Recent empirical work documents that routine jobs like assembly line work are disappearing. This question looks at a growth model in which there are two sectors, producing “routine” and “non-routine” output, where technological progress eliminates routine jobs.

**PART I: THE ROUTINE SECTOR**

Producing output in the routine sector requires performance of a continuum of tasks, indexed by \( i \) on the interval \([0,1]\). Letting \( y(i) \) denote the output of task \( i \), routine sector output \( Y_R \) is given by

\[
Y_R = \left[ \int_0^1 y(i)^{(p-1)/p} \, di \right]^{p/(p-1)}
\]

Each task \( i \) can be performed either by capital alone, or by labor alone. Performing task \( i \) using capital alone requires \( k(i) \) units of capital, while using labor alone requires \( l(i) \) units of labor. Tasks are ordered such that the capital is relatively most effective at producing task 0, while labor is relatively most effective at producing task 1. Specifically,

\[
k(i) = i/A \quad ; \quad l(i) = 1 - i
\]

Where \( A \) represents capital embodied technological progress that increases capital’s productivity relative to labor at each task. \( A \) will be time-varying; for now, I suppress the time subscript for simplicity. It is optimal for firms to choose some cutoff task \( \vartheta \) such that firms use capital alone for tasks in the interval \( 0 \leq i < \vartheta \); use labor alone for tasks in the interval \( \vartheta < i \leq 1 \); and are indifferent between using capital and labor for task \( i = \vartheta \). The cutoff task \( \vartheta \) will also be time-varying; I suppress the time subscript for simplicity.

In this part of the problem we are going to find a production function expressing \( Y_R \) as a function of overall capital \( K \) and labor \( L_R \) devoted to the routine sector. The first step is to find optimal task outputs \( y(i) \) for a given \( K, L_R, \) and \( \vartheta \). This involves maximizing (1) subject to:

\[
\int_0^\vartheta y(i)k(i)di = K
\]

\[
\int_\vartheta^1 y(i)l(i)di = L_R, \quad \text{where } k(i) \text{ and } l(i) \text{ are given by (2).}
\]

(a) [20 points] Show that optimal task outputs for \( 0 \leq i \leq \vartheta \) satisfy:

\[
y(i) = (2-p)AK(\vartheta)^{p-2}(i)^{-p}
\]

Analogous math, which you do not need to do, shows that task outputs for \( \vartheta \leq i \leq 1 \) satisfy:

\[
y(i) = (2-p)L_R(1-\vartheta)^{p-2}(1-i)^{-p}
\]
(b) [5 points]. Optimal \( \vartheta \) is such that both (5) and (6) hold for \( i = \vartheta \). Show that \( \vartheta \) satisfies

\[
(7) \quad \vartheta = \frac{(AK)^{1/2}}{[L_R^{1/2} + (AK)^{1/2}]} 
\]

(c) [25 points]. Plug (5) and (6) back into (1). Use (7) and simplify to establish the following production function for the routine sector (up to a scaling constant, which you can ignore):

\[
(8) \quad Y_R = [(AK)^{1/2} + L_R^{1/2}]^2 
\]

**PART II: MACRO MODEL**

There is a representative household with standard preferences over consumption and hours worked in the routine and non-routine sectors, where \( L = L_R + L_N \) is total hours worked:

\[
(9) \quad \text{Max} \sum_0^{\infty} \beta^t U(C_t, L_t) \text{ where } U(C, L) = \log(C) + \log(1-L) 
\]

Aggregate output is a Cobb-Douglas aggregate of output from the routine sector and the non-routine sector. Routine sector output obeys (8), while output in the non-routine sector is simply equal to hours worked in the non-routine sector (i.e. there is no capital in the non-routine sector):

\[
(10) \quad Y_t = Y_R^t \alpha \quad Y_N^{t(1-\alpha)} = [(A_tK_t)^{1/2} + L_R^{1/2}]^{2\alpha} \quad L_N^{t(1-\alpha)} 
\]

The aggregate resource constraint relates output, consumption and investment in a standard way:

\[
(11) \quad Y_t = C_t + K_{t+1} - (1-\delta)K_t 
\]

Where \( \delta \) is the depreciation rate, satisfying \( 0 < \delta < 1 \). Capital-augmenting technology \( A_t \) in the routine sector grows at a steady rate over time. This economy satisfies the welfare theorems, so a competitive equilibrium is equal to the solution of a social planner’s problem of maximizing (9) subject to (10) and (11).

(d) [5 points] Identify state and control variables of the social planner’s problem, and write down the Bellman Equation.

(e) [20 points] Take first order and envelope conditions for the problem. Derive a static equation for optimal labor supply to either sector, as well as an Euler equation for optimal intertemporal consumption. In expressing these static and Euler equations you may find it convenient to define \( w_t \) and \( R_t \) as market prices, which are set equal to marginal products of labor and capital.

(f) [25 points] Derive expressions for the shares of aggregate output representing payments to routine labor, non-routine labor and capital in terms of \( \vartheta_t \) and \( \alpha \). You will need to use (7) as well as your results from (e). What happens to the share of output going to overall labor (routine and nonroutine) in this economy as \( A \) grows over time?