, their return will fall below firms’ expected return on capital, reflecting a positive liquidity premium. Firms then find it more costly to hold assets to cover liquidity needs, which causes them to reduce production.

Firms may become borrowing constrained if a shock reduces the portion of their expected profits that they can credibly promise to investors below the quantity of funds they need to continue production. Since agency costs drive a wedge between total profits and the portion that is pledgeable to investors, such a project may be unable to secure external financing although it would be socially optimal to continue production. Thus firms would like to make arrangements to meet future financing needs should they lose access to credit markets.

Firms obtain these funds in two ways. First, firms may borrow funds when they are not borrowing constrained and purchase assets that can be liquidated in the event of a liquidity shock. If these assets are not liquidity, their proceeds can be returned to the firm’s investors. Such assets include corporate and government bonds and shares of money market funds. Since it doesn’t matter whether the firm liquidates the asset or uses it as collateral to obtain a loan, this category also includes repos and other securitized borrowing. Second, firms may lock in financing at convenient terms by purchasing a credit line from a financial intermediary. Typically the firm pays a premium to an intermediary to provide financing at specified terms up to some limit, even if such financing becomes unprofitable in expectation. This category primarily consists of credit lines issued by banks. I refer to assets that can be easily sold to meet liquidity shocks as liquid assets; I refer to the market for liquid assets as the market for liquidity; and I refer to the availability of liquidity on good terms as the supply of liquidity.

To model the market for liquidity, I build on the framework of Holmstrom and Tirole (1998). Firms raise funds in an initial period, which they invest in projects that yield a return in the final period. Projects receive a liquidity shock in an intermediate period, which require that firms inject a quantity of funds proportional to the size of the shock. If the funds are not provided, the project is lost. Since firms have no income in the intermediate period, they must obtain funds
from households to meet these shocks.

In my model, firms meet liquidity shocks in three ways. First, a firm that experiences a shock can issue new liabilities to raise funds. Second, firms can hold credit lines issued by an intermediary. These are financial arrangements that commit the intermediary to providing funds at pre-arranged terms should the firm experience a bad liquidity shock. Third, a firm can purchase non state-contingent assets that can be sold in the intermediate period. This category includes liabilities issued by other sectors of the economy, foreign lenders, and the government. I refer to assets in this last category as outside liquidity.

Liquidity can be viewed as a kind of insurance. In the intermediate period, some firms will need additional funds while other firms will not. Ex ante firms do not know which type they will be, and they would like to purchase insurance against receiving a bad shock. Under complete markets, firms could hold state-contingent assets that would provide funds exactly as needed. With incomplete markets, firms can hold non-state-contingent assets to meet these shocks, but this arrangement will not be efficient when assets are scarce because firms that experience good shocks will be left holding excess assets. Financial intermediaries can avoid this inefficiency by selling credit lines on which firms draw in the event of a shock.

I depart from Holmstrom and Tirole (1998) by supposing that intermediaries are also subject to agency costs. When lending is subject to agency costs, the supply of liquidity provided by intermediaries depends on their net worth. When banks have low net worth, they have less collateral to be seized in the event of bankruptcy and therefore have more incentive to engage in fraud, for instance by failing to exert effort in screening loan applicants, or by taking on excessively risky loans since they do not bear downside risk. When these agency constraints are binding, the economy will not be able to achieve perfect risk pooling of idiosyncratic liquidity risk even in the presence of intermediaries. Moreover, if the marginal agency cost is increasing in the quantity of funds intermediated, then the liquidity premium will be increasing in the quantity of bank financing and decreasing in bank net worth.

Since the payoffs to the initial investors are contingent on which liquidity shocks are met and on what terms, liquidity and initial financing are determined jointly. Thus disruptions in one market will affect the other. In particular, if liquidity becomes scarce this will lower the return to capital and cause investment to fall.

Why might liquidity be scarce? Why can’t firms simply buy assets that pay at the prevailing interest rate, sell them to raise funds in the event of a liquidity shock, and if no shock is realized hold them to maturity? If there were a sufficient quantity of such assets then this would indeed be possible. However, such assets may be in limited supply. This is somewhat removed from everyday experience: we are used to being able to save (buy assets) easily, while borrowing (issuing liabilities) may be difficult. Yet every asset is also a liability of someone else in the economy. Thus any friction that limits the ability of some agents to obtain loans (issue liabilities) will also limit

\[\text{\footnotesize\ref{1}}\text{Several papers use agency costs in models of banks. One notable example is Mattesini, Monnet, and Wright (2009), which argues that agents with large stakes in the continuation of the economy are better suited to serve as banks. This is analogous to my model, in which bank agency costs depend on their asset holdings.}\]
the supply of assets. The same frictions that limit borrowing also limit the economy’s capacity to create assets, and thus restrict the supply of liquidity.

In my model, credit constraints take the form of collateral requirements arising from moral hazard. Households cannot borrow at all because they do not own any collateral and thus will default on any obligation. Firms have collateral in the form of their investment projects, but they have specific expertise in operating their projects. This expertise is not contractable, and just before project completion firms are given the opportunity to enjoy a private benefit by employing this expertise elsewhere. Therefore firms must receive a sufficient share of the profits in all states in order to cooperate, and can only issue claims against the residual value of their projects rather than their full value. Following Holmstrom and Tirole (1998), I refer to this constraint as limited pledgeability.

Let \( \hat{q} \) be the fundamental price implied by the risk-adjusted expected return of the asset. Then all agents in the economy would value the asset at the price \( \hat{q} \) if it were always held to maturity. However, in the event of a liquidity shock, a firm could sell the asset to obtain funds to meet the shock. Therefore the asset has an option value to firms, and they will be willing to pay more for the asset. When liquidity is in abundant supply assets will be priced at \( \hat{q} \), which I refer to as the abundant liquidity case. However, if the economy’s stock of liquid assets is below the quantity firms desire to hold at the price \( \hat{q} \), there will be an excess demand for assets at \( \hat{q} \) and the market-clearing price will rise. I refer to the spread \( q - \hat{q} \) as the liquidity premium, and a positive liquidity premium as the scarce liquidity case.\(^2\)

There is ample evidence of positive liquidity premia. The simplest example is money, which commands a positive liquidity premium relative to safe government bonds. This is intuitive because money has some utility in transactions. It is also commonly acknowledged that the interest rates on safe government bonds are lower than one might expect relative to other assets in the economy. This suggests a positive liquidity premium on government debt. Moreover, during times of crisis interest rates on many forms of safe assets fall, a phenomenon often called a flight to quality. This may reflect falling returns to capital due to productivity shocks, but it may also indicate a rising liquidity premium due to a decrease in the supply of liquidity, which makes liquid government bonds more desirable than longer-term investments that cannot be easily liquidated.

Why might an economic downturn cause a fall in the supply of liquidity? I consider two mechanisms. First, a recession leads to losses on projects that underlie liquid assets in the economy. For instance, the 2008 financial crisis was preceded by a drop in housing prices which led to an increase in mortgage delinquencies. Since mortgages were the underlying assets behind many securities, this rise in mortgage delinquencies represented the destruction of a fraction of the economy’s stock of liquid assets. In a world with scarce liquidity, this should raise the cost of investment and raise the prices of liquid assets, for instance by reducing interest rates on government bonds. I label this mechanism the asset channel.

\(^2\)If an asset has risk-free gross interest rate \( 1 + r \) then the fundamental price is \( \hat{q} = \frac{1}{1 + r} \). Thus a positive liquidity premium corresponds to a lower interest rate.
In addition to the direct destruction of assets held by agents to meet liquidity shocks, a recession will generally involve falling asset prices and losses on bank loans. This will reduce bank net worth, tightening credit conditions and leading to a fall in investment. This mechanism is even stronger if the financial system is poorly capitalized. Leading up to the 2008 crisis, banks were highly leveraged, and so intermediaries suffered large losses due to the housing crash. This was a major source of worsening credit conditions and falling investment. I refer to this second mechanism as the balance sheet channel.

I can distinguish between these mechanisms in my model. The asset channel involves the direct destruction of liquid assets, which in my model corresponds to a fall in outside liquidity. The balance sheet channel is modeled as a fall in bank net worth. These two channels lead to different macroeconomic results and thus imply different policy prescriptions.

If the economy is initially in a region with a positive liquidity premium, then a fall in outside liquidity reduces investment and increases the liquidity premium. I find that if the economy is initially in an equilibrium with positive bank financing, then a decrease in bank net worth will tighten the supply of liquidity, leading to a fall in investment. This fall in investment will decrease demand for liquidity, which will tend to decrease the equilibrium liquidity premium. However, this fall in bank assets will reduce the supply of liquidity by making it more costly for banks to produce, which will tend to increase the liquidity premium. Which effect dominates will depend on the precise specification of agency costs.

The final step in the analysis is to consider optimal government liquidity policy. As described above, when there is a liquidity premium the government can improve the allocation by issuing liquid liabilities such as government debt. If issuing bonds were costless, then the optimal policy would always be for the government to issue bonds until liquidity is no longer scarce. However, realistically there are some costs of government debt. One cost is that debt must be repaid, and the government must levy distortionary taxes to do so. The cost of issuing debt must be weighed against the benefits of providing liquidity.

I analyze optimal government policy within the context of this model. I find that when there is a positive liquidity premium at an interior equilibrium, it will always be optimal for the government to provide some public liquidity. This result is quite interesting by itself. It implies that in an economy with a positive liquidity premium there is positive value of issuing government debt. If total government debt is sufficiently low, the issuance of additional debt will crowd in investment, and these benefits will outweigh the costs of higher debt.

I also analyze how the optimal supply of government debt changes when the economy’s initial stock of liquid assets changes. I find that when the stock of bank assets falls, the optimal supply of public liquidity increases. This result can be interpreted in several ways. One interpretation is simply that the government should issue more debt when the banking sector is poorly capitalized. Since this is likely true during recessions, this result can be taken as a justification for procyclical budget deficits. The result can also be interpreted as a need for public liquidity provision more broadly. The government can provide liquidity by lending directly to firms or financial intermedi-
aries in distress. Thus during the 2008 crisis, the Federal Reserve established a number of liquidity facilities to provide funds to intermediaries responsible for a large portion of the economy’s supply of liquidity.

By contrast, when the economy’s stock of private outside liquidity falls, the optimal supply of public liquidity also falls. This is because social welfare is convex in outside liquidity, and thus the marginal value of issuing public outside liquidity is increasing in the stock of private outside liquidity. Intuitively, public liquidity increases welfare by decreasing the price of liquidity. Since this price decrease is applied to all inframarginal units, the welfare gains of issuing outside liquidity are larger if the economy’s initial stock of outside liquidity is large.

**Related Literature.** This paper is part of the literature exploring the role of public liabilities in providing liquidity by serving as stores of value. Samuelson (1958) shows that a government bond can enable intergenerational trades that would not otherwise occur. Woodford (1990) shows that when income and investment opportunities are not synchronized, investment and liquid assets are complements, so that under some conditions the issuance of government debt will “crowd in” investment. Kiyotaki and Moore (2008) consider both public and private liquidity in a similar framework, and Farhi and Tirole (2012) introduce bubbles as a store of value to explore the interplay between public, private, and bubble liquidity. In all of these models, agents would like to transfer funds forward in time, but are unable to do so due to a market incompleteness. I build on these models by introducing financial intermediaries that supply liquidity.

Several papers have examined the role of banks in creating liquid assets. In Diamond and Dybvig (1983), banks offer liquidity insurance by pooling claims to investment projects and selling demand deposits to households. Holmstrom and Tirole (1998) models liquidity in a framework identical to my own, and show that banks can perfectly insure against idiosyncratic liquidity shocks but not aggregate liquidity shocks. Brunnermeier and Sannikov (2010) introduce intermediaries to a model similar to Kiyotaki and Moore (2008), and show that a fall in bank assets will decrease the supply of inside liquidity.\(^3\) I depart from these models by making bank lending subject to an agency cost that depends on bank assets.

The idea that bank lending is subject to agency costs has several precedents in the banking literature. Calomiris and Kahn (1991) argue that demandable debt is a mechanism to ensure the cooperation of banks. They argue that agency problems are paramount in banking, writing “...studies of banking failures give fraud a prominent place in the list of causes. Studies of 19th- and 20th-century banking indicate that fraud and conflicts of interest characterize the vast majority of bank failures for state and nationally chartered banks.” Likewise, Diamond and Rajan (2001) theorize that banks adopt fragile asset structures as a commitment device. Mattesini, Monnet, and Wright (2009) examine which agents will serve as banks in a mechanism design framework; they find that agents who have a larger stake in the system serve as banks, since exclusion from future trades

\(^3\) Other papers that discuss liquid asset creation by banks in comparison to government liabilities include Stein (2012) and Greenwood, Hanson, and Stein (2010).
is more costly for such agents. I capture this notion by making agency costs decreasing in bank assets, since bank assets serve as collateral and well-capitalized banks have more to lose if their reputations are damaged.

This paper is also related to the recent literature exploring the connection between the 2008 financial crisis and the market for liquidity. Pozsar et al. (2010) analyzes the 2008 financial crisis as a run on the “shadow banking” system, and discusses Fed policy in response to these events. Pozsar (2011) argues that the rise of shadow banking was driven by high demand for safe and liquid secured assets similar to Treasury debt, primarily driven by institutional cash pools. In the years immediately preceding the crisis, the demand for safe liquid assets exceeded the supply of government liabilities by at least $1.5 trillion, and the shadow banking sector developed to fill this need. Pozsar (2011) recommend that policy makers consider issuing a greater volume of Treasury bills to fill this demand for liquidity.\footnote{Bernanke et al. (2011), Caballero (2010), and Acharya and Schnabl (2010) also discuss the role of high demand for safe assets in the run-up to the 2008 crisis.}

There have been several empirical studies of the role of government liquidity. Krishnamurthy and Vissing-Jorgensen (2012) find that government bonds hold a liquidity premium over corporate bonds, with about half of the 100 bps average spread explained by the superior liquidity of Treasury bonds. Kashyap and Stein (2000) find evidence of a credit channel of monetary policy.

Finally, there is a large literature exploring the linkages between the financial sector and the real economy.\footnote{Prominent examples include Bernanke and Gertler (1989) and Kiyotaki and Moore (1997).} Most of this literature focuses on the role of banks in lending funds to firms for investment. I depart from this literature by considering the role of the financial sector in providing liquid assets as means of saving, not providing loans. Banks provide both loan services to borrowers and liquidity services to depositors, and I focus on the latter in this paper.

## 2 Model

### 2.1 Without Banks

I first consider the model without banks. This case is a simplified version of Holmstrom and Tirole (1998). I give the derivation in detail because it will serve as the basis for the model with banks that follows.

**Preferences, Endowment, and Technology.** I consider an economy containing three periods, labeled $t = 0, 1, 2$. The economy is populated by a unit measure of two types of agents: households and firms. There is a single good used for both consumption and investment, which is not storable between periods. All agents have linear utility over consumption across all three periods, meaning they have utility functions $u(c_0, c_1, c_2) = c_0 + c_1 + c_2$.

Households have an endowment of the good in each period that is sufficiently large that their
ability to lend to firms is never limited in the constrained case.\textsuperscript{6} I denote their period $i$ endowment by $H_i$. Households also enter period 0 holding a stock $I$ of trees that yield a unit return of the good in period 2. These trees are the economy’s stock of liquid assets, which I will refer to as outside liquidity. Firms have an initial endowment of $A$ units of the good in period 0, and no endowment in any other period.

Firms operate a linear production technology which yields a gross return of $\rho_1 > 1$ between period 0 and period 2. Thus if a firm invests $I$ units of the good in period 0, the project will produce $\rho_1 I$ in period 2 if the project is completed. During period 1, each firm receives an idiosyncratic shock $\rho$. A firm that suffers shock $\rho$ must supply an additional $\rho I$ units of the good to the project to continue its operation. If these funds are not provided, the project will produce nothing. $\rho$ is drawn from $\{\rho_L, \rho_H\}$, with $\rho_L < \rho_H < \rho_1$, and takes on the value $\rho_L$ with probability $p$, and $\rho_H$ with probability $1 - p$. Therefore in period 1 a fraction $p$ of the unit measure of firms will suffer the low shock $\rho_L$, and a fraction $1 - p$ will suffer the high shock $\rho_H$.

We can summarize the production plan of a firm by the initial investment $I$ and a continuation policy rule $\{\lambda_L, \lambda_H\}$, where $\lambda_s = 1$ means that the project continues in the event of liquidity shock $s$, and $\lambda_s = 0$ means that production does not continue.

**Unconstrained Optimum.** Before introducing credit constraints, I briefly discuss the unconstrained optimum. Consider a consumption plan $\{C_1, C_2, C_3\}$, where $C_i$ is total consumption of firms and households in period $i$. The optimal production plan maximizes total consumption subject to the economy’s resource constraints. By period, these resource constraints are

\begin{align*}
C_0 + I & \leq A + H_0 \quad (1) \\
C_1 + p\lambda_L \rho_L I + (1 - p)\lambda_H \rho_H I & \leq H_1 \quad (2) \\
H_2 + p\lambda_L \rho_1 I + (1 - p)\lambda_H \rho_1 I + l & \geq C_2 \quad (3)
\end{align*}

In period 0, households have endowment $H_0$ and firms have endowment $A$. These funds are spent on consumption $C_0 \geq 0$ and investment $I \geq 0$. In period 1, households have endowment $H_1$. Funds $C_1 \geq 0$ are used for consumption, funds $p\lambda_L \rho_L I$ are used to meet low liquidity shocks, and funds $(1 - p)\lambda_H \rho_H I$ are used to meet high liquidity shocks. In period 2, households have endowment $H_2$, and projects produce $p\lambda_L \rho_1 I + (1 - p)\lambda_H \rho_1 I$. These funds are spent on consumption $C_2 \geq 0$.

The unconstrained optimal production plan is the solution to

$$\max_{\{\lambda, I\}} C_0 + C_1 + C_2$$

subject to (1) - (3), together with non-negativity constraints $C_i \geq 0$ and $I \geq 0$, and subject to

\textsuperscript{6}At the unconstrained optimum defined below, this endowment will limit investment. The assumption here is that $H_i$ is sufficiently large relative to firm assets $A$ that for a non-trivial leverage constraint, firm borrowing will not be limited by available funds, but only by the firm’s ability to commit to repaying their investors.
\( \lambda_s \in \{0, 1\} \) for \( s \in \{L, H\} \).

**Proposition 1.** The unconstrained optimal production plan is \( \lambda_H = 1, \lambda_L = 1 \), and

\[
I = \begin{cases} 
A + \overline{H}_0 & R \geq 0 \\
0 & R < 0 
\end{cases}
\]

where \( R = p(\rho_1 - \rho_L) + (1 - p)(\rho_1 - \rho_H) - 1 \).

*Proof.* See Appendix A1. \( \square \)

Since utility is linear with no discounting, any distribution of consumption between agents is Pareto efficient as long as total consumption is maximized. Here \( R \) is the net expected return per unit of investment. Since agents are indifferent between consuming in periods 0, 1, or 2, as long as there is a positive return to investment the optimal production plan is to invest all available resources in period 0. Since both liquidity shocks are smaller than the output of the project \( \rho_1 \), it is ex post optimal to meet both shocks.

I assume for the rest of the paper that the project is profitable even if only the low shock triggers continuation, meaning

\[
p(\rho_1 - \rho_L) > 1
\]

Condition (6) together with \( \rho_1 > \rho_H \) implies \( R > 0 \), so investment yields a positive return in expectation. Thus the unconstrained optimal production plan is to invest all available resources as defined in (5).

**Limited Pledgeability.** The first-best production plan is not implementable because firm borrowing is subject to a limited pledgeability constraint. This constraint arises from moral hazard.\(^8\) At the end of period 1 each firm with a functioning project is presented with an alternative opportunity. If a firm shirks by pursuing this opportunity, the firm’s project fails and the firm earns private benefit \( BI \), where \( B > 0 \). There is no legal recourse for investors to seize any portion of this private benefit. Therefore in every state in which a firm’s project is successful, the firm must receive at least a share \( BI \) of the output in order to cooperate.

Let \( \rho_0 = \rho_1 - B \). Since a successful project produces \( \rho_1 I \) output in period 2, and since the firm must receive \( BI \) in order to operate the project, external investors may receive no more than \( \rho_1 I - BI = \rho_0 I \). Therefore \( \rho_0 I \) is the portion of a project’s output that a firm can pledge to investors.

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\(^7\)This proposition assumes that \( \overline{H}_1 \) is sufficiently large relative to \( \overline{H}_0 \) that the limiting factor on investment is the supply of funds in period 0, not funds available to meet the liquidity shocks in period 1. This is equivalent to the condition \( \overline{H}_1 \geq [p\rho_L + (1 - p)\rho_H] (A + \overline{H}_0) \).

\(^8\)There are many alternate motivations for limited pledgeability. This description corresponds to the microfoundation provided in Holmstrom and Tirole (1998).
I refer to $\rho_0$ as firm pledgeability.\footnote{It is also possible to make firms’ other asset holdings subject to an agency cost. Then firms would have to keep some of the return on these assets, and only a fraction could be used to repay the original investors. This is analogous to a tax on liquid assets purchased by firms, since firms would have to buy $(1 + \tau)$ assets in order to increase their pledgeable funds by $l$. From the perspective of the firm, this is analogous to an increase in the liquidity premium.}

I assume that $\rho_0$ satisfies

\begin{align}
\rho_L &< \rho_0 < \rho_H \quad (7) \\
\rho_0 &< 1 + p\rho_L + (1-p)\rho_H \quad (8) \\
p\rho_0 &< 1 + p\rho_L \quad (9) \\
\rho_0 &> p\rho_L + (1-p)\rho_H \quad (10)
\end{align}

Condition (7) says that pledgeability is sufficient to meet the low shock, but not the high shock. Condition (8) and (9) say that expected funds required per unit of investment exceed the pledgeable portion of the return, whether both shocks or just the low shock are met, so that projects cannot be financed solely with external funds. Thus limited pledgeability is sufficiently severe that it implies a “skin in the game” constraint that requires firms to put up some of their own capital. If either of these conditions failed to hold, the scale of investment would not be limited by pledgeability. Finally, condition (10) says that pledgeable funds are sufficient to meet the expected size of the liquidity shock, so that there are enough pledgeable funds owned by firms in the aggregate to meet all liquidity shocks in period 1. Thus liquidity will only be scarce when there is imperfect pooling of these funds.\footnote{Condition (8) together with $\rho_L < \rho_H$ implies that external financing is also insufficient if only the high shock is met.}

Households cannot borrow at all. Since they cannot offer any collateral, they will renege on any promise they make.\footnote{We can interpret household initial holdings of trees $l$ as a portion of their endowment in period 2 that is pledgeable.}

\textbf{Households.} Households supply funds by purchasing state-contingent assets from firms. Given linear utility and no discounting, households are willing to purchase any asset that promises an expected return of at least 1. Therefore the supply of funds in the economy is perfectly elastic at a gross return of 1. Households consume all of their income that is not used to purchase assets.

Let $q$ be the price of trees in period 0. Since trees yield a unit return, households will demand an infinite quantity of trees if $q < 1$ and no trees if $q > 1$. If $q = 1$ households are indifferent between any quantity of trees. Therefore household asset demand is perfectly elastic at $q = 1$.

\textbf{Firms.} Firms raise funds from households in period 0 by selling a contract offering a state-contingent return in period 2. The contract specifies payments to initial investors of $R_s^I$ in the case of shock $s \in \{L, H\}$. Firms also purchase $l$ trees from households at a price of $q$. In period 1, firms experience liquidity shocks. Firms raise funds to meet these shocks by selling new claims to households to be paid in period 2. Since households know the shock experienced by a firm, they
will require exact compensation for funds provided. I denote by $R^1_s$ the funds repaid in period 2 to period 1 investors by a firm that experiences shock $s \in \{L, H\}$.

Any equilibrium contract must satisfy a number of constraints arising from incentive compatibility of the parties to the contract. First, the initial investors must be compensated in expectation for the funds they provide. The firm requires $I$ funds in period 0 for its initial investment, plus $ql$ funds to purchase trees. Firms have initial assets $A$, and the rest must be raised from outside investors. In order for initial investors to buy this contract, their expected payments $pR^1_I + (1 - p)R^1_H$ must satisfy

$$pR^1_I + (1 - p)R^1_H \geq I + ql - A \tag{11}$$

In period 1, firms receive liquidity shocks, and those that continue production meet these shocks by issuing new liabilities to investors. I denote by $R^1_s$ the repayment in period 2 of a firm that experiences liquidity shock $s \in \{L, H\}$ to its period 1 investors. These funds must be sufficient to cover liquidity shocks, which implies constraints

$$\lambda_H\rho_HI \leq R^1_H \tag{12}$$
$$\lambda_L\rho_HI \leq R^1_H \tag{13}$$

In order for production to take place, firms need a sufficient share of the profits to cooperate. They repay their outside investors from the pledgeable portion of their output and from the return on their asset holdings $l$. This implies constraints on total repayment of

$$R^1_I + R^1_L \leq \lambda_L\rho_0I + l \tag{14}$$
$$R^1_H + R^1_H \leq \lambda_H\rho_0I + l \tag{15}$$

Note that firms do not use outside assets $l$ to pay for liquidity shocks directly. Rather, buying assets in period 0 allows firms to increase their pledgeability by the amount $l$, and therefore borrow more from period 1 investors. These two specifications are analogous, but writing the constraints in this manner will simplify the exposition.

How can liquid assets improve the allocation? We can think of this as a commitment problem on the part of households. Since projects that experience a high shock are profitable to continue ($\rho_1 > \rho_H$), it is optimal for households to provide funds to firms to meet high liquidity shocks. However, once a high liquidity shock is realized firms do not have enough pledgeable funds to pay to meet them, since they can only promise a fraction $\rho_0 < \rho_H$ of their output. Ex ante, households would like to promise to provide funds at a loss in this event. This would allow a higher level of investment and therefore higher payments in good states. Since $\rho_0 > p\rho_L + (1 - p)\rho_H$, the higher payments in good states can be made large enough to compensate households for the negative returns in bad states so that households come out ahead on average. However, by assumption households are not able to commit to providing funds at a loss once the liquidity shock is realized.
Liquid assets provide a mechanism for circumventing this commitment problem. Households can provide firms with funds in period 0 before the liquidity shock is realized that firms use to buy liquid assets. If the firm experiences a high shock these assets are sold to raise the necessary funds, and if a low shock is realized the earnings are returned to the households. Effectively, households are providing collateral for their promise to pay firms in period 1 to meet liquidity shocks. Therefore liquid assets serve as a social commitment mechanism.

**Optimal Contract.** Firms choose the profit-maximizing contract

$$\max_{R, \lambda, I, l} \left\{ p \left( \lambda_L \rho_1 I - R_L^1 - R_L^1 + l \right) + (1 - p) \left( \lambda_H \rho_1 I - R_H^1 - R_H^1 + l \right) \right\}$$

s.t. (11)-(15)

$$R_L^1 \geq 0, R_H^1 \geq 0, R_L^1 \geq 0, R_H^1 \geq 0, I \geq 0$$

where $R = \{R_L^1, R_H^1, R_L^1, R_H^1\}$ and $\lambda = \{\lambda_L, \lambda_H\}$. Non-negativity constraints on payments to households arise from the assumption of limited commitment by households. In the absence of these constraints, households might find it ex ante optimal to promise to provide funds at a loss to finance high liquidity shocks, and be compensated by promised high returns in the case of low liquidity shocks.

**Proposition 2.** At the optimum we have $\lambda_L = 1$, and constraints (11) - (14) hold with equality.

*Proof.* See appendix A2.

The intuition for this result is as follows. Since investment yields a positive return, firms will invest until constraint (11) is binding. Since repaying period 1 investors is a cost, firms will reduce these payments to the point at which (12) and (13) bind. Finally, if the pledgeability constraints were not binding, firms would be able to increase investment by borrowing more from initial investors. Since our assumptions on parameters imply that a single unit of investment yields greater than unit return, firms will borrow in order to invest until the pledgeability constraints are binding in both states.

Proposition 2 greatly simplifies the statement of the problem. Since constraints (12) - (13) and (14) - (15) hold with equality, we can substitute them directly into the various expressions in the problem. Substituting constraints (14) and (15) into the objective function, we find that firm payoffs are $p (\rho_1 - \rho_0) I$ if $\lambda_H = 0$, and $(\rho_1 - \rho_0) I$ if $\lambda_H = 1$. Since the pledgeability constraint binds, firms receive exactly the amount $(\rho_1 - \rho_0) I$ necessary for them to cooperate in equilibrium.
Also, these equations allow us to derive exact expressions for $R_1^H$ and $R_1^L$. These are

\begin{align*}
R_1^H &= \lambda_H \rho_H I \\
R_1^L &= \rho_L I \\
R_1^L &= (\rho_0 - \rho_L) I + l \\
R_1^H &= \lambda_H (\rho_0 - \rho_H) I + l
\end{align*}

Substituting these into the initial investor’s participation constraint (11), we derive the leverage constraint

\[ I \leq \frac{A - (q - 1) l}{1 - p (\rho_0 - \rho_L) - \lambda_H (1 - p) (\rho_0 - \rho_H)} \tag{16} \]

which will hold with equality at the optimum. Intuitively, the pledgeability constraint is sufficiently severe that firms must supply a fraction of the capital for investment (a skin-in-the-game constraint). This implies that investment is limited by firms’ initial assets, which can be expressed as a limit on leverage, as given in (16).

The non-negativity constraints on $R_1^H$, $R_1^L$, and $R_1^I$ are trivially satisfied. The non-negativity constraint on $R_1^I$ can be expressed as $l \geq \lambda_H (\rho_H - \rho_0) I$, which implies that holdings of outside liquidity $l$ must be sufficiently large to meet the high liquidity shock.

We can now express the optimal contracting problem of the firm as

\[
\max_{\lambda_H, I, l} \{ (\rho_1 - \rho_0) [p + (1 - p) \lambda_H] I \} \\
\text{s.t.} \quad (16) \\
\lambda_H (\rho_H - \rho_0) I \leq l 
\]

plus non-negativity constraints on $I$ and $l$. We can characterize the solution using the following proposition.

**Proposition 3.** The optimal continuation rule of firms is $\lambda_H = 1$ if and only if

\[ q - 1 \leq \frac{(1 - p) [1 - p (\rho_H - \rho_L)]}{p (\rho_H - \rho_0)} \tag{17} \]

If (17) holds with equality, the firm is indifferent between $\lambda_H = 1$ and $\lambda_H = 0$. The optimal choice of $I$ is

\[ I = \frac{A}{1 - p (\rho_0 - \rho_L) - \lambda_H (q - p) (\rho_0 - \rho_H)} \]

and the optimal choice of $l$ is

\[ l = \lambda_H (\rho_H - \rho_0) I. \]

**Proof.** See appendix A2.

Since $q - 1$ is always non-negative, (17) will never hold if the parameters satisfy

\[ p (\rho_H - \rho_L) > 1 \]
and it will never be optimal to meet the high shock. This condition is a simplification of the expression

\[ \frac{pA}{1 - p (\rho_0 - \rho_L)} > \frac{A}{1 - p (\rho_0 - \rho_L) - (1 - p) (\rho_0 - \rho_H)} \]

which says that investment when firms meet only the low shock, times the fraction \( p \) of such projects that succeed, is greater than investment when firms meet both shocks. Intuitively, if adverse liquidity shocks are sufficiently bad relative to benign shocks, it is preferable to spend limited pledgeable funds meeting low shocks only with a higher level of initial investment, rather than meeting both shocks with a lower level of investment.

When \( p (\rho_H - \rho_L) \leq 1 \), (17) defines a cutoff level of \( q \) such that, if liquidity is more expensive than this cutoff, it is optimal to meet only the low shock. Intuitively, since firms must purchase liquid assets in order to meet adverse liquidity shocks, a high liquidity premium makes meeting high shocks more expensive, and at some point it is no longer optimal to meet them.

**Equilibrium.** Proposition 3 defines a demand for liquid assets \( l \), which is a decreasing function of \( q \), and drops to zero at the cutoff defined by (17). To determine equilibrium \( q \), we use market-clearing in the liquidity market. Since households will only hold assets that yield a unit return or better, if \( q > 1 \) aggregate liquidity demand is exactly equal to liquidity demand from firms. Equilibrium is depicted in Figure 1 and characterized in Proposition 4.

**Proposition 4.** Suppose \( \bar{I} \) is given. Let \( \bar{I}_0 = A/ [1 - p (\rho_0 - \rho_L)] \) and \( \chi_1 = 1 - \rho_0 + p \rho_L + (1 - p) \rho_H \). Then,

(i) If \( p (\rho_H - \rho_L) > 1 \), equilibrium \( q = 1 \) and all firms pursue the strategy \( \lambda_H = 0, l = 0, \) and \( I = \bar{I}_0 \).

(ii) If \( p (\rho_H - \rho_L) \leq 1 \) and \( \bar{I} \geq (\rho_H - \rho_0) A/\chi_1 \), equilibrium \( q = 1 \) and firm behavior is given by \( l = A/\chi_1, l = (\rho_H - \rho_0) A/\chi_1 \), and \( \lambda_H = 1 \).

(iii) If \( p (\rho_H - \rho_L) \leq 1 \) and \((\rho_H - \rho_0) \bar{I}_0 \leq \bar{I} \leq (\rho_H - \rho_0) A/\chi_1 \), equilibrium \( q \) is given by

\[ q - 1 = \frac{A}{\bar{I}} - \frac{1 - p (\rho_H - \rho_L)}{\rho_H - \rho_0} \]

and firm behavior is given by \( \lambda_H = 1, l = \bar{I}, \) and \( I = \bar{I}/ (\rho_H - \rho_0) \).

(iv) If \( p (\rho_H - \rho_L) \leq 1 \) and \( \bar{I} < (\rho_H - \rho_0) \bar{I}_0 \), equilibrium \( q \) is given by

\[ q - 1 = \frac{1 - p (\rho_0 - \rho_L) - p [1 - p (\rho_H - \rho_L)]}{p (\rho_H - \rho_0)} \]

and firms are indifferent between \( \lambda_H = 0 \) and \( \lambda_H = 1 \). A fraction \( \zeta \) of firms choose \( \lambda_H = 1, l = p \bar{I}_0, \) and \( l = (\rho_H - \rho_0) p \bar{I}_0, \) where \( \zeta = \bar{I}/ [(\rho_H - \rho_0) p \bar{I}_0] \). The remaining fraction \( 1 - \zeta \) pursue the strategy
\[ \lambda_H = 0, l = 0, \text{ and } I = I_0. \]

**Proof.** See appendix A2.

Proposition 4 defines two cutoff levels of \( \bar{l} \).\textsuperscript{12} If outside liquidity \( \bar{l} \) is above the higher cutoff, then all firms meet both liquidity shocks and there is no liquidity premium \((q - 1 = 0)\). In this case, the economy still fails to achieve the unconstrained optimum due to the usual effect of credit constraints restricting initial investment, but there are no additional effects due to limited liquidity.

I refer to the equilibrium allocation with \( q = 1 \) as the constrained optimum. The constrained optimum achieves the highest level of investment that respects the aggregate limited pledgeability constraint. This level of investment is \( I = A/\chi_1 \), meaning that firms must provide \( \chi_1 \) of their own funds for every unit of investment, and so \( 1/\chi_1 \) is the leverage ratio in the economy. \( \chi_1 = 1 + p\rho_L + (1 - p)\rho_H - \rho_0 \) is the amount by which expected required funds per unit of investment exceed pledgeable funds. Our assumptions on \( \rho_0 \) imply \( \chi_1 \in (0, 1) \), and so \( I \in (A, \infty) \). When \( \rho_0 \) is large relative to \( 1 + p\rho_L + (1 - p)\rho_H \), most investment is financed using external funds and leverage is high. When \( \rho_0 \) is small relative relative to \( 1 + p\rho_L + (1 - p)\rho_H \), external financing is severely limited and leverage is low.

If \( \bar{l} \) lies below the higher cutoff, then the economy is liquidity constrained. Then there is insufficient liquidity for all firms to meet the high shock at the constrained optimal level of investment.

\textsuperscript{12}Note \( p(\rho_H - \rho_0)I_0 \leq (\rho_H - \rho_0)A/\chi_1 \) if and only if \( p(\rho_H - \rho_L) \leq 1 \), and so the cutoffs satisfy the assumed ordering.
Therefore equilibrium $q$ must rise until the quantity of liquidity demanded by firms equals available liquidity $\bar{l}$. Liquidity demand is downward sloping because firms' pledgeable funds are used to pay the higher price of liquidity, and so funds available to finance initial investment fall, tightening the leverage constraint.

If $\bar{l}$ lies below the lower cutoff, then there is no $q$ that will convince all firms to hold exactly $\bar{l}$, because before desired $l$ reaches $\bar{l}$ firms find it optimal to meet only the low shock. Therefore in equilibrium only a fraction of firms will hold liquid assets and meet both shocks. Since firms are identical when they make the decision whether to purchase liquid assets, firms must be indifferent between meeting only the low shock and meeting both shocks. Therefore $q$ is fixed at this level, and further declines in $\bar{l}$ will only reduce the fraction of firms that meet both shocks.

I refer to an equilibrium that lies strictly between these two cutoffs as an interior equilibrium. This is the most interesting case, because here a marginal change in available liquidity will change both the equilibrium investment and the equilibrium liquidity premium $q$. Such a shift is depicted in Figure 2, which shows that a fall in outside liquidity $\bar{l}$ will increase equilibrium $q$ and decrease equilibrium $I$.

### 2.2 With Banks

In the previous section, we showed that there may be insufficient liquidity in the economy, resulting in a fall in investment and, in some cases, a fall in the share of firms meeting high liquidity shocks. This arises from firms' inability to pool their pledgeable funds. In period 1, firms that ex-
experience a low shock have more pledgeable funds available than they need to meet their liquidity shock, whereas firms that experience a high shock have fewer pledgeable funds than they need. Since by assumption there are sufficient pledgeable funds in the aggregate to meet all liquidity shocks, firms can achieve the constrained optimum if they are able to pool their funds. Such pooling could be achieved by arranging for transfers from those firms that receive a low shock to those firms that receive a high shock.

Liquidity pooling can be achieved by introducing an intermediary (bank). Firms and banks enter into a contract (exchange state-contingent liabilities) such that firms promise to pay the bank in period 2 (when its project produces) in return for insurance against liquidity shocks. The bank promises to pay firms that received a high shock the necessary funds to meet this shock. Banks raise these funds in period 1 by issuing liabilities that are bought by households, which they repay in period 2 using funds they receive from firms. Since firms are in the aggregate able to promise enough funds in period 2 to achieve the constrained optimum, without further frictions this arrangement can achieve the constrained optimum even when the economy has no outside liquidity ($I = 0$).

The above discussion assumes that banks have no limits on their ability to commit funds. However, it is reasonable to think that banks are subject to agency costs of a similar nature to firms. Banks act as agents that invest on behalf of depositors, in this case households that provide funds in period 1. In the process of lending, banks may need to exert effort in screening firms in period 0, verifying the liquidity shock they received in period 1, and collecting funds from them in period 2. Banks also have opportunities to defraud their depositors by withholding funds. If banks do not receive a sufficient share of the profits, they will not exert effort screening or collecting loans, and may have an incentive for fraud.

Let bank agency costs be represented by the function $C\left(\frac{D}{1-p}, K\right)$, where $D$ are funds that banks receive from their borrowers in period 2, and $K$ is an endowment that banks receive in period 2.\textsuperscript{13} I assume that banks can commit to pay their endowment $K$. Thus we can think of $K$ as other bank assets that can be offered as collateral to reduce agency costs. I assume that agency costs are increasing and convex in funds intermediated, decreasing in bank assets, and satisfy the Inada conditions with respect to funds intermediated $D$. Thus the function $C(\cdot)$ satisfies $C_1(\cdot) > 0$ for $D > 0$, $C_{11}(\cdot) > 0$, $C_2(\cdot) < 0$, $C_1(\cdot) \to 0$ as $D \to 0$ and $C_1(\cdot) \to \infty$ as $D \to \infty$. I assume that $C$ is thrice continuously differentiable in both arguments. I further assume that there are no agency costs at $D = 0$, so that $C(0, K) = 0$. Together with $C_1(\cdot) > 0$, this implies $C(x, K) > 0$ for $x > 0$.

In equilibrium, banks must receive sufficient profits to cover these agency costs. Let $\pi$ be bank profits, which are bank receipts from firms in period 2 in excess of payments to households. Then in equilibrium $\pi$ must satisfy

\[
\pi \geq C\left(\frac{D}{1-p}, K\right)
\]

\textsuperscript{13}D is divided by $1 - p$ as a scaling factor for convenience.
This is the bank’s incentive compatibility constraint that ensures that the bank exerts effort screening loans.

Preferences, Endowment, Technology, and Pledgeability. Preferences, endowments, and technology are precisely the same as in section 2.1, except for the inclusion of banks. Like firms and households, there is a unit measure of banks, and banks have linear utility over consumption across all three periods, meaning they have utility function \( u(c_0, c_1, c_2) = c_0 + c_1 + c_2 \).

As described above, banks receive an endowment of \( K \) units of the final good in period 2. I assume banks can credibly promise to transfer these funds, so they serve as collateral.

Pledgeability for firms is precisely as in the case without banks.

Households. As above, households supply funds perfectly elastically at an expected return of 1. Therefore every asset will be priced to equal its expected return, or households will not purchase the asset. Households are willing to sell their supply of trees at a price \( q \geq 1 \), and will not buy any trees if \( q > 1 \).

Banks. I suppose that there is a measure 1 of banks. I assume that in equilibrium a measure of firms will borrow from each bank, so that banks are both perfectly competitive and perfectly diversified.\(^{14}\)

Banks sell credit lines to firms, which are contracts represented by \((M, R^B_L, R^B_H)\). \( M \) is the credit maximum, which are the funds the bank pays to the firm in the event of a high liquidity shock. \( R^B_L \) and \( R^B_H \) are payments from the firm to the bank in period 2 in case of a low shock and high shock respectively.

If all firms purchase credit lines and both shocks are met, then in period 1 a total of \((1 - p)M\) funds will be supplied by banks to firms that received a high shock. Banks obtain these funds by selling claims to households, which they repay in period 2. Each bank receives payments \( pR^B_L + (1 - p)R^B_H \) from firms in period 2, and it must compensate its investors by repaying the \((1 - p) M\) funds raised in period 1. Thus bank profits are

\[
\pi = pR^B_L + (1 - p) R^B_H - (1 - p) M
\]

As described above, banks must be sufficiently compensated to cooperate. Here banks repay funds \( D = (1 - p)M \) to each of their borrowers. Thus their profits must satisfy the total constraint

\[
\int_i \pi_i \geq C \left( \int_i M_i, K \right)
\]  

(18)

Since banks will not enter into any contract at an expected loss, we have \( \pi \geq 0 \) for each individual contract. I assume that all firms negotiate with all banks simultaneously, and agree on a contract that maximizes firm profits subject to overall constraint and the non-negativity constraint.

\(^{14}\)The simplest way to think of this is that there are \( N \) banks of size \( 1/N \), and \( N^2 \) firms of size \( 1/N^2 \). Then let \( N \to \infty \).
Banks and Insurance. Throughout this paper, I refer to the agents that pool risk as banks, and the credit arrangement firms make with banks as credit lines. However, those firms that experience bad liquidity shocks will not repay anything, and so the optimal contract is analogous to a standard insurance contract. In reality, firms that purchase credit lines from banks generally pay an initial fee, and must repay if they draw on the credit line. I can model an initial fee without difficulty, since firms can simply borrow from households in period 0 to pay a fee. However, in my model firms that experience a bad liquidity shock cannot repay what they borrow, because these firms do not have access to sufficient pledgeable funds. Thus the contracts I describe differ from credit lines observed in reality.

Despite this, I continue to refer to the credit arrangement between banks and households as a credit line. This is consistent with Holmstrom and Tirole (1998), who refer to the same arrangements as credit lines. Moreover, the liquidity services provided by banks are effectively a form of insurance. A primary service of banks is to pool risk by making individually risky loans and issuing safe deposits, an operation analogous to a financial insurance company. Another service is pooling liquidity by holding partially-liquid assets and issue liquid liabilities. Banks in Diamond and Dybvig (1983) are analogous to insurance companies offering insurance for liquidity shocks, a point made by Bryant (1980) in a similar model.

The insurance provided by banks in my model is the inverse of the standard liquidity insurance, since it is purchased by firms rather than by banks. In the traditional models banks pool returns from investment projects from firms and sell the resulting contracts to households. Here banks pool scarce liquidity in the form of limited pledgeability from households and sell the resulting contracts to firms.

Firms. As before, firms choose production plan \( \{I, \lambda\} \), with \( I \geq 0 \) investment in period 0, and project continuation policy rule \( \lambda_s \) for state \( s \in \{L, H\} \). Firms raise funds in period 0 from households by selling a contract offering a state-contingent return to be paid in period 2. The contract specifies payments to initial investors of \( R^I_s \) in the case of shock \( s \in \{L, H\} \). Firms also purchase \( l \) trees from households at a price of \( q \). Finally, firms purchase credit lines from banks, which involves choosing a credit limit \( M \) to be received in period 1 in the event of a high liquidity shock, and payments to banks \( R^B_s \) in period 2 in the event of shock \( s \).

In period 1, firms experience their liquidity shocks. Firms raise funds to meet these shocks by selling new claims to households to be paid in period 2, and if they receive a high shock they also draw on their credit line from banks. Since households know the shock experienced by a firm, they will require exact compensation for funds provided. I denote by \( R^I_s \) the funds repaid in period 2 to period 1 investors by a firm that experiences shock \( s \).

The equilibrium contract must satisfy a number of constraints arising from incentive compatibility for all parties to the contract. First, the initial investors must be compensated in expectation for the funds they provide. The firm requires \( I \) funds in period 0 for its initial investment, plus \( ql \) funds to purchase trees. Firms have initial assets \( A \), and the rest must be raised
from outside investors. In order for initial investors to buy this contract, their expected payments
\[ pR_I^L + (1 - p)R_I^H \] must satisfy
\[ pR_I^L + (1 - p)R_I^H \geq I + ql - A \] (19)

In period 1, firms receive liquidity shocks, and those that continue production meet these shocks by issuing new liabilities to investors, and by drawing on their credit lines if they received the high shock. I denote by \( R_s^1 \) the repayment in period 2 of a firm that experiences liquidity shock \( s \) to its period 1 investors. Together with the credit line, these funds must be sufficient to cover liquidity shocks, which implies constraints
\[ \lambda_H \rho_H I \leq R_I^H + M \] (20)
\[ \lambda_L \rho_L I \leq R_I^L \] (21)

In order for production to take place, firms need to receive a sufficient share of the profits in period 2 to cooperate. They repay their outside investors from the pledgeable portion of their output and from the return on their asset holdings \( l \). This implies constraints on total repayment of
\[ R_I^L + R_I^H + R_B^L \leq \lambda_L \rho_0 I + l \] (22)
\[ R_I^H + R_I^H + R_B^H \leq \lambda_H \rho_0 I + l \] (23)

Finally, firms’ choices of \((M, R_B^L, R_B^H)\) must satisfy the bank incentive compatibility constraint (18), which depends on the credit lines purchased by all firms. I assume that each firm takes the decisions of all other firms as given in their negotiations with banks, and that firms hold all bargaining power relative to banks. Then (18) defines a schedule of possible credit lines for firm \( j \), which we can express using the function \( \pi_j(M) \). Thus the overall constraint (18) implies the credit line \((\pi, M)\) chosen by firm \( j \) must satisfy
\[ \pi \geq \pi_j(M) \] (24)
\( \pi_j(\cdot) \) is equal to 0 for \( M \) small enough that (18) does not bind. For larger \( M \), \( \pi_j(\cdot) \) is defined implicitly by (18) holding with equality. Applying the implicit function theorem we find
\[ \pi'_j(M) = C_1 \left( \int M_i, K \right) \]

Since \( M_j \) is an infinitesimal portion of \( \int M_i \), we conclude that \( \pi_j(\cdot) \) is linear when (18) is binding. \( \pi_j(\cdot) \) is depicted in figure 3.

Optimal Contract. Firms choose the profit-maximizing contract
\[
\text{max}_{R_i, \lambda, \rho} \left\{ p \left( \lambda \rho I - R^1_L - R^B_L + I \right) + (1 - p) \left( \lambda \rho H - R^1_H - R^B_H + I \right) \right\} \\
\text{s.t.} \quad \text{(19)-(24)} \\
R^1_L \geq 0, R^1_H \geq 0, R^B_L \geq 0, R^B_H \geq 0, I \geq 0, M \geq 0, \pi \geq 0
\]

where \( R = \{ R^1_L, R^1_H, R^B_L, R^B_H \} \). Non-negativity constraints on payments to households arise from the assumption of limited commitment by households. In the absence of these constraints, households might find it ex ante optimal to promise to provide funds at a loss to finance high liquidity shocks, and be compensated by promised high returns in the case of low liquidity shocks.

**Proposition 5.** At the solution, \( \lambda_L = 1 \), and constraints (19) - (24) hold with equality.

**Proof.** See appendix A3.

The intuition for this result is similar to Proposition 2. Since investment yields a positive return, firms will invest until constraint (19) is binding. Since repaying period 1 investors is a cost, firms will reduce these payments to the point at which (20) and (21) bind. If the pledgeability constraints were not binding, firms would be able to increase investment by borrowing more from initial investors. Since our assumptions on parameters imply that a single unit of investment yields greater than unit return, firms will borrow in order to invest until the pledgeability constraints are binding in both states. Finally, firms will reduce their payments to banks until the...
bank participation constraint (24) or the bank profits non-negativity constraint binds.

Proposition 5 greatly simplifies the statement of the problem. Since constraints (20) - (21) and (22) - (23) hold with equality, we can substitute them directly into the various expressions in the problem. Substituting constraints (22) and (23) into the objective function, we find that firm payoffs are \( p (\rho_1 - \rho_0) I \) if \( \lambda_H = 0 \), and \( (\rho_1 - \rho_0) I \) if \( \lambda_H = 1 \). Since the pledgeability constraint binds, firms receive exactly the amount \( (\rho_1 - \rho_0) I \) necessary for them to cooperate in equilibrium. Also, these equations allow us to derive exact expressions for \( R^1_H \) and \( R^1_L \). These are

\[
R^1_H = \lambda_H \rho_H I - M \\
R^1_L = \rho_L I
\]

We don’t have precise values yet for \( R^1_I \) and \( R^1_H \), because we do not know \( R^B_L \) and \( R^B_H \). However, we can express the constraints we have as a leverage constraint analogous to (16). We can combine (22) and (23) to derive the expression

\[
p \left[ R^I_L + \rho_H I + R^B_L \right] + (1 - p) \left[ R^I_H + \lambda_H \rho_H I - M + R^B_H \right] = p [\rho_0 I + l] + (1 - p) [\lambda_H \rho_0 I + l]
\]

Since this expression contains \( p R^I_L + (1 - p) R^I_H \) and \( p R^B_L + (1 - p) R^B_H - (1 - p) M \), we can substitute in (19) and the definition of \( \pi \). This allows us to derive the leverage constraint

\[
I \leq \frac{A - (q - 1) l - \pi}{1 - p (\rho_0 - \rho_L) - \lambda_H (1 - p) (\rho_0 - \rho_H)}
\] (25)

which will hold with equality at the solution. Intuitively, the pledgeability constraint is sufficiently severe that firms must supply a fraction of the capital for investment (a skin-in-the-game constraint). This implies that investment is limited by firms’ initial assets, which can be expressed as a constraint on leverage, as given in (25).

We can now rewrite the other constraints in the problem. The non-negativity constraints on \( R^I_L, R^I_H, R^I_L \) and \( R^B_L \) are trivially satisfied, given \( M \geq 0 \). Bank participation implies \( \pi \geq 0 \). The non-negativity constraints on \( R^I_H \) and \( R^B_H \) imply

\[
\lambda_H \rho_H I \leq \lambda_H \rho_0 I + M + l
\] (26)

Together with the bank agency constraint facing an individual firm (24), we can express the optimal contracting problem facing a given firm as

\[
\max_{\lambda_H, I, L, M, \pi} \left\{ (\rho_1 - \rho_0) \left[ p + (1 - p) \lambda_H \right] I \right\} \\
\text{s.t. } (24), (25), \text{ and } (26)
\]

plus non-negativity constraints on \( I, M, \) and \( \pi \). By Proposition 5, (24) and (25) hold with equality at the solution. Optimal firm behavior is given in Proposition 6.
Proposition 6. Let \( q \geq 1 \), \( K > 0 \), and the credit lines of all other firms \((M, \pi)\) be given, and let \( \chi_1 = 1 - \rho_0 + pp_L + (1 - p) \rho_H \). Let \( \overline{M}_1 \) be defined implicitly by \( \overline{M}_1 \chi_1 = (\rho_H - \rho_0) \left[ A - \pi \left( \overline{M}_1 \right) \right] \), and let \( \overline{M}_2 \) be defined implicitly as the smallest value of \( M \) such that \( q - 1 \) is a subgradient of \( \pi \left( M \right) \). Then optimal firm behavior is as follows:

(i) If \( q - 1 \leq \pi' \left( 0 \right) \) and \( p \left( \rho_H - \rho_0 \right) \left( q - 1 \right) \leq 1 - p \left( \rho_0 - \rho_L \right) - p \chi_1 \), then \( \lambda = 1 \), \( M = 0 \), \( \pi = 0 \), \( I = A \left/ \left[ \chi_1 + (\rho_H - \rho_0) \left( q - 1 \right) \right] \right. \), and \( I = (\rho_H - \rho_0) \left/ \chi_1 \right. \).

(ii) If \( \overline{M}_1 \leq \overline{M}_2 \) and \( \left[ 1 - p \left( \rho_0 - \rho_L \right) \right] \overline{M}_1 \geq Ap \left( \rho_H - \rho_0 \right) \), then \( \lambda = 1 \), \( M = \overline{M}_1 \), \( \pi = \pi \left( \overline{M}_1 \right) \), \( I = 0 \), and \( I = \left[ A - \pi \left( \overline{M}_1 \right) \right] \left/ \chi_1 \right. \).

(iii) If \( \overline{M}_1 > \overline{M}_2 \), and

\[
pA \leq \frac{A - \pi \left( \overline{M}_2 \right) \left( q - 1 \right) \overline{M}_2}{(\rho_H - \rho_0) \left( q - 1 \right) + \chi_1} \left[ 1 - p \left( \rho_0 - \rho_L \right) \right]
\]

then \( \lambda = 1 \). If \( \pi' \left( M \right) = q - 1 \) for \( M > \overline{M}_2 \), then any \( M \in [\overline{M}_2, \overline{M}_1] \) is a solution, while if \( \pi' \left( M \right) > q - 1 \) for \( M > \overline{M}_2 \), then only \( M = \overline{M}_2 \) is a solution. For each \( M \), optimal behavior satisfies \( \pi = \pi \left( M \right) \), and

\[
I = \frac{A - \pi \left( M \right) \left( q - 1 \right) M}{(\rho_H - \rho_0) \left( q - 1 \right) + \chi_1}
\]

\[
l = \frac{(\rho_H - \rho_0) \left[ A - \pi \left( M \right) \right] - M \chi_1}{(\rho_H - \rho_0) \left( q - 1 \right) + \chi_1}
\]

(iv) If (i)-(iii) do not hold, we have \( \lambda = 0 \), \( I = 0 \), \( M = 0 \), and \( I = A \left/ \left[ 1 - p \left( \rho_0 - \rho_L \right) \right] \right. \).

Proof. See appendix A3.

Proposition 6 characterizes the optimal contract for given \( q \), \( K \), and \( \pi \left( \cdot \right) \). \( \overline{M}_2 \) is the smallest credit line such that the marginal agency cost of additional bank financing exceeds the marginal cost of purchasing outside liquidity \( l \), which is the liquidity premium \( q - 1 \). \( \overline{M}_1 \) corresponds to firm’s choice of credit limit if they do not purchase any outside liquidity, which will happen if \( q - 1 \) is sufficiently high.

Intuitively, if a firm finds it optimal to meet the high liquidity shock, it must choose between financing this shock by purchasing a credit line from a bank, holding outside liquidity, or both. If \( \pi' \left( 0 \right) \geq q - 1 \), then any credit line is more expensive than outside liquidity, and so the firm will choose to finance the high liquidity shock using outside liquidity exclusively. If \( \overline{M}_1 \leq \overline{M}_2 \), the marginal cost of financing the firm’s desired level of investment using credit lines is less than the
marginal cost of holding outside liquidity, and so the firm holds no outside liquidity. If instead
$M_2 < M_1$, the marginal cost of a credit line crosses the marginal cost of outside liquidity from
below, so that the firm chooses a credit line of up to this point, and then purchases outside liquidity
to finance the remainder.

The other conditions ensure that it is in fact optimal to meet the high liquidity shock given the
prevailing price of liquidity.

**Equilibrium.** Proposition 6 defines a firm’s optimal continuation rule $\lambda_H$, investment $I$, outside
liquidity holdings $l$, and credit line $(\pi, M)$ given cost of liquidity $q$, bank assets $K$, and the bank
contracts of all other firms $(\pi_i, M_i)$. That is, it defines $(\lambda_H, I, l, \pi, M)$ given $(q, K, \pi_i, M_i)$. An
equilibrium is a set of $(\lambda_{H,i}, I_i, l_i, \pi_i, M_i)$ for each firm $i$ such that each firm’s decisions.

Since firms are identical and negotiate with banks simultaneously, I restrict attention to equi-
libria in which a fraction $\zeta \in [0, 1]$ of firms purchase contracts from banks, and these firms all
purchase the same contract $(\pi, M)$. We need a condition that guarantees that for given $K$ and
$q - 1$, there will be exactly one such symmetric contract. This will hold true for $\zeta = 1$, but we need
a further condition to ensure uniqueness for $\zeta < 1$. The bank agency cost function must satisfy

$$M \left[ C_1 (M, K) + \bar{I}C_{11} (M, K) \right] \geq C (M, K) + \bar{I}C_1 (M, K)$$

(27)

This condition guarantees that $C (M \zeta, K) / \zeta + \bar{I}C_1 (M \zeta, K) / \zeta$ is weakly increasing in $\zeta$. That is,
holding the size of credit lines $M$ fixed, an increase in the fraction of firms buying credit lines $\zeta$
will increase pledgeable funds available to firms.

Proposition 6 together with the assumption of symmetric credit lines uniquely defines optimal
firm behavior $(\zeta, I, l, M, \pi)$ for given $(q, K)$. To define equilibrium, we need one further condition
to determine $q$, for which we use the outside liquidity market clearing condition. Households
desire to hold outside liquidity $l$ only when $q = 1$, and so either $q = 1$ and firms may hold any $l$
such that $\zeta l \leq \bar{l}$, or else $q > 1$ and firms must hold $\zeta l = \bar{l}$. We can express this market clearing
condition as

$$(q - 1) \left( \bar{l} - \zeta l \right) = 0$$

(28)

This equilibrium is described in Proposition 7.

**Proposition 7.** Let $K > 0$ be given and let $\chi_1 = 1 - \rho_0 + pp_L + (1 - p) \rho_H$. Let $M^*$ be defined by

$$\bar{I}C_1 (M^*, K) + C (M^*, K) = A - \frac{M^* + \bar{l}}{\rho_H - \rho_0} \chi_1$$

and let

$$I^* = \frac{M^* + \bar{l}}{\rho_H - \rho_0}$$
then,

(i) If \( p (\rho_H - \rho_L) > 1 \), then \( \lambda_H = 0 \), \( M = \pi = l = 0 \), and \( I = A / [1 - p (\rho_0 - \rho_L)] \).

(ii) If \( p (\rho_H - \rho_L) \leq 1 \) and \( \bar{l} \geq (\rho_H - \rho_0) A / \chi_1 \), then \( \lambda_H = 1 \), \( M = \pi = 0 \), \( I = A / \chi_1 \), \( l = (\rho_H - \rho_0) A / \chi_1 \), and \( q = 1 \).

(iii) If \( p (\rho_H - \rho_L) \leq 1 \), \( \bar{l} < (\rho_H - \rho_0) A / \chi_1 \), and \( I^* \geq p I_0 \), then \( \lambda_H = 1 \), \( l = \bar{l} \), \( M = M^* \), \( \pi = C (M^*, K) \), \( I = I^* \), and \( q = 1 + C_1 (M^*, K) \).

(iv) If \( p (\rho_H - \rho_L) \leq 1 \), \( \bar{l} < (\rho_H - \rho_0) A / \chi_1 \), and \( I^* < p I_0 \), then a fraction \( \zeta \) of firms choose \( \lambda_H = 1 \), \( l = p I_0 \), \( M = (\rho_H - \rho_0) I - \bar{l} / \zeta \), and \( l = \bar{l} / \zeta \), and the remaining fraction \( 1 - \zeta \) of firms choose \( \lambda_H = 0 \), \( M = \pi = l = 0 \), and \( I = A / [1 - p (\rho_0 - \rho_L)] \). The fraction \( \zeta \) satisfies

\[
C (\zeta M, K) + C_1 (\zeta M, K) \bar{l} = \left[ \frac{A - \chi_1 I}{(\rho_H - \rho_0) I} \right] (\zeta M + \bar{l})
\]

which specifies a unique value of \( \zeta M \). Together with \( \zeta = \frac{M + \bar{l}}{(\rho_H - \rho_0) I} \) from the definition of \( M \), this defines separate values for \( M \) and \( \zeta \).

Proof. See appendix A3.

\[
I = \frac{A - \bar{l} C_1 (M, K)}{\chi_1} \quad I = \frac{M + \bar{l}}{\rho_H - \rho_0}
\]

Proposition 7 shows that if there is insufficient outside liquidity \( \bar{l} \), there will be a positive liquidity premium and lower investment in equilibrium compared to the constrained optimum. Thus in the presence of bank agency costs, the addition of banks to the model does not make outside liquidity unnecessary. This point is worth emphasizing because it is the point of departure from Holmstrom and Tirole (1998). Holmstrom and Tirole (1998) do not have bank agency costs, and thus the introduction of banks allows the economy to achieve the constrained optimum. In order to allow for a non-trivial discussion of liquidity and optimal policy in the presence of banks, Holmstrom and Tirole (1998) need to introduce an aggregate liquidity shock. By including bank agency costs, I find that liquidity can be scarce in the presence of banks even when there is no aggregate liquidity shock.

Moreover, I find that the liquidity premium will be equal to marginal agency costs in equilibrium. This is intuitive. We can think of banks as a sector of the economy that produces liquid
assets, or equivalently that manufactures commitment. Bank agency costs are the cost function of this sector, and therefore the equilibrium price of liquidity will be equal to the marginal cost of production. Thus the precise nature of agency costs in the banking sector determine the liquidity premium, which is a component of the overall cost of the asset, along with its riskiness and return. Thus anything that affects agency costs in the banking sector will affect the capacity of the economy to commit funds, which will in turn affect investment and output.

Finally, it is worth emphasizing that the liquidity premium depends on the marginal agency costs, whereas investment depends on both total and marginal agency costs. This interplay between total and marginal agency costs in equilibrium will be essential to determining the effects of changes in bank assets and outside liquidity on equilibrium quantities.

### 2.3 Comparative Statics

I now explore the characteristics of the equilibrium defined in Proposition 7. Given our discussion of the role of liquidity, we are interested in how changes in the supply of liquidity affect equilibrium values, notably investment \( I \) and the liquidity premium \( q - 1 \). I explore these questions by deriving and discussing the comparative statics of the equilibrium with respect to changes in outside liquidity \( \tilde{l} \) and bank assets \( K \).

In the following discussion, we will restrict attention to the case in which \( \tilde{l} < \frac{(\rho_H - \rho_0)A}{x_1} \) so that liquidity is scarce in equilibrium, in which \( I^* > \rho I_0 \) so that all firms meet both liquidity shocks, and in which \( \tilde{l} > 0 \). I will refer to such a point as an interior equilibrium. Thus equilibrium investment is \( I^* \) as defined in Proposition 7.

There is no closed-form representation of \( I^* \) for general agency costs \( C(\cdot) \). However, we can use the implicit function theorem to derive expressions for the marginal change in equilibrium \( I^* \) from a change in outside liquidity \( \tilde{l} \) or bank assets \( K \). These are given in the Proposition 8.

**Proposition 8.** At an interior equilibrium, \( I \) is \( I^* \) and \( M \) is \( M^* \) as defined in Proposition 7, and a marginal change in \( \tilde{l} \) or \( K \) result in a marginal change in equilibrium \( I^* \) according to

\[
\frac{dI^*}{d\tilde{l}} = \frac{\tilde{l}C_{11}(\cdot)}{(\rho_H - \rho_0) \left[ \tilde{l}C_{11}(\cdot) + C_1(\cdot) \right] + \chi_1}
\]

\[
\frac{dI^*}{dK} = -\frac{\tilde{l}C_{12}(\cdot) + C_2(\cdot)}{(\rho_H - \rho_0) \left[ \tilde{l}C_{11}(\cdot) + C_1(\cdot) \right] + \chi_1}
\]

**Proof.** See appendix A4.

**Corollary 9.** At an interior equilibrium,

(i) An increase in \( \tilde{l} \) will result in an increase in \( I^* \).

(ii) An increase in \( K \) will result in an increase in \( I^* \) if and only if \( C_{12}(\cdot) < -C_2(\cdot)/\tilde{l} \).

(iii) A sufficient condition for an increase in \( K \) to result in an increase in \( I^* \) is \( C_{12}(\cdot) < 0 \).
These results are quite intuitive. We would expect that, if the economy is initially at an interior solution and liquidity constrained, a marginal increase in liquidity would result in an increase in investment. This is because we can think of liquidity as an input into the production of investment, and a decrease in the cost of an input will tend to increase optimal production. This is always the case for outside liquidity, and is also the case for bank assets with a small refinement. This refinement simply guarantees that an increase in bank assets will lower not only the cost of liquidity, but also the marginal cost of liquidity, thus ruling out a few rather strange cases.

I depict these shifts using a simple diagram. By Proposition 7, equilibrium \( I^\star \) satisfies

\[
\chi_1 I^\star = A - C_1 (\cdot) \bar{I} - C (\cdot)
\]

This expression gives investment as a function of pledgeable assets \( A \) minus the additional pledgeable funds needed to secure sufficient liquidity to sustain the given level of investment, which I will refer to as the cost of liquidity. We can depict equilibrium \( I^\star \) as the intersection of two curves representing the two sides of this expression, as shown in Figure 4. The upward sloping curve depicts the total internal assets the firm must have in period 0 to obtain funds for investment, which is \( \chi_1 I^\star \); and the downward sloping curve corresponds to the pledgeable assets of the firm that are available to finance initial investment, which are total assets \( A \) minus the cost of liquidity \( C_1 (\cdot) \bar{I} + C (\cdot) \). We can interpret changes in \( \bar{I} \) or \( K \) as shifts in the latter curve.

A marginal increase in \( \bar{I} \) will result in a marginal decrease in liquidity costs at every level of \( I \).
We can see this by decomposing the shift into two parts. First, an increase in $\bar{I}$ will increase the quantity of outside liquidity that the firm is holding, resulting in a marginal increase in liquidity costs $q - 1 = C_1 (\cdot)$. However, an increase in $\bar{I}$ holding $I$ fixed implies a decrease the necessary credit maximum $M$ by an equal amount. A marginal decrease in $M$ will decrease the cost of liquidity $C_1 (\cdot)\bar{I} + C (\cdot)$ by the marginal amount $C_{11} (\cdot)\bar{I} + C_1 (\cdot)$. Since this is larger than the increase in the cost of liquidity due to holding more $\bar{I}$, the cost of liquidity will fall. This results in an upward shift in the asset schedule at every $I$, resulting in higher equilibrium $I^\star$. This is depicted in Figure 5.

A marginal increase in $K$ will result in a marginal change in the cost of liquidity $C_1 (\cdot)\bar{I} + C (\cdot)$ equal to $C_{12} (\cdot)\bar{I} + C_2 (\cdot)$. Since by assumption $C_2 (\cdot) < 0$, an increase in bank assets $K$ will decrease total agency costs at every $I$. However, we did not make any assumptions on $C_{12} (\cdot)$, and so it is possible that this increase in $K$ results in an increase in $C_1 (\cdot)$, and therefore an increase in $q - 1$, that is sufficient to increase the cost of liquidity. The additional assumption $C_{12} (\cdot) < 0$ is sufficient to rule out this possibility, and is in fact stronger than necessary. Then an increase in $K$ will result in a decrease in liquidity costs $C_1 (\cdot)\bar{I} + C (\cdot)$, causing an upward shift in the asset schedule resulting in higher equilibrium $I^\star$, as again also depicted in Figure 5.

**The liquidity premium.** I now investigate the effect of a change in outside liquidity $\bar{I}$ or bank assets $K$ on the equilibrium liquidity premium $q - 1$. These are given in Proposition 10.

**Proposition 10.** At an interior equilibrium, a marginal change in $\bar{I}$ or $K$ result in a marginal change in
equilibrium \( q - 1 \) according to

\[
\frac{dq}{d\bar{l}} = -\frac{C_{11}(\cdot) [\rho_H - \rho_0] C_1(\cdot) + \chi_1}{(\rho_H - \rho_0) [\bar{I} C_{11}(\cdot) + C_1(\cdot)] + \chi_1}
\]

\[
\frac{dq}{dK} = \frac{(\rho_H - \rho_0) [C_{12}(\cdot) C_1(\cdot) - C_{11}(\cdot) C_2(\cdot)] + \chi_1 C_{12}(\cdot)}{(\rho_H - \rho_0) [\bar{I} C_{11}(\cdot) + C_1(\cdot)] + \chi_1}
\]

**Proof.** See appendix A4.  

**Corollary 11.** At an interior equilibrium,

(i) An increase in \( \bar{l} \) results in a decrease in equilibrium \( q \).

(ii) An increase in \( K \) results in a decrease in equilibrium \( q \) if and only if

\[
(\rho_H - \rho_0) [C_{11}(\cdot) C_2(\cdot) - C_1(\cdot) C_{12}(\cdot)] > \chi_1 C_{12}(\cdot)
\]

**Proof.** See appendix A4.  

The first result is quite intuitive. We would expect that an increase in outside liquidity would decrease the cost of outside liquidity. We can illustrate this using the diagram for equilibrium in the market for outside liquidity. A decrease in \( \bar{l} \) will result in a leftward shift in the supply of outside liquidity, and since demand for outside liquidity is downward sloping, this shift will increase equilibrium \( q \). This is illustrated in Figure 6.

The second result implies that for some parameters an increase in \( K \) results in an increase in \( q \). This result may seem surprising, but it has an intuitive explanation. We can think of \( M \) and \( l \) as two inputs into production. A decrease in the cost of one input will tend to decrease demand for the other input due to the substitution effect. However, a decrease in the cost of one input will make production cheaper, and therefore increase demand for investment. Since we are initially at an interior solution where both inputs are used, an increase in investment demand will increase demand for both inputs. The net change in the liquidity premium \( q \) will depend on which of these effects dominates.\(^{15}\)

These two effects are illustrated in Figure 7. The upward sloping schedule is the arbitrage condition

\[
q - 1 = C_1(M, K)
\]

and the point \( M^\star \) is the equilibrium credit line, which is equal to \( (\rho_H - \rho_0) I^\star - \bar{l} \). An increase in \( K \) will everywhere lower the cost of bank financing, which is a downward shift in the arbitrage

\(^{15}\)This is analogous to the ambiguity in a consumption problem when the price of one good falls. Consumption of the other goods will tend to fall due to the substitution effect, but will tend to rise due to the income effect. The net result will depend on which effect dominates.
condition. This will tend to decrease equilibrium \( q \) holding \( M^* \) fixed. However, the decrease in agency costs will increase investment demand \( I^* \), which results in a rightward shift in the equilibrium credit maximum \( M^* \), which will tend to raise equilibrium \( q \).\(^{16}\) These two shifts are depicted in Figure 7.

Condition (29) specifies when the substitution effect dominates. Since we assumed \( C_1 > 0 \), \( C_{11} > 0 \), and \( C_2 < 0 \), \( q \) will never be decreasing in \( K \) if \( C_{12} > 0 \). This is intuitive, because when \( C_{12} > 0 \), marginal agency costs are increasing in \( K \), and so an increase in \( K \) will shift the arbitrage condition upwards instead of downwards, and the substitution effect reverses direction. Suppose that \( C_{12} < 0 \). Then the terms \( C_{11} C_2 \) and \( C_1 C_{12} \) are both negative, and we can rewrite condition (29) as

\[
\left| \frac{C_{12}}{C_2} \right| > \frac{(\rho_H - \rho_0) C_{11}}{\chi_1 + (\rho_H - \rho_0) C_1}.
\]

Therefore the substitution effect will dominate when \( C_{12} \) is large in absolute value relative to \( C_2 \). Intuitively, \( C_{12} \) gives the amount by which marginal agency costs fall due to an increase in bank assets, which determines the size of the shift in the arbitrage condition in Figure 7, which corresponds to the substitution effect. \( C_2 \) will determine the total increase in pledgeable income due to an increase in bank assets, which is a key determinant of investment demand and so determines the size of the shift in the equilibrium credit line \( M^* \) in Figure 7.

Since by the arbitrage condition \( C_{12} = \partial q / \partial K \), this condition states that the substitution effect dominates as long as the effect of an increase in \( K \) on the liquidity premium \( q \) is large relative to

\(^{16}\)This assumes that \( C_{12} \left( \cdot \right) < -C_2 \left( \cdot \right) / \ell \), so that \( \frac{dI}{dK} > 0 \), as given in Corollary 9.
the effect on total agency costs $C$.

3 Government Policy

The previous sections took the supply of liquid assets in the economy as given. However, many government policies directly affect the supply of liquid assets. When central banks engage in conventional monetary policy, they do so by buying and selling assets that differ in their liquidity. Likewise, when governments issue new bonds they are increasing the economy’s store of liquid assets. Finally, during times of crisis governments often provide liquidity to firms and banks directly.

We can capture these activities in our model by allowing the government to issue bonds. Suppose that the government issues $x$ bonds with a face value of 1 in period 0. The funds from the sale of these bonds are returned to households via a lump sum transfer and consumed immediately. In period 2, the government levies a tax on the households in order to raise funds that it uses to repay the bonds.

Since government bonds are perfect substitutes for outside liquid assets $\bar{I}$, government bonds will sell at the same equilibrium price $q$. From the perspective of firms and households, it is as though the stock of outside liquidity had increased from $\bar{I}$ to $\bar{I} + x$, and so equilibrium will be exactly the same as given in Proposition 7, except that $\bar{I}$ is replaced by $\bar{I} + x$. Likewise the comparative statics outlined in Proposition 8 still hold with $\bar{I}$ replaced by $\bar{I} + x$, and $dI/dx = dI/d\bar{I}$.

By assumption, the government has perfect credibility, and therefore is able to commit to re-
paying its debt. Since the government can raise taxes, it can promise a sufficiently large quantity of pledgeable funds in period 1 to cover any potential liquidity shock. By Proposition 7, there is a level of outside liquidity that achieves the constrained optimum, and so the government can issue bonds in order to achieve the constrained optimum level of investment. If there were no costs to taxation, then this would be the optimal policy. However, I assume that the government can only raise funds using a distortionary tax, although it can disburse funds to households in a lump-sum transfer. Suppose that the deadweight loss from raising $x$ funds is given by the function $D(x)$, which I assume is increasing and convex in $x$, and satisfies $D'(0) = 0$ and $D'(x) \to \infty$ as $x \to \infty$.

**General agency costs.** The total social surplus if both shocks are met will be

$$RI - D(x)$$

where $R = \rho_1 - p\rho_L - (1 - p)\rho_H - 1$, since all other aspects of the allocation represent transfers.\(^{18}\)

Then at an interior solution the necessary condition for optimality is

$$R \frac{dI}{dx} = D'(x)$$

which will imply a unique optimum if $\frac{d^2I}{dx^2} < 0$ for all $x$. However, given the expression for $\frac{dI}{dl}$ derived previously in Proposition 8, the sign of $\frac{d^2I}{dx^2}$ will depend on $C_{111}(\cdot)$, about which we have made no assumptions. Therefore we cannot say whether $\frac{dI}{dx}$ is increasing or decreasing in $x$, and we cannot give an expression that uniquely defines optimal policy. We can, however, say a few things about optimal policy in general.

**Proposition 12.** Suppose that in the absence of government policy the economy is initially at an interior equilibrium. Then,

(i) The optimal supply of government liquid assets $x^\star$ will be positive.

(ii) If optimal $x^\star$ implies that the economy is still at an interior equilibrium, and if the current choice of $x^\star$ is unique, then a marginal change in $\tilde{I}$ or $K_0$ will shift the optimal point according to

$$\frac{dx^\star}{d\tilde{I}} = -\frac{R \frac{\partial^2 I}{\partial x^2}}{R \frac{\partial^2 I}{\partial x^2} - D''(x)}$$

$$\frac{dx^\star}{dK_0} = -\frac{R \frac{\partial^2 I}{\partial x^2}}{R \frac{\partial^2 I}{\partial x^2} - D''(x)}$$

\(^{17}\)I assume here that the government must raise funds by taxing activity in some unmodeled market.

\(^{18}\)This is with the no-production case normalized to 0. Additional liquidity has no value unless both shocks are met, so we will restrict attention to this case.
(iii) At an interior optimal point $x^\star$, we have $R \frac{d^2 l}{dx^2} < D''(x)$.

(iv) We have $\frac{dx}{d\ell} > 0 \iff \frac{d^2 l}{dx^2} > 0$ and $\frac{dx}{dK_0} > 0 \iff \frac{d^2 l}{dxK_0} > 0$.

Proof. See appendix A5. □

The first result in Proposition 12 is that if the economy is at an interior equilibrium in the absence of any policy, it will always be optimal to choose $x^\star > 0$. Thus when liquidity is scarce it will in general be optimal for the government to issue bonds solely for their value in providing liquidity, even though taxation is distortionary. The intuition for this result is that at an interior solution with a positive liquidity premium there is a positive marginal value of issuing bonds because they serve as liquid assets. Since the marginal cost of taxation is zero at $x = 0$, the marginal value of issuing bonds at $x = 0$ is greater than the marginal cost, and therefore it will be always be optimal to issue some positive quantity of bonds.

This intuition is depicted in Figure 8. The line that passes through the origin is the marginal cost of raising $x$ in funds via taxation, and the other line is the marginal value of increasing the supply of liquid assets $R \frac{dI}{dx}$. Moreover, there is a point $\bar{x} = \frac{\Lambda (\rho_1 - \rho_0)}{\lambda_1} - \bar{I}$, such that if the government chooses $x \geq \bar{x}$ the economy achieves the constrained optimum, and the marginal value of increasing $x$ above $\bar{x}$ drops to zero. Therefore there will be some point at which the two curves intersect, and this will be at some $x > 0$.

Results (ii) - (iv) in Proposition 12 have a similarly straightforward intuition. Suppose that there is an interior unique solution at the point $x^\star$. Then at $x^\star$ the marginal cost and marginal
benefit curves intersect, as depicted in Figure 8. Since \( x^* \) is a local maximum, the marginal benefit curve crosses the marginal cost curve from above, as claimed in statement (iii). A change in \( I \) or \( K \) will not shift the marginal cost curve. Since the equation for the marginal benefit curve is \( R \frac{dI}{dx} \), an increase in \( I \) will shift the marginal benefit curve upwards if \( \frac{d^2 I}{dx^2} > 0 \), which will increase the point of intersection \( x^* \). Similarly if \( \frac{d^2 I}{dx^2} K > 0 \), an increase in \( K \) will shift the marginal benefit curve upwards, which will again increase the point of intersection \( x^* \), and the reverse. Such a shift is depicted in Figure 9.

Quadratic agency costs. In order to say more about optimal policy, we need to impose further restrictions on agency costs and deadweight loss. In particular, suppose that the deadweight loss from taxation is a quadratic function of revenue raised:\(^{19}\)

\[
D(x) = \frac{\sigma}{2} x^2
\]

and that bank agency costs are quadratic in \( M \) and linearly decreasing in \( K \)

\[
C(M, K) = \frac{\psi M^2}{2K}
\]

Then optimal policy is summarized by Proposition 13.

---

\(^{19}\)If supply and demand curves are linear in the market that the government is taxing, then deadweight loss will be quadratic in \( x \).
Proposition 13. Suppose that in the absence of government policy the economy is initially at an interior equilibrium. Then,

(i) If

\[
\bar{I} \geq \left[ 1 - \frac{\psi \chi_1 R}{\sigma (\rho_H - \rho_0)^2 A \psi + \sigma \chi_1 K} \right] \left( \rho_H - \rho_0 \right) \frac{A}{\chi_1}
\]

(30)

the optimal policy is

\[ x^* = \frac{A(\rho_H - \rho_0)}{\chi_1} - \bar{I}. \]

(ii) If (30) does not hold, optimal policy \( x^* > 0 \) is uniquely defined by

\[
\frac{R}{\sigma} \phi \left( \frac{\bar{I} + x}{x} \right) = \psi (\rho_H - \rho_0)^2 I + \chi_1 K
\]

(iii) If (30) does not hold, then a marginal change in \( \bar{I} \) or \( K \) will shift the optimal supply of public debt according to

\[
\frac{dx^*}{d\bar{I}} = \frac{(R \phi)^2}{x} - \psi^2 (\rho_H - \rho_0)^2 \frac{\bar{I}}{x (R \phi)^2 + \psi^2 (\rho_H - \rho_0)^2}
\]

\[
\frac{dx^*}{dK} = -\frac{\psi^2 (\rho_H - \rho_0)^2 \left[ (\rho_H - \rho_0)^2 I^2 - \left( \bar{I} + x \right)^2 \right] \left( \frac{x}{x + \bar{I}} \right) (1 R \phi) + \chi_1 \frac{R \phi}{\sigma}}{\bar{I} \left( \frac{R \phi}{\sigma} \right)^2 + \psi^2 (\rho_H - \rho_0)^2}
\]

which have signs \( \frac{dx}{d\bar{I}} > 0 \) and \( \frac{dx}{dK} < 0 \).

Proof. See appendix A5.

Consistent with Proposition 12, Proposition 13 shows that if the economy is at an interior equilibrium in the absence of any policy, it will always be optimal to choose \( x^* > 0 \). However, now we have an expression that defines a unique value of \( x^* \). Moreover, we have a condition for when it will be optimal to provide \( x^* = \frac{A(\rho_H - \rho_0)}{\chi_1} - \bar{I} \), which is the level of assets that eliminate the liquidity premium so that the economy achieves the constrained optimum production plan.20

One implication of Proposition 13 is that for some parameters it will be optimal to provide public liquidity, but not to entirely eliminate the liquidity premium. Therefore for some parameters we will have an interior solution, and we can explore the effects of shifts change in \( \bar{I} \) and \( K \) on this optimal quantity \( x^* \).

20Although not the constrained optimum level of welfare, because of the deadweight loss of positive taxation.
Proposition 13 further establishes the signs of \(dx^\ast/dl\) and \(dx^\ast/dK\). \(dx^\ast/dl > 0\) implies that at an interior solution a fall in outside liquidity \(\bar{l}\) will decrease the optimal supply of public liquidity \(x^\ast\). This is somewhat counterintuitive, since we might think that private and public liquidity are substitutes. However, the marginal value of additional public liquidity is increasing in the quantity of private liquidity \(l\). As we showed in Proposition 10, an increase in \(\bar{l}\) will result in a decrease in \(q\). This \(q\) will apply not only to the marginal unit of outside liquidity, but also to the inframarginal units. Therefore an increase in outside public liquidity will have a larger benefit if there is already a large quantity of private outside liquidity in the economy, and will have a relatively small benefit if there is low private outside liquidity. Thus a fall in private outside liquidity \(\bar{l}\) will decrease the marginal value of public liquidity, since the marginal unit of \(x\) will no longer be lowering the price of as many inframarginal assets \(\bar{l}\). This will decrease the optimal supply of outside liquidity.

By contrast, we have \(dx/dK < 0\), which is the expected sign. This implies that the optimal policy response to a decrease in bank assets \(K\) will be to increase the supply of public liquidity in order to make up for the shortfall. Intuitively, a fall in bank assets will raise marginal agency costs, which will tend to increase the liquidity premium \(q\). A higher liquidity premium implies a higher return to issuing liquid assets to decrease this liquidity premium, and therefore optimal supply of public liquidity will increase.

It is worth emphasizing that these result do not assume any sort of aggregate liquidity shock as in Holmstrom and Tirole (1998). Here the provision of public liquidity acts as a substitute to bank financing, and therefore decreases the cost of liquidity in the economy, which is positive because of bank agency costs. Thus while Holmstrom and Tirole (1998) discusses countercyclical policy, the cycle in question is variation in an aggregate liquidity shock. By contrast, my discussion applies to variations in bank assets and the stock of liquid assets in the economy. My model exhibits cyclical variation in the economy’s capacity to meet liquidity shocks, rather than cyclical variation in liquidity shocks themselves.

The benefit of my approach is that variations in bank net worth can be observed and are procyclical in nature. Thus my model provides both a mechanism for the propagation of shocks and a justification for countercyclical public liquidity provision.

4 Conclusion

I analyze the supply and demand for liquidity in the economy using a simple model of investment and liquidity provision. I find that if there is insufficient liquidity, equilibrium investment will be below the constrained optimum. This effect can be reduced by introducing intermediaries that pool liquidity and insure against shocks. When banks are subject to agency costs that are convex in funds intermediated and decreasing in bank assets, firms will face a tradeoff between financing their liquidity shocks by holding scarce liquidity, and financing by holding credit lines issued by banks. At an interior equilibrium, the liquidity premium will equal the marginal agency costs of
banks, liquid assets will sell at a premium over their expected return because of their scarcity, and investment will still be lower than the constrained optimum.

The government can provide liquidity by issuing public liabilities such as bonds. If public liabilities were costless to issue, the government could achieve the constrained optimum. However, if funds can only be raised through distortionary taxation, the government may choose an optimal supply of liquidity that does not fully eliminate the liquidity premium. Nevertheless, if the economy is initially at an interior equilibrium with scarce liquidity, the government will find it optimal to issue a strictly positive quantity of bonds.

I consider two comparative static exercises: a fall in bank assets, and a fall in outside liquidity. We can think of these as two channels by which shocks propagate through the financial system via the supply of liquidity. A fall in bank assets could represent losses to the economy’s productive factors, or simply accounting losses due to a credit bust. A fall in outside liquidity could represent asset destruction (such as a credit freeze) or a sharp drop in liquid assets due to capital flight or a decision by the government to decrease its supply of liquidity.

I find that for a plausible parameterization, a fall in bank assets will decrease investment. The liquidity premium may rise or fall, depending on whether the fall in liquidity demand (due to lower investment) or the fall in liquidity supply (due to less liquidity) dominates. The optimal policy response is to increase provision of public liquidity in order to partially offset the fall in investment.

I find that a fall in outside liquidity will decrease investment and raise the liquidity premium. The optimal policy response is to decrease provision of public liquidity, since the marginal value to the economy of public liquidity is increasing in outside liquidity.
References


Caballero, R.J. 2010. “The "Other" Imbalance and the Financial Crisis.”


A1: Unconstrained Optimum

**Proposition.** The unconstrained optimal production plan is

\[
I = \begin{cases} 
A + \overline{H}_0 & R \geq 0 \\
0 & R < 0 
\end{cases}
\]

\[
\lambda_L = 1 \\
\lambda_H = 1
\]

(31)

where \( R = p (\rho_1 - \rho_L) + (1 - p) (\rho_1 - \rho_H) - 1 \).

**Proof.** We can solve (4) using the Lagrangian

\[
L = \sum_{i=0}^{2} C_i + \mu_0 \left[ A + \overline{H}_0 - C_0 - I \right] + \mu_1 \left[ \overline{H}_1 - C_1 - p \lambda_L \rho_L I - (1 - p) \lambda_H \rho_H I \right] \\
+ \mu_2 \left[ \overline{H}_2 + p \lambda_L \rho_L I + (1 - p) \lambda_H \rho_H I + I - C_2 \right] + \sum_{i=0}^{2} \nu_i C_i + \nu_3 I
\]

The first-order condition with respect to \( C_i \) is

\[
\mu_i = 1 + \nu_i
\]

which implies \( \mu_i > 0 \), and so the period budget constraints hold with equality. Substituting them directly into the objective function, we obtain the new problem

\[
\max_{\lambda_L, \lambda_H, I} \left\{ A + \overline{H}_0 + \overline{H}_1 + \overline{H}_2 + p \lambda_L (\rho_1 - \rho_L) I + (1 - p) \lambda_H (\rho_1 - \rho_H) I - I \right\}
\]

s.t.

\[
\overline{H} \geq I - A
\]

(33)

\[
\overline{H} \geq p \lambda_L \rho_L I + (1 - p) \lambda_H \rho_H I
\]

(34)

\[
I \geq 0
\]
where \( \lambda_s \in \{0, 1\} \) for \( s \in \{L, H\} \). The constraints (33) and (34) are the non-negativity constraints on \( C_0 \) and \( C_1 \) respectively. The Lagrangian of (32) is

\[
L = A + \bar{P}_0 + \bar{P}_1 + \bar{P}_2 + p\lambda_L (\rho_1 - \rho_L) I + (1 - p)\lambda_H (\rho_1 - \rho_H) I - I + \mu_1 [A + \bar{P} - I] + \mu_2 [\bar{P} - p\lambda_L \rho_L I - (1 - p)\lambda_H \rho_H I] + \mu_3 I
\]  

(35)

The first-order condition of (35) with respect to \( I \) is

\[ p\lambda_L (\rho_1 - \rho_L) + (1 - p)\lambda_H (\rho_1 - \rho_H) - 1 = \mu_2 [p\lambda_L \rho_L + (1 - p)\lambda_H \rho_H] + \mu_1 - \mu_3 \]

By assumption, there will always be sufficient funds in period 1 to meet liquidity shocks, meaning \( \bar{P}_1 \geq p\lambda_L \rho_L I + (1 - p)\lambda_H \rho_H I \), and so we have \( \mu_2 = 0 \). Thus the first-order condition reduces to

\[ p\lambda_L (\rho_1 - \rho_L) + (1 - p)\lambda_H (\rho_1 - \rho_H) - 1 = \mu_1 - \mu_3 \]

The left-hand side \( p\lambda_L (\rho_1 - \rho_L) + (1 - p)\lambda_H (\rho_1 - \rho_H) - 1 \) is the net return on investment. If the net return is positive, then \( \mu_1 > 0 \), the non-negativity constraint on \( C_0 \) binds, and the economy invests all available funds in period 0. If the net return is negative, then \( \mu_3 > 0 \), the non-negativity constraint on \( I \) binds, and the economy does not invest anything. We can express this investment rule as

\[
I = \begin{cases} 
A + \bar{P}_0 & p\lambda_L (\rho_1 - \rho_L) + (1 - p)\lambda_H (\rho_1 - \rho_H) > 1 \\
[0, A + \bar{P}_0] & p\lambda_L (\rho_1 - \rho_L) + (1 - p)\lambda_H (\rho_1 - \rho_H) = 1 \\
0 & p\lambda_L (\rho_1 - \rho_L) + (1 - p)\lambda_H (\rho_1 - \rho_H) < 1 
\end{cases}
\]

Now we just need to determine the optimal choices of \( \lambda_L \) and \( \lambda_H \). Increasing \( \lambda_L \) and \( \lambda_H \) from 0 to 1 results in increases in the objective function of \( p\lambda_L (\rho_1 - \rho_L) I \) and \( (1 - p)\lambda_H (\rho_1 - \rho_H) I \) respectively. Given our assumption that \( \rho_1 > \rho_H > \rho_L \), both of these terms are positive for \( I > 0 \). Therefore it is optimal to choose \( \lambda_L = 1 \) and \( \lambda_H = 1 \). Since the return from a project is greater than the additional cost of bringing the project to completion after the liquidity shock is realized, the optimal continuation policy is to continue in all cases.

We can now define the unconstrained optimum. The solution is the set \( \{C_0, C_1, C_2, I, \lambda\} \) with

\[
\begin{align*}
C_0 &= A + \bar{P}_0 - I \\
C_1 &= \bar{P}_1 - p\rho_L I - (1 - p)\rho_H I \\
C_2 &= \bar{P}_2 + p\rho_L I + (1 - p)\rho_H I \\
\lambda_L &= 1 \\
\lambda_H &= 1 \\
I &= \begin{cases} 
A + \bar{P}_0 & R \geq 0 \\
0 & R < 0 
\end{cases}
\]
\]

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where
\[ R = p (\rho_1 - \rho_L) + (1 - p) (\rho_1 - \rho_H) - 1 \]

A2: Equilibrium with No Banks

**Proposition.** At the solution we have \( \lambda_L = 1 \), and constraints (11) - (14) hold with equality.

**Proof.** We need to show that it is always worthwhile to increase \( R_1 \) in order to increase \( I \) as long as the constraint on pledgeable income in the \( s \) state does not bind. If this is true, then it follows that in any equilibrium with positive external financing, limited pledgeability binds, meaning that firms receive exactly the amount necessary for them to cooperate, and no more.

The first step is to argue that the period 1 investors will be paid exactly the funds necessary to finance meeting the liquidity shocks, and no more. This should not be controversial, since there is no other benefit to increasing payments to period 1 investors. To show this, we take the Lagrangian of the problem
\[
L = p \left( \lambda_L \rho_1 I - R_1^L - R_1^I + I \right) + (1 - p) \left( \lambda_H \rho_1 I - R_1^H - R_1^I + I \right) \\
+ \mu_1 \left[ R_1^L - \lambda_L \rho_1 I \right] + \mu_2 \left[ R_1^H - \lambda_H \rho_1 I \right] \\
+ \mu_3 \left[ pR_1^L + (1 - p)R_1^H - I - ql + A \right] \\
+ \mu_4 \left[ \lambda_H \rho_0 I + I - R_1^H - R_1^I \right] + \mu_5 \left[ \lambda_L \rho_0 I + I - R_1^L - R_1^I \right]
\]
and differentiate with respect to \( R_1^L \) and \( R_1^H \). This yields conditions
\[
\begin{align*}
\mu_1 & \leq \mu_4 + p \\
\mu_2 & \leq \mu_5 + 1 - p
\end{align*}
\]
which hold with equality if the corresponding \( R_1^s > 0 \). We also have \( R_1^L \geq \lambda_L \rho_1 I \) and \( R_1^H \geq \lambda_H \rho_1 I \), which indicates that if \( I > 0 \) and \( s > 0 \), we have \( R_1^s > 0 \). So we can conclude that in fact we have \( R_1^s = \lambda_s \rho_s I \), for \( s \in \{L, H\} \). Substituting these terms directly, we can rewrite the Lagrangian as
\[
L = p \left( \lambda_L \rho_1 I - R_1^L - \lambda_L \rho_1 I + I \right) + (1 - p) \left( \lambda_H \rho_1 I - R_1^H - \lambda_H \rho_1 I + I \right) \\
+ \mu_3 \left[ pR_1^L + (1 - p)R_1^H - I - ql + A \right] \\
+ \mu_4 \left[ \lambda_H \rho_0 I + I - \lambda_H \rho_1 I - R_1^H \right] + \mu_5 \left[ \lambda_L \rho_0 I + I - R_1^L - \lambda_L \rho_1 I \right]
\]

Now we derive conditions for optimal \( \lambda_s \). From the Lagrangian, we find that the marginal
values of increasing $\lambda_L$ and $\lambda_H$ are respectively

\[
p(\rho_1 - \rho_L) I + \mu_5 (\rho_0 - \rho_L) I \\
(1 - p) (\rho_1 - \rho_H) I + \mu_5 (\rho_0 - \rho_H) I
\]

The optimal choice of $\lambda_s$ is 1 if the marginal value is positive, and 0 if the marginal value is negative. Since by assumption we have $\rho_1 > \rho_L$ and $\rho_0 > \rho_L$, the first condition is strictly positive, which means that we have $\lambda_L = 1$. The sign of the second is ambiguous since $\rho_0 < \rho_H$, and so the value of $\lambda_H$ is not clear.

The next step is to observe that a pledgeability constraint will bind if (1) that shock is met, and (2) the multiplier on investment is greater than 1, i.e. $\mu_3 > 1$. This is logical because the direct cost of an increase in $R_I$ is 1, so if the value of increasing investment is greater than 1, the optimal solution will be to increase $R_I$ until some other constraint binds. We can see this from the first-order conditions with respect to $R_I$, which are respectively

\[
\mu_5 \geq p (\mu_3 - 1) \\
\mu_4 \geq (1 - p) (\mu_3 - 1)
\]

These show that the pledgeability constraints bind if and only if $\mu_3 > 1$.

Now we need only show that $\mu_3 > 1$ to establish the claim. To show this, we differentiate the Lagrangian with respect to $I$, which yield

\[
\mu_3 \leq [p (\rho_1 - \rho_L) + \mu_5 (\rho_0 - \rho_L)] \lambda_L + [(1 - p) (\rho_1 - \rho_H) + \mu_4 (\rho_0 - \rho_H)] \lambda_H
\]

which holds with equality when $I > 0$. Since we have $\lambda_H = 0$ whenever $(1 - p) (\rho_1 - \rho_H) + \mu_4 (\rho_0 - \rho_H) < 0$, the term $[(1 - p) (\rho_1 - \rho_H) + \mu_4 (\rho_0 - \rho_H)] \lambda_H$ is non-negative. Since $\lambda_L = 1$ and since by assumption $p (\rho_1 - \rho_L) > 1$, the term $[p (\rho_1 - \rho_L) + \mu_5 (\rho_0 - \rho_L)] \lambda_L > 1$. Therefore we have $\mu_3 > 1$ as long as $I > 0$, which implies $\mu_4 > 0$ and $\mu_5 > 0$ and establishes the claim.

**Proposition.** The optimal continuation rule of firms is $\lambda_H = 1$ if and only if

\[
q - 1 \leq \frac{(1 - p) [1 - p (\rho_H - \rho_L)]}{p (\rho_H - \rho_0)}
\]

The optimal choice of $I$ is given by

\[
I = \frac{A}{1 - p (\rho_0 - \rho_L) - \lambda_H (q - p) (\rho_0 - \rho_H)}
\]

and the optimal choice of $I$ is $I = \lambda_H (\rho_H - \rho_0) I$. 

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Proof. We first establish that the constraints

$$I \leq \frac{A - (q - 1) l}{1 - p (\rho_0 - \rho_L) - (1 - p) \lambda_H (\rho_0 - \rho_H)}$$

and $\lambda_H (\rho_H - \rho_0) I \leq l$ hold with equality. Consider the Lagrangian

$$L = p (\rho_1 - \rho_L) I + (1 - p) \lambda_H (\rho_1 - \rho_H) I + \mu_1 [l - \lambda_H (\rho_H - \rho_0) I]$$

$$+ \mu_2 [A - (q - 1) l - [1 - p (\rho_0 - \rho_L) - (1 - p) \lambda_H (\rho_0 - \rho_H)] I]$$

The first-order condition with respect to $l$ yields

$$\mu_1 \leq (q - 1) \mu_2$$

which holds with equality when $l > 0$. Since we have $l \geq \lambda_H (\rho_H - \rho_0) I$, as long as we have $\lambda_H = 1$ and $I > 0$, we will have $l > 0$. We proved in Proposition 2 that (11) binds at the solution, so we have $\mu_2 > 0$, which implies that $\mu_1 > 0$ as long as we have $q - 1 > 0$, $\lambda_H = 1$, and $I > 0$. If we have $q - 1 = 0$, then there is no cost of holding unnecessary $l$, and so the optimal level of $l$ is undetermined. We assume without loss of generality that $l = \lambda_H (\rho_H - \rho_0) I$ holds with equality.

Now we substitute this expression for $l$ into the leverage constraint, and derive an expression for $l$

$$I = \frac{A}{1 - p (\rho_0 - \rho_L) - \lambda_H (q - p) (\rho_0 - \rho_H)}$$

which will hold for $\lambda_H = 1$. This same expression will also hold for $\lambda_H = 0$ and $l = 0$ from the leverage constraint with $l = 0$, so this will hold in either case.

All that is left is to determine when $\lambda_H = 1$ will be optimal. This will be true as long as

$$\frac{(\rho_1 - \rho_0) [p + (1 - p) \lambda_H] I}{1 - p (\rho_0 - \rho_L) - \lambda_H (q - p) (\rho_0 - \rho_H)} \geq \frac{(\rho_1 - \rho_0) [p + (1 - p) \lambda_H] A}{1 - p (\rho_0 - \rho_L) - \lambda_H (q - p) (\rho_0 - \rho_H)}$$

which simplifies to

$$q - 1 \leq \frac{(1 - p) [1 - p (\rho_H - \rho_0)]}{p (\rho_H - \rho_0)}$$

$\square$

Proposition. Suppose $I$ is given. Let $I_0 = A/ [1 - p (\rho_0 - \rho_L)]$ and $\chi_1 = 1 - \rho_0 + p \rho_L + (1 - p) \rho_H$. Then,

(i) If $p (\rho_H - \rho_L) > 1$, equilibrium $q = 1$ and all firms pursue the strategy $\lambda_H = 0$, $I = 0$, and $I = I_0$.

(ii) If $p (\rho_H - \rho_L) \leq 1$ and $I \geq (\rho_H - \rho_0) A/\chi_1$, equilibrium $q = 1$ and firm behavior is given by $I = A/\chi_1$, $I = (\rho_H - \rho_0) A/\chi_1$, and $\lambda_H = 1$.  

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(iii) If \( p (\rho_H - \rho_L) \leq 1 \) and \((\rho_H - \rho_0) T_0 \leq T \leq (\rho_H - \rho_0) A / \chi_1\), equilibrium \( q \) is given by

\[
q - 1 = \frac{A}{T} - \frac{1 - p (\rho_H - \rho_L)}{\rho_H - \rho_0}
\]

and firm behavior is given by \( \lambda_H = 1, l = T, \) and \( I = I / (\rho_H - \rho_0) \).

(iv) If \( p (\rho_H - \rho_L) \leq 1 \) and \( T < (\rho_H - \rho_0) I_0 \), equilibrium \( q \) is given by

\[
q - 1 = \frac{1 - p (\rho_0 - \rho_L) - p [1 - p (\rho_H - \rho_L)]}{p (\rho_H - \rho_0)}
\]

and firms are indifferent between \( \lambda_H = 0 \) and \( \lambda_H = 1 \). A fraction \( \zeta \) of firms choose \( \lambda_H = 1, I = p T_0 \), and \( l = (\rho_H - \rho_0) p T_0 \), where \( \zeta = \frac{T}{[ (\rho_H - \rho_0) p T_0] \}. The remaining fraction \( 1 - \zeta \) pursue the strategy \( \lambda_H = 0, I = l \), and \( I = \bar{T}_0 \).

**Proof.** (i) follows directly from Proposition 3.

For (ii), we simply look at the demand for outside assets \( I \) at \( q = 1 \). If \( I < \bar{T} \), then we know from market clearing that equilibrium \( q = 0 \) because households must hold some of \( \bar{T} \) in equilibrium. Substituting \( q = 1 \) into the expression for \( I \) in Proposition 3, we find that \( I = \frac{A}{1 - \rho_0 + p \rho_L + (1 - p) \rho_H} \). Since firms need to hold at least \( l = (\rho_H - \rho_0) l_0 \) in order to meet both shocks, we find that firm must hold at least \( l = \frac{(\rho_H - \rho_0) A}{1 - \rho_0 + p \rho_L + (1 - p) \rho_H} \), which is feasible if \( \bar{T} \geq \frac{(\rho_H - \rho_0) A}{1 - \rho_0 + p \rho_L + (1 - p) \rho_H} \), thus proving the statement.

For (iii), we want to find the range of \( \bar{T} \) for which all firms meet both shocks and \( q > 0 \). Since \( q > 0 \), households do not hold any liquid assets, and so all \( \bar{T} \) are held by firms. Since all firms meet both shocks and hold liquid assets, we have \( l = l \) by market clearing. Then we use \( I = \frac{A}{1 - p (\rho_0 - \rho_L) - (q - p) (\rho_0 - \rho_H)} \) and \( I = (\rho_H - \rho_0) l \) from Proposition 3 to solve for implied \( q \). This calculation yields

\[
q - 1 = \frac{A}{\bar{T}} - \frac{1 - \rho_0 + p \rho_L + (1 - p) \rho_H}{\rho_H - \rho_0}
\]

This will be the equilibrium as long as firms are willing to meet both shocks at this level of \( q \). By Proposition 3, firms are willing meet any shocks as long as

\[
q - 1 \leq \frac{(1 - p) [1 - p (\rho_H - \rho_L)]}{p (\rho_H - \rho_0)}
\]

Combined with the condition above, this yields the necessary level of \( \bar{T} \) for this to be an equilibrium

\[
\bar{T} \geq \frac{p (\rho_H - \rho_0) A}{1 - p (\rho_0 - \rho_L)}
\]
If \( \bar{l} \) is above this threshold, then \( l = \bar{l}, I = \frac{\bar{l}}{\rho_H - \rho_0}, \) and \( \lambda_1 = 1. \)

For (iv), if \( \bar{l} \) is below this threshold then there is insufficient liquidity for all firms to meet all shocks. However, no firms meeting both shocks would not be an equilibrium, because in this case the available outside liquidity \( \bar{l} \) would need to be held by households or firms that do not need it, which they would only do if we had \( q = 0. \) But if we have \( q = 0, \) then firms would prefer to buy outside liquidity at this price and meet both shocks. Therefore in equilibrium we must have a fraction of firms meeting both shocks, while the rest only meet the low shock. In order for this to be an equilibrium the firms must be indifferent between these two strategies. From Proposition 3, this will be true if

\[
q - 1 = \frac{(1 - p) [1 - p (\rho_H - \rho_L)]}{p (\rho_H - \rho_0)}
\]

Substituting this \( q \) into the expression for \( I \) from Proposition 3, we obtain

\[
I = \frac{pA}{1 - p (\rho_0 - \rho_L)}
\]

The corresponding amount of outside liquidity held by each firm must satisfy \( l = (\rho_H - \rho_0) I, \) so we have

\[
l = \frac{pA (\rho_H - \rho_0)}{1 - p (\rho_0 - \rho_L)}
\]

Let the fraction of firms that meet both shocks be \( \zeta. \) Since \( q > 0, \) by market clearing the firms that meet both shocks must hold total liquidity \( \bar{l}. \) Therefore we have \( \zeta \bar{l} = \bar{l}, \) and so \( \zeta = \frac{\bar{l}}{I}, \) or

\[
\zeta = \frac{[1 - p (\rho_0 - \rho_L)] \bar{l}}{pA (\rho_H - \rho_0)}
\]

The firms that meet only the low shock will follow the same strategy as when \( p (\rho_H - \rho_L) > 1, \) earning the same profits as the firms that meet both shocks. This proves the final statement. \( \square \)

**A3: Equilibrium with Banks**

**Proposition.** At the solution \( \lambda_L = 1, \) and constraints (18) - (23) hold with equality.

**Proof.** The first step is to show that statements (20) and (21) bind, meaning that period-1 investors will be paid exactly enough funds to meet the liquidity shock. Write the Lagrangian for the opti-
mal contracting problem as

\[
L = p \left( \lambda_L \rho_I - R^1_L - R^B_L + I \right) + (1-p) \left( \lambda_H \rho_I - R^1_H - R^B_H + I \right)
\]

\[
+ \mu_1 \left[ R^1_L - \lambda_L \rho_L I \right] + \mu_2 \left[ R^1_H + M - \lambda_H \rho_H I \right]
\]

\[
+ \mu_3 \left[ pR^1_L + (1-p)R^1_H - I - qI + A \right]
\]

\[
+ \mu_4 \left[ \lambda_L \rho_0 I + I - R^1_L - R^B_L \right] + \mu_5 \left[ \lambda_H \rho_0 I + I - R^1_H - R^B_H \right]
\]

\[
+ \mu_6 \left[ \pi - \pi^\prime (M) \right]
\]

where \(\pi = pR^B_L + (1-p) R^B_H - (1-p) M\). Differentiating with respect to \(R^1_L\) and \(R^1_H\), we derive

\[
\mu_1 \leq p + \mu_4
\]

\[
\mu_2 \leq (1-p) + \mu_5
\]

which hold with equality if \(R^1_L > 0\) or \(R^1_H > 0\) respectively. From (20) we have \(R^1_L \geq \lambda_L \rho_L I\), and so either \(R^1_L > 0\), which implies \(\mu_1 = p + \mu_4 > 0\), or \(R^1_L = 0\) in which case \(\lambda_L \rho_L I = 0\) (since \(\lambda_L \rho_L I \geq 0\)). In either case, the constraint (20) holds with equality. From (21) we have \(R^1_H + M \geq \lambda_H \rho_H I\). However it is possible that we have \(R^1_H = 0\) and \(M > 0\), and that (21) does not hold with equality. We need an additional argument to establish that this is not the case.

To do so, we differentiate the Lagrangian with respect to \(M\). This yields

\[
\mu_2 \leq \mu_6 \left[ 1 - p + \pi^\prime (M) \right]
\]

which holds with equality if \(M > 0\). We know that \(\pi^\prime (M) \geq 0\). From this we can conclude that either (1) \(M = 0\), or (2) \(\mu_2 > 0\) and \(\mu_6 > 0\), or (3) \(\mu_2 = 0\) and \(\mu_6 = 0\).

Now I argue that (24) holds with equality. Differentiating the Lagrangian with respect to \(R^B_L\) and \(R^B_H\), we obtain

\[
\mu_6 \leq 1 + \frac{\mu_4}{p}
\]

\[
\mu_6 \leq 1 + \frac{\mu_5}{1-p}
\]

which hold with equality if \(R^B_L > 0\) or \(R^B_H > 0\) respectively. From this we conclude that either \(\mu_6 > 0\) or \(R^B_L = R^B_H = 0\). If the latter, then from the constraint that \(\pi \geq 0\) we have \(M = 0\), and so (24) holds with equality. So we conclude that either \(M = 0\) or \(\mu_6\) is strictly positive.

If \(M = 0\), then (21) becomes \(R^1_H \geq \lambda_H \rho_I H\), and so (21) holds with equality. If \(\mu_6 > 0\), then \(\mu_2 > 0\) and (21) holds with equality. So we can conclude that in either case (21) holds with equality.

We now have expressions for \(R^1_L\) and \(R^1_H\). Substituting these directly into the Lagrangian, we
Therefore we have

\[ L = p \left( \lambda_L \rho_L I - R_L^I - \lambda_L \rho_L I - R_L^\beta + I \right) + (1 - p) \left( \lambda_H \rho_H I - R_H^I - \lambda_H \rho_H I + M - R_H^\beta + I \right) \\
+ \mu_3 \left[ p R_L^I + (1 - p) R_H^I - I - q I + A \right] \\
+ \mu_4 \left[ \lambda_L \rho_0 I + l - R_L^I - \lambda_L \rho_L I - R_L^\beta \right] + \mu_5 \left[ \lambda_H \rho_0 I + l - R_H^I - \lambda_H \rho_H I + M - R_H^\beta \right] \\
+ \mu_6 \left[ \pi - \pi(M) \right] \]

We would now like to argue that \( \lambda_L = 1 \). Since \( \lambda_L, \lambda_H \in \{0, 1\} \), we can differentiate the Lagrangian with respect to each \( \lambda_s \), and conclude that if this derivative is non-negative then \( \lambda_s = 1 \). The derivatives of the Lagrangian with respect to \( \lambda_L \) and \( \lambda_H \) are respectively

\[ \left[ p (\rho_1 - \rho_L) + \mu_4 (\rho_0 - \rho_L) \right] I \]
\[ \left[ p (\rho_1 - \rho_H) + \mu_5 (\rho_0 - \rho_H) \right] I \]

Since by assumption \( \rho_1 > \rho_L \) and \( \rho_0 > \rho_L \), the first expression is positive for \( I > 0 \), and so we have \( \lambda_L = 1 \). The second expression may not be positive because \( \rho_0 < \rho_H \), so we conclude that \( \lambda_H = 1 \) if and only if \( [p (\rho_1 - \rho_H) + \mu_5 (\rho_0 - \rho_H)] I \geq 0 \).

Differentiating the Lagrangian with respect to \( R_L^I \) and \( R_H^I \) yields

\[ \mu_4 \geq p (\mu_3 - 1) \]
\[ \mu_5 \geq (1 - p) (\mu_3 - 1) \]

which hold with equality when \( R_L^I > 0 \) or \( R_H^I > 0 \) respectively. Therefore (22) and (23) hold with equality as long as \( \mu_3 > 1 \).

Now to finish proving the proposition, it is enough to establish that \( \mu_3 > 1 \), since this would prove that (22), (23) and (19) hold with equality. Differentiating the Lagrangian with respect to \( I \), we obtain

\[ \mu_3 \geq p (\rho_1 - \rho_L) + \mu_4 (\rho_0 - \rho_L) + \lambda_H [(1 - p) (\rho_1 - \rho_H) + \mu_5 (\rho_0 - \rho_H)] \]

By assumption, we have \( p (\rho_1 - \rho_L) > 1 \), so we have

\[ \mu_3 > 1 + \mu_4 (\rho_0 - \rho_L) + \lambda_H [(1 - p) (\rho_1 - \rho_H) + \mu_5 (\rho_0 - \rho_H)] \]

Since \( \mu_4 \geq 0 \) and \( \rho_0 > \rho_L \), the term \( \mu_4 (\rho_0 - \rho_L) \geq 0 \). As we found above, either \( \lambda_H = 0 \) or \( (1 - p) (\rho_1 - \rho_H) + \mu_5 (\rho_0 - \rho_H) \geq 0 \), so the term \( \lambda_H [(1 - p) (\rho_1 - \rho_H) + \mu_5 (\rho_0 - \rho_H)] \geq 0 \). Therefore we have \( \mu_3 > 1 \), and therefore \( \mu_4 > 0 \) and \( \mu_5 > 0 \). This proves that (22), (23) and (19) hold with equality.

**Proposition.** Let \( q \geq 1, K > 0 \), and the credit lines of all other firms \((M, \pi)\) be given, and let \( \chi_1 = \)
$1 - \rho_0 + pp_L + (1 - p) \rho_H$. Let $\overline{M}_1$ be defined implicitly by $\overline{M}_1 \chi_1 = (\rho_H - \rho_0) [A - \pi (\overline{M}_1)]$, and let $\overline{M}_2$ be defined implicitly as the smallest value of $M$ such that $\pi' (M) \geq q - 1$. Then optimal firm behavior is as follows:

(i) If $p (\rho_H - \rho_L) < 1$, $q - 1 \leq \pi' (0)$, and $p (\rho_H - \rho_0) (q - 1) \leq 1 - p (\rho_0 - \rho_L) - p \chi_1$, then $\lambda_H = 1$, $M = 0$, $\pi = 0$, $l = A / [\chi_1 + (\rho_H - \rho_0) (q - 1)]$, and $I = (\rho_H - \rho_0) I$.

(ii) If $p (\rho_H - \rho_L) < 1$, $\overline{M}_1 \leq \overline{M}_2$, and $[1 - p (\rho_0 - \rho_L)] \overline{M}_1 \geq Ap (\rho_H - \rho_0)$, then $\lambda_H = 1$, $M = \overline{M}_1$, $\pi = \pi (\overline{M}_1)$, $l = 0$, and $I = [A - \pi (\overline{M}_1)] / \chi_1$.

(iii) If $p (\rho_H - \rho_L) < 1$, $\overline{M}_1 > \overline{M}_2$, and

$$pA \leq \left[ \frac{A - \pi (\overline{M}_2) + (q - 1) \overline{M}_2}{(\rho_H - \rho_0) (q - 1) + \chi_1} \right] \left[ 1 - p (\rho_0 - \rho_L) \right]$$

then $\lambda_H = 1$. If $\pi' (\overline{M}_2) = q - 1$, then any $M \in [\overline{M}_2, \overline{M}_1]$ is a solution, while if $\pi' (\overline{M}_2) > q - 1$ then only $M = \overline{M}_2$ is a solution. For each $M$, optimal behavior satisfies $\pi = \pi (M)$, and

$$I = \frac{A - \pi (M) + (q - 1) M}{(\rho_H - \rho_0) (q - 1) + \chi_1}$$

$$l = \frac{(\rho_H - \rho_0) [A - \pi (M)] - M \chi_1}{(\rho_H - \rho_0) (q - 1) + \chi_1}$$

(iv) If (i)-(iii) do not hold, we have $\lambda_H = 0$, $l = 0$, $M = 0$, and $I = A / [1 - p (\rho_0 - \rho_L)]$.

Proof. I solve this problem in two stages. In the first stage, I suppose that $\lambda_H = 1$ and solve for the optimal way to finance investment. In the second stage I check whether it is in fact optimal to meet both shocks.

Assuming $\lambda_H = 1$, the problem of an individual firm given price of bank financing $\pi (M)$ and price of outside liquidity $q - 1$ reduces to

$$\max_{I,M,l} I$$

s.t. $I \leq \frac{A - (q - 1) l - \pi (M)}{\chi_1}$

$I \leq \frac{M + l}{\rho_H - \rho_0}$

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The lagrangian is

\[ L = I + \lambda \left( \frac{A - (q - 1) I - \pi (M)}{\chi_1} - I \right) + \mu \left( \frac{M + I}{\rho_H - \rho_0} - I \right) \]

The first-order conditions with respect to \( I, M, \) and \( l \) are

\[ \begin{align*}
1 & \leq \lambda + \mu (\rho_H - \rho_0) \\
\chi_1 \mu & \leq \lambda \pi' (M) \\
\chi_1 \mu & \leq \lambda (q - 1)
\end{align*} \]

which hold with equality if \( I > 0, M > 0, \) and \( l > 0 \) respectively. Note that \( \pi (M) \) is in fact not everywhere differentiable, since it has a kink at the point at which the bank agency cost first bind. I therefore interpret \( \pi' (M) \) as a subgradient of \( \pi (\cdot) \) at this point.

Since (25) and (26) hold with equality at the solution, if we know the value of \( M \), we have that \( l \) satisfies

\[ l = \frac{(\rho_H - \rho_0) [A - \pi (M)] - M \chi_1}{\chi_1 + (\rho_H - \rho_0) (q - 1)} \quad (36) \]

Suppose \( I > 0 \). Then one of \( l \) or \( M \) must be positive. There are three possibilities:

**Case 1:** \( l = 0 \) and \( M > 0 \). Then we have \( \pi' (M) \leq q - 1 \). Then \( l = 0 \) together with (36) implies

\[ M \chi_1 = (\rho_H - \rho_0) [A - \pi (M)] \quad (37) \]

Since \( \pi (M) \) is increasing in \( M \), the righthand side is decreasing in \( M \) while the lefthand side is increasing in \( M \). Thus (37) implicitly defines a unique value of \( M \), call it \( \overline{M}_1 \). Together with \( l = 0 \) this implies

\[ I_1 = \frac{A - \pi (\overline{M}_1)}{\chi_1} \]

**Case 2:** \( l > 0 \) and \( M > 0 \). Then both first-order conditions bind and we have \( q - 1 = \pi' (M) \). Let \( \overline{M}_1 \) be implicitly defined as the smallest value of \( M \) s.t. \( \pi' (M) \geq q - 1 \). Since \( \pi (M) \) is equal to zero up to some value of \( M \) and then is linear thereafter, there are three possibilities. If the slope of the linear portion of \( \pi (\cdot) \) is steeper than \( q - 1 \), then \( \overline{M}_1 \) corresponds to the location of the kink in \( \pi (\cdot) \), and \( q - 1 \) is the slope of one of the subdifferentials of \( \pi (\cdot) \) at \( \overline{M}_1 \). If the slope of the linear portion of \( \pi (\cdot) \) is exactly equal to \( q - 1 \), then \( \overline{M}_1 \) is once again the location of the kink in \( \pi (\cdot) \), but now any \( M \geq \overline{M}_1 \) satisfies \( \pi' (M) = q - 1 \). Finally, if the slope of the linear portion of \( \pi (\cdot) \) is flatter than \( q - 1 \), then the value \( \overline{M}_1 \) does not exist and case 2 will never hold.

Suppose \( M \) satisfies \( \pi' (M) = q - 1 \). Then (36) implies

\[ l = \frac{(\rho_H - \rho_0) [A - \pi (M)] - M \chi_1}{(\rho_H - \rho_0) (q - 1) + \chi_1} \]
and I satisfies

\[ I = \frac{A - \pi(M) + (q - 1) M}{(\rho_H - \rho_0) (q - 1) + \chi_1} \]

Since we must have \( I \geq 0 \), this is only a potential solution if

\[ (\rho_H - \rho_0) [A - \pi(M)] \geq M \chi_1 \]

Since \( \bar{M}_1 \) was defined by \( (\rho_H - \rho_0) [A - \pi(\bar{M}_1)] = \bar{M}_1 \chi_1 \), if \( \bar{M}_2 > \bar{M}_1 \) then \( I < 0 \), which violates the non-negativity constraint. Thus case 2 can only occur when there exists \( M \leq \bar{M}_1 \) such that \( \pi'(M) = q - 1 \).

**Case 3:** \( I > 0 \) and \( M = 0 \). This implies \( q - 1 \leq \pi'(0) \). Then the solution is

\[ I = \frac{(\rho_H - \rho_0) A}{\chi_1 + (\rho_H - \rho_0) (q - 1)} \]

Now I calculate when each of these cases holds. First suppose that \( q - 1 < \pi'(0) \). Then case 3 holds, because \( \pi'(M) \) is weakly increasing in \( M \), and so the conditions of case 1 and case 2 will never hold. If instead \( q - 1 = \pi'(0) \), then all three cases can hold, and the relative quantities \( I \) and \( M \) are indeterminate. In all cases, \( I \) takes the same value, namely \( I = A / [\chi_1 + (\rho_H - \rho_0) (q - 1)] \), while \( M \) and \( l \) can be any values s.t. \( M \geq 0, I \geq 0, M + l = (\rho_H - \rho_0) I \), and \( \pi'(M) = q - 1 \).

Now suppose that \( q - 1 > \pi'(0) \), as will be the case if \( q > 1 \) and \( \bar{M}_2 > 0 \). Then case 3 does not hold and one of case 1 or case 2 holds. If \( \bar{M}_1 \leq \bar{M}_2 \), then case 2 cannot hold because this implies \( \bar{l}_2 \leq 0 \), and thus the assumption of \( l > 0 \) in case 2 is violated. If \( \bar{M}_2 < \bar{M}_1 \), then we must consider two possibilities. If \( \pi'(M) > q - 1 \) for \( M > \bar{M}_2 \), meaning that the slope of the linear portion of \( \pi(\cdot) \) is steeper than \( q - 1 \), then case 2 holds, and moreover is uniquely specified, with \( M = \bar{M}_2 \) and \( l > 0 \). By contrast, if \( \pi'(M) = q - 1 \) for \( M > \bar{M}_2 \), then cases 1 and 2 result in the same level of investment. The firm is indifferent between \( M \in [\bar{M}_2, \bar{M}_1] \), and \( l = \bar{M}_1 - M \).

Now we need only determine under what conditions firms prefer to meet both shocks. Firms prefer to meet the high shock if investment exceeds \( pA / [1 - p (\rho_0 - \rho_L)] \), which is the investment when only the low shock is met times the probability of the low shock. When \( q - 1 \leq \pi'(0) \) so that case 3 holds, firms meet both shocks as long as

\[ q - 1 \leq \frac{1 - p (\rho_L - \rho_0)}{p (\rho_H - \rho_0)} \]

When \( q - 1 > \pi'(0) \) and \( \bar{M}_1 \leq \bar{M}_2 \) so that case 1 holds, firms meet both shocks as long as

\[ \bar{M}_1 \geq \frac{Ap (\rho_H - \rho_0)}{1 - p (\rho_0 - \rho_L)} \]
When \( q - 1 > \pi' (0) \) and \( \overline{M}_1 > \overline{M}_2 \) so that case 2 holds, firms meet both shocks as long as

\[
\frac{pA}{1 - p(\rho_0 - \rho_L)} \leq \frac{A - \pi (\overline{M}_2) + (q - 1) \overline{M}_2}{(\rho_H - \rho_0) (q - 1) + \chi_1}
\]

\( \Box \)

**Proposition.** Let \( K > 0 \) be given and let \( \chi_1 = 1 - \rho_0 + p\rho_L + (1 - p) \rho_H \). Let \( M^* \) be defined by

\[
\overline{C}_1 (M^*, K) + C (M^*, K) = A - \frac{(M^* + \overline{I}) \chi_1}{\rho_H - \rho_0}
\]

and let

\[
I^* = \frac{M^* + \overline{I}}{\rho_H - \rho_0}
\]

then,

(i) If \( p (\rho_H - \rho_L) > 1 \), then \( \lambda_H = 0 \), \( M = \pi = l = 0 \), and \( I = A/ [1 - p (\rho_0 - \rho_L)] \).

(ii) If \( p (\rho_H - \rho_L) \leq 1 \) and \( \overline{I} \geq (\rho_H - \rho_0) A/\chi_1 \), then \( \lambda_H = 1 \), \( M = \pi = 0 \), \( I = A/\chi_1 \), \( l = (\rho_H - \rho_0) A/\chi_1 \), and \( q = 1 \).

(iii) If \( p (\rho_H - \rho_L) \leq 1 \), \( \overline{I} < (\rho_H - \rho_0) A/\chi_1 \), and \( I^* \geq pI_0 \), then \( \lambda_H = 1 \), \( l = \overline{I} \), \( M = M^* \), \( \pi = C (M^*, K) \), \( I = I^* \), and \( q = 1 + C_1 (M^*, K) \).

(iv) If \( p (\rho_H - \rho_L) \leq 1 \), \( \overline{I} < (\rho_H - \rho_0) A/\chi_1 \), and \( I^* < pI_0 \), then a fraction \( \zeta \) of firms choose \( \lambda_H = 1 \), \( l = p\overline{I}_0 \), \( M = (\rho_H - \rho_0) I - \overline{I}/\zeta \), and \( I = \overline{I}/\zeta \), and the remaining fraction \( 1 - \zeta \) of firms choose \( \lambda_H = 0 \), \( M = \pi = l = 0 \), and \( I = A/ [1 - p (\rho_0 - \rho_L)] \). The fraction \( \zeta \) satisfies

\[
C (\zeta M, K) + C_1 (\zeta M, K) \overline{I} = \left[ \frac{A - \chi_1 I}{(\rho_H - \rho_0) I} \right] (\zeta M + \overline{I})
\]

which specifies a unique value of \( \zeta M \). Together with \( \zeta = \frac{M^* + \overline{I}}{(\rho_H - \rho_0) I} \) from the definition of \( M \), this defines separate values for \( M \) and \( \zeta \).

**Proof.** We define equilibrium by combining firm behavior as given in Proposition 6 with the market clearing condition \( \zeta l \leq \overline{I} \) and \( (\tilde{z} l - \tilde{I}) (q - 1) = 0 \). This says that firms who meet the high shock must hold all available outside liquidity \( \overline{I} \) unless \( q = 1 \), in which case households are willing to hold some and firms hold less than \( \overline{I} \). Together with the assumption of symmetric credit lines, these are sufficient to define equilibrium.
Now suppose that \( p (\rho_H - \rho_L) > 1 \). Then I claim that in all equilibria in which a fraction \( \zeta > 0 \) of firms purchase credit lines from banks, these credit lines satisfy \( C_1 (\zeta M, K) = q - 1 \). Note that \( C_1 (\zeta M, K) \) is equal to \( \pi' (M) \) faced by each individual firm. By Proposition 6, there are three cases to consider: \( \pi' (0) \leq q - 1, M_1 \leq M_2 \), and \( M_2 > M_1 \).

Suppose that \( \pi' (0) \geq q - 1 \). Then \( C_1 (0, K) \geq q - 1 \), but since we assumed \( C_1 (0, K) = 0 \), this will only hold true if \( q = 1 \), in which case \( C_1 (0, K) = q - 1 \).

Suppose instead that \( \pi' (0) < q - 1 \). Suppose that we have \( M_1 \leq M_2 \). Then firms will choose \( M = M_1 > 0 \), and we must have \( \pi' (M_1) \leq q - 1 \). This implies \( q - 1 \geq C_1 (\zeta M_1, K) > 0 \), and so \( q > 1 \). Yet in this case firms will choose \( l = 0 \). \( q > 1 \) and \( \zeta l = 0 \) together contradict market clearing, since they imply that households hold a positive quantity of liquid assets despite \( q > 1 \).

Finally, suppose that \( \pi' (0) < q - 1 \). Suppose that \( M = M_2 \), or any \( M \in [M_2, M_1] \) for which \( \pi' (M) = q - 1 \). Suppose that for all \( M > M_2 \), we have \( \pi' (M) \geq q - 1 \). Then all firms will choose credit maximum \( M_2 \), which is at the kink in their functions \( \pi (\cdot) \), at which point \( \pi (\cdot) \) is equal to 0. Then we have \( \pi (M) = 0 \) for at the firm’s choice of \( M \), which violates the bank agency cost constraint \( \zeta \pi \geq C (\zeta M, K) \). Thus we conclude that at the choice of \( M \), we have \( C_1 (\zeta M, K) = q - 1 \).

Let’s first consider equilibria in which \( q = 1 \). In this case \( C_1 (\zeta M, K) = q - 1 \) implies \( M = 0 \). Then \( \pi = 0 \), and so (25) implies \( I = A / \chi_1 \). Since \( M = 0 \), the high liquidity shock is entirely financed by holding outside liquidity, and so (26) implies \( l = (\rho_H - \rho_0) A / \chi_1 \). Since \( p (\rho_H - \rho_L) > 1 \), all firms will choose to meet both shocks if \( q = 1 \), and so \( \zeta = 1 \). Therefore each firm’s choice of \( l \) cannot exceed \( \bar{I} \). This equilibrium will attain as long as \( \bar{I} \geq (\rho_H - \rho_0) A / \chi_1 \).

Suppose instead that \( p (\rho_H - \rho_L) > 1 \) and \( \bar{I} < (\rho_H - \rho_0) A / \chi_1 \). Then \( q = 1 \) will not be an equilibrium, and so \( q > 1 \). Then firms must hold all outside liquidity, which implies \( \zeta = \bar{I} \). Combined with (18), (25), and (26), plus \( q - 1 = C_1 (\zeta M, K) \) obtained above, we have equilibrium conditions:

\[
\begin{align*}
\zeta l &= \bar{I} \\
q - 1 &= C_1 (\zeta M, K) \\
\zeta \pi &= C (\zeta M, K) \\
(\rho_H - \rho_0) I &= M + l \\
\chi_1 I &= A - \pi - (q - 1) l
\end{align*}
\]

We can simplify these conditions by replacing \( l, q, \) and \( \pi \) to obtain

\[
\begin{align*}
(\rho_H - \rho_0) I &= M + \bar{I} / \zeta \\
\chi_1 I &= A - C (\zeta M, K) / \zeta - C_1 (\zeta M, K) \bar{I} / \zeta
\end{align*}
\]

yielding two equations in three unknowns \((\zeta, M, I)\). We need one more expression to specify
equilibrium. We use the fact that firms prefer to meet both shocks as long as

$$ I \geq pA / [1 - p (\rho_0 - \rho_L)] $$

(40)

If (40) holds as a strict inequality, firms strictly prefer to meet both shocks and so $$\zeta = 1$$. Thus if firms meet both shocks, we have either $$\zeta = 1$$ or (40) holds with equality.

Suppose first that $$\zeta = 1$$. Then $$M$$ and $$I$$ satisfy

$$ (\rho_H - \rho_0) I = M + I $$

$$ \chi_1 I = A - C (M, K) - C_1 (M, K) I $$

Combining these expressions, we find that $$M$$ satisfies

$$ C (M, K) + C_1 (M, K) I = A - \chi_1 \left( \frac{M + I}{\rho_H - \rho_0} \right) $$

(41)

Note that the lefthand side of (41) is strictly increasing in $$M$$ and the righthand side is strictly decreasing in $$M$$. Moreover, for $$M = 0$$ the lefthand side of (41) is equal to 0, and the righthand side equals $$A - \chi_1 I \rho_H / (\rho_H - \rho_0)$$, which is positive by assumption. Finally, as $$M \to \infty$$ the righthand side of (41) goes to $$-\infty$$ while the lefthand side goes to $$\infty$$. Therefore (41) defines a unique value of $$M$$, which is the $$M^\ast$$ given in Proposition 7. Given $$M = M^\ast$$, we immediately have $$I^\ast = \left( M^\ast + I \right) / (\rho_H - \rho_0)$$. It only remains to check whether in fact it is preferable to meet both shocks. This will be true as long as

$$ I^\ast \geq pA / [1 - p (\rho_0 - \rho_L)] $$

Now suppose that $$I^\ast < pA / [1 - p (\rho_0 - \rho_L)]$$, so the $$\zeta = 1$$ is not an equilibrium. Then investment must satisfy

$$ I = \frac{pA}{1 - p (\rho_0 - \rho_L)} $$

Combining with (38) and (39), we obtain

$$ \zeta M \chi_1 (\rho_H - \rho_0) + \left[ C (\zeta M, K) + C_1 (\zeta M, K) I \right] = \zeta A - \frac{\chi_1 I}{(\rho_H - \rho_0)} $$

(42)

I claim that (42) uniquely specifies a value of $$M$$ for a given $$\zeta$$, as long as $$\zeta > \chi_1 I / [A (\rho_H - \rho_0)]$$. This latter condition requires that banks are necessary even if only a fraction $$\zeta$$ of firms meet both shocks. Then observe that if $$M = 0$$, the lefthand side of (42) is zero while the right-hand side is positive. Likewise the lefthand side is increasing in $$M$$, and approaches $$\infty$$ as $$M \to \infty$$. Thus (42) specifies a unique value of $$M$$ for given $$\zeta > \chi_1 I / A$$. Call this implicit function $$M (\zeta)$$. We can calculate $$M' (\zeta)$$ using the implicit function theorem, we find

$$ M' (\zeta) = - \frac{M \chi_1 + (\rho_H - \rho_0) \left[ M (C_1 + I C_{11}) - A \right]}{\zeta \chi_1 + \zeta (\rho_H - \rho_0) (C_1 + I C_{11})} $$
Now we have an expression for \( I \) in terms of \( \zeta \), which is

\[
I = \frac{M(\zeta) + \bar{I}/\zeta}{(\rho_H - \rho_0)}
\]

which uniquely specifies a value of \( I \) given \( \zeta \). Since in equilibrium we have \( I = pA / [1 - p(\rho_0 - \rho_L)] \), we obtain an expression for equilibrium \( \zeta \) which is

\[
M(\zeta) + \bar{I}/\zeta = \frac{pA(\rho_H - \rho_0)}{1 - p(\rho_0 - \rho_L)} \tag{43}
\]

The righthand side of (43) is a positive constant. I claim that the lefthand side of (43) is strictly decreasing in \( \zeta \). Differentiating the lefthand side, we obtain

\[
M'(\zeta) - \frac{I}{\zeta^2} = -\frac{M_C + (\rho_H - \rho_0) \left[ M \left( C_1 + \bar{I}C_{11} \right) - A \right]}{\zeta M_C + (\rho_H - \rho_0) \left( C_1 + \bar{I}C_{11} \right)} - \frac{\bar{I}}{\zeta^2}
\]

Further simplification shows that this expression is negative as long as

\[
(\rho_H - \rho_0) \left[ M \left( C_1 + \bar{I}C_{11} \right) - C - \bar{I}C_1 \right] + \left[ \frac{\bar{I}}{\zeta} (\rho_H - \rho_0) \left( C_1 + \bar{I}C_{11} \right) \right] > 0
\]

which will hold because the second term is strictly positive, and the first term is weakly positive since we assumed in (27) that

\[
M \left( C_1 + \bar{I}C_{11} \right) \geq C + \bar{I}C_1
\]

Therefore we conclude that if (43) defines a \( \zeta \), it will be unique.

We still need to establish that (43) in fact defines a value of \( \zeta \). First observe that for \( \zeta = 1 \), the \( I \) implied by (38) and (39) was \( I^* < pA / [1 - p(\rho_0 - \rho_L)] \). We know that as \( \zeta \) decreases, \( I(\zeta) \) increases.

Now consider what happens at \( \zeta = \chi_1 \bar{I} / [A(\rho_H - \rho_0)] \). This value lies on the interval \((0, 1)\). At this value of \( \zeta \), there is sufficient outside liquidity for all firms that choose to meet the high shock to finance their optimal level of investment entirely by holding outside liquidity. Substituting this value into (38), we obtain

\[
I = \frac{A}{\chi_1} + \frac{M}{(\rho_H - \rho_0)}
\]

and substituting this into (39), we obtain

\[
\frac{M\chi_1}{(\rho_H - \rho_0)} = -C(\zeta M, K)/\zeta - C_1(\zeta M, K)\bar{I}/\zeta
\]

which will only hold at \( M = 0 \). Then we have \( I = A/\chi_1 \), which by our initial assumption \( p(\rho_H - \rho_L) > 1 \) will be greater than \( pA / [1 - p(\rho_0 - \rho_L)] \). Therefore there is a unique value of
\[ \zeta \in \left[ \chi_1 I/[A (\rho_H - \rho_0)], 1 \right] \text{ at which } I(\zeta) = pA/[1 - p (\rho_0 - \rho_L)]. \]

From this we conclude that (43) specifies a unique equilibrium \( \zeta \). Combining (38), (39), and (43), we find that \( \zeta M \) satisfies

\[
C(\zeta M, K) + C_1(\zeta M, K) \bar{I} = \left[ \frac{A - \chi_1 I}{(\rho_H - \rho_0) I} \right] \left( \bar{\zeta} M + \bar{I} \right)
\]

thus proving the proposition.

**A4: Comparative Statics with Banks**

**Proposition.** At an interior equilibrium, \( I^* \) and \( M^* \) as defined in Proposition 7, and a marginal change in \( \bar{I} \) or \( K \) result in a marginal change in equilibrium \( I^* \) according to

\[
\begin{align*}
\frac{dI^*}{d\bar{I}} &= \frac{\bar{I}C_{11}(\cdot)}{(\rho_H - \rho_0) \left[ \bar{I}C_{11}(\cdot) + C_1(\cdot) \right] + \chi_1} \\
\frac{dI^*}{dK} &= -\frac{\bar{I}C_{12}(\cdot) + C_2(\cdot)}{(\rho_H - \rho_0) \left[ \bar{I}C_{11}(\cdot) + C_1(\cdot) \right] + \chi_1}
\end{align*}
\]

**Proof.** \( I^* \) is defined implicitly by

\[
f(I, K, \bar{I}) = \chi_1 I - A + \bar{I}C_1 \left( (\rho_H - \rho_0) I - \bar{I}, K \right) + C \left( (\rho_H - \rho_0) I - \bar{I}, K \right) = 0
\]

\( f(\cdot) \) has partial derivatives

\[
\begin{align*}
\frac{\partial f}{\partial I} &= (\rho_H - \rho_0) \left[ \bar{I}C_{11}(\cdot) + C_1(\cdot) \right] + \chi_1 \\
\frac{\partial f}{\partial \bar{I}} &= -\bar{I}C_{11}(\cdot) \\
\frac{\partial f}{\partial K} &= \bar{I}C_{12}(\cdot) + C_2(\cdot)
\end{align*}
\]

By the implicit function theorem, the effects of changes in \( \bar{I} \) or \( K \) on equilibrium \( I \) are given by

\[
\begin{align*}
\frac{dI}{d\bar{I}} &= \frac{\bar{I}C_{11}(\cdot)}{(\rho_H - \rho_0) \left[ \bar{I}C_{11}(\cdot) + C_1(\cdot) \right] + \chi_1} \\
\frac{dI}{dK} &= -\frac{\bar{I}C_{12}(\cdot) + C_2(\cdot)}{(\rho_H - \rho_0) \left[ \bar{I}C_{11}(\cdot) + C_1(\cdot) \right] + \chi_1}
\end{align*}
\]
Corollary. At an interior equilibrium:

(i) An increase in $\bar{l}$ will result in an increase in $I^\star$.
(ii) An increase in $K$ will result in an increase in $I^\star$ iff $C_{12} (\cdot) < - \frac{C_{2} (\cdot)}{l}$.
(iii) A sufficient condition for an increase in $K$ to result in an increase in $I^\star$ is $C_{12} (\cdot) < 0$.

Proof. The first follows directly from the expression for $\frac{dI^\star}{dl}$ given in Proposition 8 once we observe that $C_{1} > 0$ and $C_{11} > 0$. The second follows directly from the expression for $\frac{dI^\star}{dK}$ given in Proposition 8 once we further observe that $C_{2} < 0$. The third follows by further observing that $- \frac{C_{2} (\cdot)}{l} > 0$, and so if $C_{12} (\cdot) < 0$ then $C_{12} < - \frac{C_{2} (\cdot)}{l}$.

Proposition. At an interior equilibrium, a marginal change in $\bar{l}$ or $K$ result in a marginal change in equilibrium $q - 1$ according to

\[
\frac{dq}{dl} = - \frac{C_{11} (\cdot) ((\rho_{H} - \rho_{0}) C_{1} (\cdot) + \chi_{1})}{(\rho_{H} - \rho_{0}) [I C_{11} (\cdot) + C_{1} (\cdot)] + \chi_{1}}
\]

\[
\frac{dq}{dK} = \frac{(\rho_{H} - \rho_{0}) [C_{12} (\cdot) C_{1} (\cdot) - C_{11} (\cdot) C_{2} (\cdot)] + \chi_{1} C_{12} (\cdot)}{(\rho_{H} - \rho_{0}) [I C_{11} (\cdot) + C_{1} (\cdot)] + \chi_{1}}
\]

Proof. Equilibrium $q$ satisfies the arbitrage condition $q - 1 = C_{1} \left( (\rho_{H} - \rho_{0}) I - \bar{l}, K \right)$. Differentiating this yields expressions

\[
\frac{dq}{dl} = C_{11} (\cdot) \left( (\rho_{H} - \rho_{0}) \frac{dI}{dl} - 1 \right)
\]

\[
\frac{dq}{dK} = (\rho_{H} - \rho_{0}) C_{11} (\cdot) \frac{dI}{dK} + C_{12} (\cdot)
\]

Substituting the expressions for $\frac{dI}{dl}$ and $\frac{dI}{dK}$ derived in Proposition 8, we obtain

\[
\frac{dq}{dl} = C_{11} (\cdot) \left[ \frac{(\rho_{H} - \rho_{0}) I C_{11} (\cdot)}{(\rho_{H} - \rho_{0}) [I C_{11} (\cdot) + C_{1} (\cdot)] + \chi_{1}} - 1 \right]
\]

\[
\frac{dq}{dK} = C_{12} (\cdot) - \frac{(\rho_{H} - \rho_{0}) C_{11} (\cdot) [I C_{12} (\cdot) + C_{2} (\cdot)]}{(\rho_{H} - \rho_{0}) [I C_{11} (\cdot) + C_{1} (\cdot)] + \chi_{1}}
\]

Simplification yields the given expressions.

Corollary. At an interior equilibrium:

(i) An increase in $\bar{l}$ results in a decrease in equilibrium $q$.
(ii) An increase in $K$ results in a decrease in equilibrium $q$ if and only if

\[
(\rho_{H} - \rho_{0}) [C_{11} (\cdot) C_{2} (\cdot) - C_{1} (\cdot) C_{12} (\cdot)] > \chi_{1} C_{12} (\cdot)
\]
Proof. From Proposition 10 we have expressions

\[
\frac{dq}{dl} = -\frac{C_{11}(\cdot)[(\rho_H - \rho_0)C_1(\cdot) + \chi_1]}{\rho_H - \rho_0}[C_{11}(\cdot) + C_1(\cdot)] + \chi_1
\]

\[
\frac{dq}{dK} = \frac{(\rho_H - \rho_0)[C_{12}(\cdot)C_1(\cdot) - C_{11}(\cdot)C_2(\cdot)] + \chi_1C_{12}(\cdot)}{\rho_H - \rho_0}[C_{11}(\cdot) + C_1(\cdot)] + \chi_1
\]

Since \( C_1(\cdot) > 0 \) and \( C_{11}(\cdot) > 0 \), the first expression implies \( \frac{dq}{dl} < 0 \), which proves (i). Since \( C_1(\cdot) > 0 \) and \( C_{11}(\cdot) > 0 \), the denominator of the second expression is positive, and therefore \( \frac{dq}{dK} < 0 \) iff the numerator of this expression is negative, which will be true when

\[
(\rho_H - \rho_0)[C_{12}(\cdot)C_1(\cdot) - C_{11}(\cdot)C_2(\cdot)] + \chi_1C_{12}(\cdot) < 0
\]

Which can be simplified to obtain (ii).

A5: Optimal Policy

Proposition. Suppose that in the absence of government policy the economy is initially at an interior equilibrium. Then,

(i) The optimal supply of government liquid assets \( x^* \) will be positive.

(ii) If optimal \( x^* \) implies that the economy is still at an interior equilibrium, and if the current choice of \( x^* \) is unique, then a marginal change in \( \bar{I} \) or \( K \) will shift the optimal point according to

\[
\frac{dx}{d\bar{I}} = -\frac{R\frac{\partial^2 I}{\partial x^2}}{R\frac{\partial^2 I}{\partial x^2} - D''(x)}
\]

\[
\frac{dx}{dK_0} = -\frac{R\frac{\partial^2 I}{\partial x^2}}{R\frac{\partial^2 I}{\partial x^2} - D''(x)}
\]

(iii) At an interior optimal point \( x^* \), we have \( R\frac{\partial^2 I}{\partial x^2} < D''(x) \).

(iv) We have \( \frac{dx}{d\bar{I}} > 0 \iff \frac{\partial^2 I}{\partial x^2} > 0 \) and \( \frac{dx}{dK_0} > 0 \iff \frac{\partial^2 I}{\partial x^2} > 0 \).

Proof. In turn,

(i) Suppose not. Then \( RI - C(x) \) is maximized at \( x = 0 \). Let \( f(x) = RI|_x \). Then since the function \( f(x) - C(x) \) achieves a maximum at \( x = 0 \), \( f \) and \( C \) must satisfy

\[
f(0) - C(0) \geq f(h) - C(h)
\]
for every $h > 0$. Rearranging and dividing by $h$, we can write this as
\[
\frac{C(h) - C(0)}{h} \geq \frac{f(h) - f(0)}{h}
\]
this inequality will be preserved under taking limits
\[
\lim_{h \to 0} \frac{C(h) - C(0)}{h} \geq \lim_{h \to 0} \frac{f(h) - f(0)}{h}
\]
which is just the definition of the derivatives of $C$ and $f$ at $x = 0$. Therefore we have
\[
C'(0) \geq f'(0)
\]
Since we have $C'(0) = 0$, this implies that $f'(0) = R \frac{dI}{dx} \bigg|_{x=0} \leq 0$. But from Proposition 8 we have $\frac{dI}{dx} > 0$ for $I > 0$, so this is a contradiction.

(ii) Any interior solution satisfies the optimality condition
\[
R \frac{dI}{dx} = D'(x)
\]
which is a zero of the function
\[
f(\cdot) = R \frac{dI}{dx} - D'(x) = 0
\]
The partial derivatives of $f$ are
\[
\frac{\partial f}{\partial x} = R \frac{d^2 I}{dx^2} - D''(x)
\]
\[
\frac{\partial f}{\partial l} = R \frac{d^2 I}{dx}
\]
\[
\frac{\partial f}{\partial K} = R \frac{d^2 I}{dx dK}
\]
making use of the fact that $\frac{d^2 I}{dldx} = \frac{d^2 I}{dxdx}$. Since this is a unique maximum of a continuously differentiable function the implicit function theorem is valid. Therefore we have
\[
\frac{dx}{dl} = -\frac{R \frac{d^2 I}{dx^2}}{R \frac{d^2 I}{dx^2} - D''(x)}
\]
\[
\frac{dx}{dK} = -\frac{R \frac{d^2 I}{dx dK}}{R \frac{d^2 I}{dx^2} - D''(x)}
\]
(iii) If the point $x^*$ is an interior solution, then it must also be a local maximum. At a local maximum of a global function, the function is locally concave, meaning that the second derivative is negative. Here the function we are maximizing is $RI - D(x)$. Since this function is twice continuously differentiable, it will be concave at the point $x^*$ iff the second derivative $R \frac{d^2 I}{dx^2} - D''(x) < 0$, 

which immediately implies the given condition.

(iv) Since we have \( R \frac{\partial^2 I}{\partial x^2} - D''(x) < 0 \), the denominators in the expressions for \( dx/d\bar{l} \) and \( dx/dK \) are negative. Therefore each expression will be positive iff the numerator is positive. Therefore we obtain the given statement.

**Proposition.** Suppose that in the absence of government policy the economy is initially at an interior equilibrium. Then,

(i) If

\[
\bar{I} \geq \left[ 1 - \frac{\psi \chi_1 R}{\sigma (\rho_H - \rho_0)^2} A \psi + \sigma \chi_1^2 K \right] \left( \rho_H - \rho_0 \right) \frac{A}{\chi_1}
\]

(44)

the optimal policy is \( x^* = \frac{\chi_1 (\rho_H - \rho_0)}{\psi} - \bar{I} \).

(ii) If (44) does not hold, optimal policy \( x^* > 0 \) is uniquely defined by

\[
\frac{R}{\sigma} \psi \left( \frac{\bar{I} + x}{x} \right) = \psi (\rho_H - \rho_0)^2 \bar{I} + \chi_1 K
\]

(iii) If (44) does not hold, then a marginal change in \( \bar{I} \) or \( K \) will shift the optimal supply of point according to

\[
\frac{dx}{d\bar{l}} = \frac{\left( \frac{R \psi}{\sigma x} \right)^2 - \psi^2 (\rho_H - \rho_0)^2}{\bar{I} \left( \frac{R \psi}{\sigma x} \right)^2 + \psi^2 (\rho_H - \rho_0)^2}
\]

\[
\frac{dx}{dK} = -\frac{\psi^2 (\rho_H - \rho_0)^2 \left[ (\rho_H - \rho_0)^2 \bar{I}^2 - \left( \frac{\bar{l} + x}{x+\bar{l}} \right) \left( \frac{1}{x+\bar{l}} \right) \left( \frac{1}{\bar{l}} \right) + \chi_1 \frac{R \psi}{\sigma} \right]}{\bar{I} \left( \frac{R \psi}{\sigma x} \right)^2 + \psi^2 (\rho_H - \rho_0)^2}
\]

which have signs \( \frac{dx}{d\bar{l}} > 0 \) and \( \frac{dx}{dK} < 0 \).

**Proof.** We are maximizing the expression

\[
RI - \frac{\sigma}{2} x^2
\]

where \( I \) is defined by Proposition 7, with \( \bar{l} \) replaced by \( x + \bar{l} \), and with \( C (M, K) = \frac{\psi M^2}{2} \). We restrict our attention to cases in which there is an interior equilibrium if \( x = 0 \). We have from Proposition 12 that the optimal choice of \( x \) will be positive. Then there are two types of potential solutions.
First, we must consider cases that satisfy the necessary condition

\[ R \frac{dI}{dx} = \sigma x \]

and then we must consider the boundary solution that \( x = \bar{x} \), where \( \bar{x} = (\rho_H - \rho_0) \frac{A}{\chi_1} - \bar{I} \) is the level of public liquidity \( x \) that will allow the economy to achieve the constrained optimum production plan.

First I derive an expression for the first order necessary condition. By proposition 8, at every interior point we have

\[
\frac{dI}{dx} = \frac{\left( \bar{I} + x \right) C_{11}(\cdot)}{(\rho_H - \rho_0) \left[ \left( \bar{I} + x \right) C_{11}(\cdot) + C_1(\cdot) \right] + \chi_1}
\]

since for \( C(M, K) = \frac{\psi M^2}{2K} \) we have

\[
C_1(\cdot) = \frac{\psi M}{K} \\
C_{11}(\cdot) = \frac{\psi}{K}
\]

we have

\[
\frac{dI}{dx} = \frac{\psi \left( x + \bar{I} \right)}{\psi (\rho_H - \rho_0)^2 I + \chi_1 K}
\]

Substituting this into the necessary condition yields

\[
\frac{R \psi}{\sigma} \left( \frac{x + \bar{I}}{x} \right) = \psi (\rho_H - \rho_0)^2 I + \chi_1 K
\]

Written in this form, it is clear that the left-hand side approaches \( \infty \) as \( x \to \infty \) and is strictly decreasing in \( x \), whereas the right-hand side has some positive finite value at \( x = 0 \) and is strictly increasing in \( x \), since \( I \) is increasing in \( x \). Therefore if there is an intersection it will be a unique intersection.

The only remaining question is whether this unique intersection lies above or below \( \bar{x} \), since \( dI/dx \) drops to zero at \( \bar{x} \). To determine this, we can simply check whether the left-hand side of the optimality expression is greater than the right-hand side at \( \bar{x} \). Thus if

\[
\frac{R \psi}{\sigma} \left( \frac{\bar{x} + \bar{I}}{\bar{x}} \right) \geq \psi (\rho_H - \rho_0)^2 I + \chi_1 K
\]

then the intersection lies to the right of \( \bar{x} \), and so the optimal level is \( x = \bar{x} \).

The next step is to determine the comparative statics at an interior solution. The interior solution will be the zero of
\[ f = \frac{R}{\sigma} \psi \left( \bar{I} + x \right) - \left[ \psi (\rho_H - \rho_0)^2 I + \chi_1 K \right] x = 0 \]

Partial derivatives of \( f \) are

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{R}{\sigma} \psi - \left[ \psi (\rho_H - \rho_0)^2 \frac{dI}{dx} \right] x - \left[ \psi (\rho_H - \rho_0)^2 I + \chi_1 K \right] x \\
\frac{\partial f}{\partial l} &= \frac{R}{\sigma} \psi - \left[ \psi (\rho_H - \rho_0)^2 \frac{dI}{dl} \right] x \\
\frac{\partial f}{\partial K} &= - \left[ \psi (\rho_H - \rho_0)^2 \frac{dI}{dK} + \chi_1 \right] x
\end{align*}
\]

We have derived an expression for \( \frac{dI}{dx} = \frac{dI}{dl} \) above. We have an expression for \( \frac{dI}{dK} \) from Proposition 8 for generic agency cost function \( C(\cdot) \). Using \( C_1(\cdot) \) and \( C_{11}(\cdot) \) derived above along with

\[
\begin{align*}
C_2(\cdot) &= -\frac{\psi}{2} \left( \frac{M}{K} \right)^2 \\
C_{12}(\cdot) &= -\frac{\psi M}{K^2}
\end{align*}
\]

we obtain

\[
\frac{dI}{dK} = \frac{\psi}{2} \left[ \frac{(\rho_H - \rho_0) I + \bar{I} + x}{(\rho_H - \rho_0)^2 \psi I + \chi_1 K} \right] \left( \frac{M}{K} \right)
\]

Substituting these into the partial derivatives of \( f \), we obtain

\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{R}{\sigma} \psi - \left[ \psi (\rho_H - \rho_0)^2 \frac{\psi (\rho_H - \rho_0)^2 (x + \bar{I})}{\psi (\rho_H - \rho_0)^2 I + \chi_1 K} \right] x - \left[ \psi (\rho_H - \rho_0)^2 I + \chi_1 K \right] x \\
\frac{\partial f}{\partial l} &= \frac{R}{\sigma} \psi - \left[ \psi (\rho_H - \rho_0)^2 \frac{\psi (\rho_H - \rho_0)^2 (x + \bar{I})}{\psi (\rho_H - \rho_0)^2 I + \chi_1 K} \right] x \\
\frac{\partial f}{\partial K} &= - \left[ \frac{\psi^2 (\rho_H - \rho_0)^2 \left[ (\rho_H - \rho_0) I + \bar{I} + x \right]}{(\rho_H - \rho_0)^2 \psi I + \chi_1 K} \right] \left( \frac{M}{K} \right) + \chi_1 \right] x
\end{align*}
\]

Since at the optimum we have \( \frac{R\psi}{\sigma} \left( \frac{x + \bar{I}}{x} \right) = \psi (\rho_H - \rho_0)^2 I + \chi_1 K \), these expressions can be simpli-
\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{R}{\sigma} \psi - \left[ \frac{\psi^2 (\rho_H - \rho_0)^2}{\frac{R\phi}{\sigma x}} \right] x - \left[ \frac{R\psi}{\sigma} \left( \frac{x+\bar{I}}{x} \right) \right] \\
\frac{\partial f}{\partial \bar{I}} &= \frac{R}{\sigma} \psi - \left[ \frac{\psi^2 (\rho_H - \rho_0)^2}{\frac{R\phi}{\sigma x}} \right] x \\
\frac{\partial f}{\partial K} &= -\left[ \frac{\frac{\psi^2}{2} (\rho_H - \rho_0)^2 \left( (\rho_H - \rho_0)^2 I^2 - (\bar{I} + x)^2 \right) \left( \frac{1}{K} + \chi_1 \right)}{\frac{R\phi}{\sigma} \left( \frac{x+\bar{I}}{x} \right)} + \chi_1 \frac{R\phi}{\sigma} \right] x
\end{align*}
\]

Applying the implicit function theorem, we obtain

\[
\frac{dx}{d\bar{I}} = \left( \frac{\frac{R\phi}{\sigma x}}{\frac{R\phi}{\sigma x}} + \frac{\psi^2 (\rho_H - \rho_0)^2}{\frac{R\phi}{\sigma x}} \right) x
\]

\[
\frac{dx}{dK} = \frac{-\frac{\psi^2}{2} (\rho_H - \rho_0)^2 \left( (\rho_H - \rho_0)^2 I^2 - (\bar{I} + x)^2 \right) \left( \frac{x}{x + \bar{I}} \right) \left( \frac{1}{K} + \chi_1 \frac{R\phi}{\sigma} \right)}{\frac{R\phi}{\sigma} \left( \frac{x+\bar{I}}{x} \right) + \psi^2 (\rho_H - \rho_0)^2}
\]

The final step is to determine the signs of these expressions. We have \( \frac{dx}{d\bar{I}} > 0 \) iff

\[
\left( \frac{\frac{R\phi}{\sigma x}}{\frac{R\phi}{\sigma x}} \right)^2 > \psi^2 (\rho_H - \rho_0)^2
\]

Which can be written as

\[
\frac{R\phi}{\sigma x} > \psi (\rho_H - \rho_0)
\]

At an optimum, we have \( \frac{\frac{R\phi}{\sigma x}}{\frac{R\phi}{\sigma x}} = \frac{\psi(\rho_H - \rho_0)^2 I + \chi_1 K}{x + \bar{I}} \), so this expression becomes

\[
\psi (\rho_H - \rho_0)^2 I + \chi_1 K > \psi (\rho_H - \rho_0) \left( x + \bar{I} \right)
\]

which we can write as

\[
\psi (\rho_H - \rho_0) \left[ (\rho_H - \rho_0) I - \left( x + \bar{I} \right) \right] + \chi_1 K > 0
\]

The term \( (\rho_H - \rho_0) I - \left( x + \bar{I} \right) \) is just the credit line \( M > 0 \), and so this condition is satisfied and we have \( \frac{dx}{d\bar{I}} > 0 \).

Likewise, we have \( \frac{dx}{dK} < 0 \) iff

\[
\frac{\psi^2}{2} (\rho_H - \rho_0)^2 \left( (\rho_H - \rho_0)^2 I^2 - (\bar{I} + x)^2 \right) \left( \frac{x}{x + \bar{I}} \right) \left( \frac{1}{K} + \chi_1 \frac{R\phi}{\sigma} \right) > 0
\]

The term \( \frac{\psi^2}{2} (\rho_H - \rho_0)^2 \left( (\rho_H - \rho_0)^2 I^2 - (\bar{I} + x)^2 \right) \left( \frac{x}{x + \bar{I}} \right) \left( \frac{1}{K} + \chi_1 \frac{R\phi}{\sigma} \right) > 0 \)
Since \((a^2 - b^2) = (a - b)(a + b)\), the term
\[
\left[ (\rho_H - \rho_0)^2 I^2 - (\bar{I} + x)^2 \right] = \left[ (\rho_H - \rho_0) I - (\bar{I} + x) \right] \left[ (\rho_H - \rho_0) I + (\bar{I} + x) \right]
\]
which is equal to \(M \left[ (\rho_H - \rho_0) I + (\bar{I} + x) \right] > 0\). Therefore we have
\[
\frac{\psi^2}{2} (\rho_H - \rho_0)^2 M \left[ (\rho_H - \rho_0) I + (\bar{I} + x) \right] \left( \frac{x}{x + \bar{I}} \right) \left( \frac{1}{\bar{K}} \right) + \chi_1 R\psi > 0
\]
and so we have \(\frac{dx}{d\bar{K}} < 0\). \(\square\)