Exercise 4
(Sketch of Solutions)

Part a

If $F = v = 0$, then $C(L_{-1}, L) = 0$. Given that there is no cost of adjustment, the problem can be solved period-by-period. The problem of the firm then becomes

$$\max_{L,H} L = R(A, L, H) - \omega(L, H)$$

subject to

$$R(A, L, H) = A(LH)^\alpha$$

$$\omega(L, H) = L(w_0 + w_1 H^\zeta)$$

or

$$\max_{L,H} L = A(LH)^\alpha - L(w_0 + w_1 H^\zeta)$$

FOC:

$$\frac{\partial L}{\partial L} \left[ \alpha A(LH)^{\alpha - 1} H - (w_0 + w_1 H^\zeta) = 0 \right]$$

$$\frac{\partial H}{\partial H} \left[ \alpha A(LH)^{\alpha - 1} L - \zeta L w_1 H^{\zeta - 1} = 0 \right]$$

Dividing equation (1) by equation (2)

$$\frac{\alpha A(LH)^{\alpha - 1} H}{\alpha A(LH)^{\alpha - 1} L} = \frac{w_0 + w_1 H^\zeta}{\zeta L w_1 H^{\zeta - 1}}$$

$$\frac{H}{L} = \frac{w_0 + w_1 H^\zeta}{\zeta L w_1 H^{\zeta - 1}}$$

Rearranging

$$\zeta w_1 H^\zeta = w_0 + w_1 H^\zeta$$

$$H^* = \left( \frac{w_0}{(\zeta - 1) w_1} \right)^{\frac{1}{\zeta}}$$

Rearranging equation (1)
\[ L = \left( \frac{w_0 + w_1 H^\zeta}{\alpha A H^\alpha} \right)^{\frac{1}{1-\alpha}} \] (4)

Replacing equation (3) into equation (4) we get

\[ L^* = \left( \frac{w_0 + w_1 \left( \frac{w_0}{w_1} \right)^{\frac{1}{\zeta-1}}}{\alpha A \left( \frac{w_0}{w_1} \right)^{\frac{1}{\zeta-1}}} \right)^{\frac{1}{1-\alpha}} \]

For \( \omega(L, H) \) to be convex in \( H \), it is sufficient to assume that \( \zeta > 1 \)
\[ \left[ \frac{\partial \omega(L, H)}{\partial H^2} \right] = L \zeta (\zeta - 1) w H^{\zeta-2} > 0 \]. If this is the case \( H^* > 0 \).

The optimum has the property that the firm never adjusts hours worked to changes in the profit shock \( A \) (\( H^* \) is independent of \( A \)). Firms only adjusts employment to profit shocks \( A \). As long as \( \zeta > 1 \), an increase in \( \zeta \) leads to a decrease in \( H \).

**Part b**

When the firm faces costs of adjustment the problem cannot be solved in a period by period basis. The value function when \( v > 0 \) and \( F = 0 \) is

\[ V(A, L_{-1}) = \max_{L, H} \left\{ A(LH)^\alpha - L(w_0 + w_1 H^\zeta) - \frac{v}{2} \left( \frac{L - L_{-1}}{L_{-1}} \right)^2 + \beta E[V(A', L)] \right\} \]

\[ \text{FOC} \]

\[ \frac{\partial V}{\partial L} = \alpha A(LH)^{\alpha-1} H - (w_0 + w_1 H^\zeta) - v \left( \frac{L - L_{-1}}{L_{-1}} \right) + \beta E[V_2(A', L)] = 0 \] (5)

\[ \text{H} \]

\[ \alpha A(LH)^{\alpha-1} L - \zeta Lw_1 H^{\zeta-1} = 0 \] (6)

The Envelope Condition is

\[ V_2(A, L_{-1}) = -v \left( \frac{L - L_{-1}}{L_{-1}} \right) \left( \frac{L - L_{-1}}{L_{-1}} \right) = v \left( \frac{L - L_{-1}}{L_{-1}} \right) \left( \frac{L}{L_{-1}^2} \right) \] (7)

Equation (6) implies

\[ \zeta w_1 H^{\zeta-1} = \alpha A H^{\alpha-1} L^\alpha \]
\[
H^{\zeta - \alpha} = \frac{\alpha A}{\zeta w_1 L^{1-\alpha}} \\
H^* = \left( \frac{\alpha A}{\zeta w_1 L^{1-\alpha}} \right)^{\frac{1}{\zeta - \alpha}}
\]

\(H^*\) now depends on \(L\) and it is not insulated from \(A\) shocks. \(H^*\) and \(L\) are negatively related. It is still the case that an increase in \(\zeta\) leads to a decrease in \(H^*\), all else constant.

To get the Euler equation for employment dynamics (recall that the dynamic behavior of \(L\) affects \(H\) via the last equation above, something that did not happen in part a), update equation (7) one period

\[
V_2(A', L) = v \left( \frac{L' - L}{L} \right) \left( \frac{L'}{L^2} \right)
\]

Replacing equation (9) into equation (5)

\[
\alpha A(LH)^{\alpha - 1} H - (w_0 + w_1 H^*) - v \left( \frac{L - L_{-1}}{L_{-1}} \right) + \beta E \left[ v \left( \frac{L' - L}{L} \right) \left( \frac{L'}{L^2} \right) \right] = 0
\]

Replacing \(H^*\) in

\[
\alpha AL^{\alpha - 1} \left( \frac{\alpha A}{\zeta w_1 L^{1-\alpha}} \right)^{\frac{\zeta - \alpha}{\zeta - 1}} - (w_0 + w_1 \left( \frac{\alpha A}{\zeta w_1 L^{1-\alpha}} \right)^{\frac{\zeta}{\zeta - \alpha}}) - v \left( \frac{L - L_{-1}}{L_{-1}} \right) + \beta E \left[ v \left( \frac{L' - L}{L} \right) \left( \frac{L'}{L^2} \right) \right] = 0
\]

This equation fully describes the evolution of employment over time. The model feature only smoothing incentives and if adjusting employment is very expensive, hour will fluctuate more in response to profitability shocks.

**Part c**

If \(v = 0\) and \(F > 0\), then adjustment costs are purely non-convex.

The value function in this case is

\[
V(A, L_{-1}) = \max \{ V^A(A, L_{-1}), V^N(A, L_{-1}) \}
\]

where

\[
V^A(A, L_{-1}) = \max_{L, H} \left\{ A(LH)^{\alpha} - L(w_0 + w_1 H^*) - F(1) + \beta E[V(A', L)] \right\}
\]

and

\[
V^A(A, L_{-1}) = \max_H \left\{ A(LH)^{\alpha} - L(w_0 + w_1 H^*) + \beta E[V(A', L_{-1})] \right\}
\]
Note that given the max operator in equation (10), the value function \( V(A, L_{-1}) \) is not differentiable everywhere with respect to employment. We then cannot take derivatives with respect to \( L \). The optimal hours worked is still given by equation (8).

The solution has the feature that firms only adjusts employment if the profitability shocks are big enough. If employment does not change, only hours absorb the \( A \) shocks. When the \( A \) shock is big enough, both employment and hours are used to absorb \( A \) shocks.

**Part d**

If \( v > 0 \) and \( F > 0 \), there is a mix of smoothing and non-convex adjustment in the model. The value function in this case is

\[
V(A, L_{-1}) = \max \{ V^A(A, L_{-1}), V^{NA}(A, L_{-1}) \}
\]

where

\[
V^A(A, L_{-1}) = \max_{H,L} \left\{ A(LH) - L(w_0 + w_1H) - F(1) - v \left( \frac{L - L_{-1}}{L_{-1}} \right) + \beta E[V(A', L)] \right\}
\]

and

\[
V^{NA}(A, L_{-1}) = \max_{H} \left\{ A(LH) - L(w_0 + w_1H) + \beta E[V(A', L_{-1})] \right\}
\]

Again, given that the value function \( V(A, L_{-1}) \) is not differentiable everywhere with respect to employment, we cannot differentiate with respect to \( L \). The optimal hours worked is still given by equation (8). The solution has the same features as the one in part c, with the only difference that changes in employment are now smaller when the firm decides to adjust, given the presence of convex costs of adjustment.