Question 1

Part a

State variables: $a_t$, $h_t$, $w_t$, $y_t$
Choice variables: $a_{t+1}$, $h_{t+1}$, $e_t$, $c_t$

$$V(a_t, h_t) = \max_{c_t, e_t} \{ u(c_t) + \beta E_t V[R(a_t + w_t h_t^\alpha - c_t - e_t), (1 - \delta)h_t + e_t] \}$$

Part b

First Order Conditions

$$u'(c_t) - \beta RE_t \frac{\partial V(\cdot)}{\partial a_{t+1}} = 0 \quad (1)$$

$$-\beta RE_t \frac{\partial V(\cdot)}{\partial a_{t+1}} + \beta E_t \frac{\partial V(\cdot)}{\partial h_{t+1}} = 0 \quad (2)$$

Envelope Conditions

$$\frac{\partial V(\cdot)}{\partial a_t} = \beta RE_t \frac{\partial V(\cdot)}{\partial a_{t+1}} \quad (3)$$

$$\frac{\partial V(\cdot)}{\partial h_t} = \alpha w_t h_t^{\alpha - 1} \beta RE_t \frac{\partial V(\cdot)}{\partial a_{t+1}} + (1 - \delta) \beta E_t \frac{\partial V(\cdot)}{\partial h_{t+1}} \quad (4)$$
Part c

Combining equation (2) and (4)

\[
\frac{\partial V(\cdot)}{\partial h_t} = \alpha w_t h_t^{\alpha-1} \beta R E_t \frac{\partial V(\cdot)}{\partial a_{t+1}} + (1 - \delta) \beta R E_t \frac{\partial V(\cdot)}{\partial a_{t+1}}
\]

\[
\frac{\partial V(\cdot)}{\partial h_t} = [\alpha w_t h_t^{\alpha-1} + (1 - \delta)] \beta R E_t \frac{\partial V(\cdot)}{\partial a_{t+1}} \tag{5}
\]

Replacing equation (1) into equation (5)

\[
\frac{\partial V(\cdot)}{\partial h_t} = [\alpha w_t h_t^{\alpha-1} + (1 - \delta)] u'(c_t)
\]

or

\[
\frac{\partial V(\cdot)}{\partial h_t} = R_{ht} u'(c_t) \tag{6}
\]

where

\[
R_{ht} = \alpha w_t h_t^{\alpha-1} + (1 - \delta) \tag{7}
\]

Note that \(R_{ht}\) is a function of parameters, endogenous state variables and \(w_t\). Therefore, \(R_{ht}\) is stochastic only if \(w_t\) is.

Part d

Combining equation (1) and (3)

\[
\frac{\partial V(\cdot)}{\partial a_t} = u'(c_t) \tag{8}
\]

Forwarding equation (8) one period and plugging it into equation (1) we get one Euler equation

\[
u'(c_t) = \beta R E_t u'(c_{t+1}) \tag{9}\]

Combining equation (1) and (2)

\[
u'(c_t) = \beta E_t \frac{\partial V(\cdot)}{\partial h_{t+1}} \tag{10}\]

Forwarding equation (6) one period and plugging it into equation (10) we get the second Euler equation

\[
u'(c_t) = \beta E_t R_{ht+1} u'(c_{t+1}) \tag{11}\]
Part e

Equation (11) can be rewritten as

\[ u'(c_t) = \beta E_t (R_{ht+1} + 1) u'(c_{t+1}) + \beta \text{cov} (R_{ht+1}, u'(c_{t+1})) \] (12)

Subtracting equation (12) from equation (9)

\[ 0 = \beta R E_t u'(c_{t+1}) - \beta E_t (R_{ht+1} + 1) u'(c_{t+1}) - \beta \text{cov} (R_{ht+1}, u'(c_{t+1})) \]

\[ \beta E_t (R_{ht+1} + 1) u'(c_{t+1}) - \beta R E_t u'(c_{t+1}) = -\beta \text{cov} (R_{ht+1}, u'(c_{t+1})) \]

\[ E_t (R_{ht+1} + 1) E_t u'(c_{t+1}) - R E_t u'(c_{t+1}) = -\text{cov} (R_{ht+1}, u'(c_{t+1})) \]

\[ E_t [R_{ht+1} + 1 - R] = -\frac{\text{cov} (R_{ht+1}, u'(c_{t+1}))}{E_t [u'(c_{t+1})]} \] (13)

We show below that if \( w_t \) is non-stochastic, the equity premium equals zero. Combining equation (9) and (11) we get

\[ RE_t u'(c_{t+1}) = E_t R_{ht+1} u'(c_{t+1}) \]

Given that R is fixed, the equation above implies that \( R_{ht+1} \) and \( u'(c_{t+1}) \) must be negatively correlated. Then \( \text{cov} (R_{ht+1}, u'(c_{t+1})) < 0 \) which implies a positive equity premium.

Part e

If \( w_t \) is non-stochastic, equation (9) and (11) imply that \( R_{ht+1} = R \) which implies a zero equity premium. Also note that if \( R_{ht+1} \) is fixed, \( \text{cov} (R_{ht+1}, u'(c_{t+1})) = 0 \). Equation (13) then implies a zero equity premium.

Now, equation (7) replacing into \( R_{ht+1} = R \)

\[ R = \alpha w_{t+1} h_{t+1}^{\alpha - 1} + (1 - \delta) \]

\[ h_{t+1} = \left( \frac{R - 1 + \delta}{\alpha w_{t+1}} \right)^{\frac{1}{\alpha - 1}} \] (14)

Replacing equation (14) into the equation of movement of human capital we get
\[ e_t = (1 - \delta) h_t + \left( \frac{R - 1 + \delta}{\alpha w_{t+1}} \right)^{\frac{1}{\alpha}} \]  

(15)

**Exercise 2**

Note that current period profit is \( \pi_t = P_t R_t = R_t^{1-\alpha} \). The Bellman’s equation can then be written as

\[ V(S_t) = \max_{R_t} \left\{ R_t^{1-\alpha} + \frac{1}{1 + r} V(S_t - R_t) \right\} \]

The first order condition is

\[ (1 - \alpha) R_t^{-\alpha} - \frac{1}{1 + r} V'(S_{t+1}) = 0 \]  

(16)

The Envelope Condition is

\[ V''(S_t) = \frac{1}{1 + r} V''(S_{t+1}) \]  

(17)

Combining equation (16) and (17)

\[ V'(S_t) = (1 - \alpha) R_t^{-\alpha} \]  

(18)

Forward equation (18) one period and replace it into equation (16)

\[ (1 - \alpha) R_t^{-\alpha} = \frac{1}{1 + r} (1 - \alpha) R_t^{-\alpha} \]

\[ R_t^{-\alpha} = \frac{1}{1 + r} R_t^{-\alpha} \]

\[ R_{t+1} = \left[ \frac{1}{1 + r} \right]^{\frac{1}{\alpha}} \]  

(19)

The growth rate of prices is given by

\[ \frac{P_{t+1}}{P_t} - 1 = \frac{R_{t+1}^{1-\alpha}}{R_t^{1-\alpha}} - 1 = \left( \frac{R_{t+1}}{R_t} \right)^{-\alpha} - 1 \]  

(20)

Replacing equation (19) into equation (20) we get

\[ \frac{P_{t+1}}{P_t} - 1 = \left( \left[ \frac{1}{1 + r} \right]^{\frac{1}{\alpha}} \right)^{-\alpha} - 1 \]
\[
\frac{P_{t+1}}{P_t} - 1 = \left[ \frac{1}{1 + r} \right]^{-1} - 1
\]
\[
\frac{P_{t+1}}{P_t} - 1 = 1 + r - 1
\]
\[
\frac{P_{t+1}}{P_t} - 1 = r
\]

Interpretation: the monopolist has the option of extracting today and buying a bond with the proceeds or extracting tomorrow. If the price increase is less than \( r \), it is not optimal to wait to extract. If the price increase is greater than \( r \), it is not optimal to extract today.

To find \( P_0 \) note that equation (19) can be rewritten as
\[
R_{t+1} = AR_t
\]
where \( A = \left[ \frac{1}{1 + r} \right]^{\frac{1}{2}} < 1 \). Iterating backwards on equation (21) one gets that
\[
R_t = A^t R_0 \quad (22)
\]
Replacing equation (22) into the resource constraint \( \sum_{i=0}^{\infty} R_t = S_0 \)
\[
\sum_{i=0}^{\infty} A^t R_0 = S_0 \]
\[
\frac{1}{1 - A} R_0 = S_0 \]
\[
R_0 = \left( 1 - \left[ \frac{1}{1 + r} \right]^{\frac{1}{2}} \right) S_0
\]
Replacing this last equation into the demand curve for period 0, \( P_0 = R_0^{-\alpha} \)
\[
P_0 = \left[ \left( 1 - \left[ \frac{1}{1 + r} \right]^{\frac{1}{2}} \right) S_0 \right]^{-\alpha}
\]