Question 1: Parental Income and Children’s Education

Consider a person who lives two periods and has the following preferences:

\[ U = \log(C_1) + \log(C_2) + \log(Y_2), \]

where \( C_1 \) and \( C_2 \) are consumption in periods 1 (youth) and 2 (old age), and \( Y_2 \) is the income of the person’s child, realized in period 2. The agent is thus altruistic and gets pleasure from seeing his child have a high income.

The agent receives an exogenous initial income \( w_1 \) in period 1, which he can devote to three uses: first-period consumption, saving \( (s_1) \) at a positive net interest rate \( r \), or spending on his child’s education \( (e_1) \). In period 2, the agent gets no additional flow income, and spends his accumulated savings on second-period consumption and monetary bequests \( (b_2) \) to his child. At this point, I will allow either for positive or negative bequests, where a negative bequest can be interpreted as the agent dying in debt and leaving his child to pay off the debt. The agent’s flow budget constraints are therefore:

\[ C_1 + e_1 + s_1 = w_1 \]

\[ C_2 + b_2 = (1+r)s_1 \]

Finally, income of the child in period 2 depends on the bequest and on wage income \( w_2 \), which is an increasing concave function of education \( e_1 \) satisfying \( w_2(0) = 0, w'(0) = 8 \) and \( w'(8) = 0 \). Thus:

\[ Y_2 = w_2(e_1) + b_2 \]

The model is deterministic; there is no uncertainty.

(a) Use the first order conditions of the constrained maximization to show that parents’ optimal schooling investment \( e_1^* \) is independent of parental income \( w_1 \). Call the child’s wage implied by the optimal level of schooling \( w_2^* \).

(b) Derive a closed-form solution for optimal \( C_1, C_2, \) and \( b_2 \) in terms of \( w_1, e_1^*, w_2^*, \) and \( r \).
(c) Show that the optimal bequest will be negative iff

\[ w_1 < e_1^* + (2/1+r)w_2^* \]

Provide an intuition for why the optimal bequest will be negative at low levels of parental income \( w_1 \).

Now suppose that we do not allow negative bequests, so that \( b_2 > 0 \). The nonnegativity constraint will be binding for families whose initial income is below the threshold given by the right-hand-side of (5).

(d) Prove that for families with low incomes (in the sense of (5)), optimal schooling will be an increasing function of parental income \( w_1 \). To do this, use the first-order conditions for an alternative version of the initial maximization problem in which we impose \( b_2 = 0 \) (which will be the case for low-income families). You do not need to establish a closed-form solution for \( e_1^* \) in this problem in order to establish that \( e_1 \) will be an increasing function of \( w_1 \).