Question 1

Part a

The state variables are whether the worker is employed or unemployed and asset holdings. Control variables are consumption and the probability of finding a job.

Part b

\[
V_{ut}(A_t) = \max_{c_t, s_t} \{ u(c_t) - C(s_t) + \beta s V_{ct+1}(a_{t+1}) + \beta(1 - s)V_{ut+1}(A_{t+1}) \}
\]

subject to \( A_{t+1} = (1 + r)(A_t + b - c_t) \).

\[
V_{ct}(A_t) = \max_{c_t} \{ u(c_t) + \beta V_{ct+1}(A_{t+1}) \}
\]

subject to \( A_{t+1} = (1 + r)(A_t + w - c_t) \).

Part c

Employed Worker

FOC

\[
u'(c_{ct}) - \beta(1 + r)V_{ct+1}'(A_{t+1}) = 0
\]  

(1)

Envelope Condition

\[
V_{ct}'(A_t) = \beta(1 + r)V_{ct+1}'(A_{t+1})
\]  

(2)

Combining equations (1) and (2)

\[
V_{ct}'(A_t) = u'(c_{ct})
\]  

(3)

Replacing equation (3) into equation (1)

\[
u'(c_{ct}) = \beta(1 + r)u'(c_{ct+1})
\]  

(4)
Unemployed Worker

\[ V_{ut}(A_t) = \max_{c_t, s_t} \{ u(c_t) - C(s_t) + \beta s V_{et+1}(a_{t+1}) + \beta(1-s)V_{ut+1}(A_{t+1}) \} \]

subject to \( A_{t+1} = (1 + r)(A_t + b - c_t) \).

FOC

\[ c_t \]

\[ u'(c_{ut}) - \beta(1 + r)s V'_{et+1}(A_{t+1}) - \beta(1 + r)(1 - s)V'_{ut+1}(A_{t+1}) = 0 \quad (5) \]

\[ s_t \]

\[ -C'(s_t) + \beta V_{et+1}(A_{t+1}) - \beta V_{ut+1}(A_{t+1}) = 0 \quad (6) \]

Envelope

\[ V'_{ut}(A_t) = \beta(1 + r)s V'_{et+1}(A_{t+1}) + \beta(1 + r)(1 - s)V'_{ut+1}(A_{t+1}) \quad (7) \]

Combining equation (5) and (7)

\[ V'_{ut}(A_t) = u'(c_{ut}) \quad (8) \]

Replacing equation (8) into (5)

\[ u'(c_{ut}) = \beta(1 + r)[s u'(c_{et+1}) + (1 - s)u'(c_{ut+1})] \quad (9) \]

Part d

Differentiating equation (6) with respect to \( A_t \)

\[ -C''(s_t) \frac{ds_t}{dA_t} + \beta V''_{et+1}(A_{t+1}) \frac{dA_{t+1}}{dA_t} - \beta V''_{ut+1}(A_{t+1}) \frac{dA_{t+1}}{dA_t} = 0 \]

\[ C''(s_t) \frac{ds_t}{dA_t} = \beta \frac{dA_{t+1}}{dA_t} [V'_{et+1}(A_{t+1}) - V'_{ut+1}(A_{t+1})] \]

Using equations (3) and (8)

\[ \frac{ds_t}{dA_t} = \frac{\beta}{C''(s_t)} \frac{dA_{t+1}}{dA_t} [u'(c_{et+1}) - u'(c_{ut+1})] \]

Given that \( C''(s_t) > 0 \), \( \frac{dA_{t+1}}{dA_t} > 0 \) and \( u'(c_{ut+1}) - u'(c_{et+1}) < 0 \), then \( \frac{ds_t}{dA_t} < 0 \).
Part e

If $\beta(1 + r) = 1$ equation (9) becomes

$$u'(c_{ut}) = su'(c_{et+1}) + (1 - s)u'(c_{ut+1})$$

Given that the utility function is strictly concave and that for any $A_t c_{et} > c_{ut}$, then

$$u'(c_{ut}) = su'(c_{et+1}) + (1 - s)u'(c_{ut+1}) < u'(c_{ut+1})$$

Given strict concavity, $u'(c_{ut}) < u'(c_{ut+1})$ implies $c_{ut} > c_{ut+1}$.

Part f

When $\beta(1 + r) = 1$, equation (4) implies

$$c_{et} = c_{et+1} = c$$

For assets not to go to zero or infinity over time, constant consumption implies that assets also have to be constant, which implies

$$A = (1 + r)(w + A - c)$$

$$\frac{A_t}{1 + r} = A = w - c$$

$$c = w + \frac{r}{1 + r}A_t$$

The value of being employed with asset holdings equal to $A_t$ is then

$$V_{et}(A_t) = \sum_{t=0}^{\infty} \beta^t u(w + \frac{r}{1 + r} A_t)$$

$$V_{et}(A_t) = \frac{1}{1 - \beta} u(w + \frac{r}{1 + r} A_t)$$
Question 2

Part a

Utility function is \( U(c_{it}) = -\frac{1}{a} \exp(-ac_{it}) \)

The household \( i \) problem is to maximize

\[
V \equiv -\frac{1}{a} \exp(-ac_{i1}) + \beta E_t \left[ -\frac{1}{a} \exp(-ac_{i2}) \right] \text{ subject to } c_{i2} = y_{i2} + R(y_{i1} - c_{i1})
\]

The Lagrangian can then be written as

\[
L = -\frac{1}{a} \exp(-ac_{i1}) + \beta E_t \left[ -\frac{1}{a} \exp(-ac_{i2}) \right] + \lambda \left[ y_{i2} + R(y_{i1} - c_{i1}) - c_{i2} \right]
\]

**FOC**

\[
c_{i1} \quad \exp(-ac_{i1}) - \lambda R = 0
\]

\[
c_{i2} \quad \beta E \left[ \exp(-ac_{i2}) \right] - \lambda = 0
\]

Combining these two FOC we get

\[
\exp(-ac_{i1}) = R\beta E \left[ \exp(-ac_{i2}) \right]
\]

Replacing \( c_{i2} = y_{i2} + R(y_{i1} - c_{i1}) \) into the equation above

\[
\exp(-ac_{i1}) = R\beta E \left[ \exp(-a [y_{i2} + R(y_{i1} - c_{i1})]) \right]
\]

\[
\exp(-ac_{i1}) = R\beta \exp \left( -aR(y_{i1} - c_{i1}) \right) E \left[ \exp(-ay_{i2}) \right]
\]

Given that \( y_{i2} \) is log normally distributed, \( E \left[ \exp(-ay_{i2}) \right] = \exp(-ay_{i2} + \frac{a^2}{2} \sigma^2) \). Replacing into the equation above

\[
\exp(-ac_{i1}) = R\beta \exp \left( -aR(y_{i1} - c_{i1}) \right) \exp(-ay_{i2} + \frac{a^2}{2} \sigma^2)
\]

\[
\exp(-ac_{i1}) = R\beta \exp \left( -aR(y_{i1} - c_{i1}) - ay_{i2} + \frac{a^2}{2} \sigma^2 \right)
\]

\[
-ae_{i1} = \ln R\beta - aR(y_{i1} - c_{i1}) - ay_{i2} + \frac{a^2}{2} \sigma^2
\]

\[
e_{i1} = -a^{-1} \ln R\beta + R(y_{i1} - c_{i1}) + y_{i2} - \frac{a}{2} \sigma^2
\]

\[
e_{i1} = -\frac{a^{-1}}{1+R} \ln R\beta + \frac{R}{1+R} y_{i1} + \frac{y_{i2}}{1+R} - \frac{a}{2} \frac{\sigma^2}{1+R}
\]
Part b
Aggregate consumption is

$$c_1 = \int c_1 \, di = \int \left[ -\frac{a^{-1}}{1+\sigma} \ln R\beta + \frac{R}{1+\sigma} y_1 + \frac{\sigma}{2} \frac{\sigma^2}{1+\sigma} \right] di$$

Assuming a measure one of agents

$$c_1 = -\frac{a^{-1}}{1+\sigma} \ln R\beta + \frac{R}{1+\sigma} y_2 - \frac{a}{2} \frac{\sigma^2}{1+\sigma}$$

For this economy to be in equilibrium $c_1 = y_1 = \int y_1 \, di$

$$y_1 = -\frac{a^{-1}}{1+\sigma} \ln R\beta + \frac{R}{1+\sigma} y_2 - \frac{a}{2} \frac{\sigma^2}{1+\sigma}$$

$$\frac{1}{1+\sigma} y_1 = -\frac{a^{-1}}{1+\sigma} \ln R\beta + \frac{1}{1+\sigma} y_2 - \frac{a}{2} \frac{\sigma^2}{1+\sigma}$$

$$y_1 = -a^{-1} \ln R\beta + y_2 - \frac{a}{2} \sigma^2$$

$$\ln R\beta = a (y_2 - y_1 - \frac{a}{2} \sigma^2)$$

$$R = \beta^{-1} \exp \left[ a \left( y_2 - y_1 - \frac{a}{2} \sigma^2 \right) \right]$$

Comparative statics

$$\frac{dR}{d\beta} = -\beta^{-2} \exp \left[ a \left( y_2 - y_1 - \frac{a}{2} \sigma^2 \right) \right] < 0$$

$$\frac{dR}{dy_1} = -a \beta^{-1} \exp \left[ a \left( y_2 - y_1 - \frac{a}{2} \sigma^2 \right) \right] < 0$$

$$\frac{dR}{dy_2} = a \beta^{-1} \exp \left[ a \left( y_2 - y_1 - \frac{a}{2} \sigma^2 \right) \right] > 0$$

$$\frac{dR}{d\sigma^2} = -\frac{a}{2} \beta^{-1} \exp \left[ a \left( y_2 - y_1 - \frac{a}{2} \sigma^2 \right) \right] < 0$$

Part c

$$\frac{dR}{da} = \exp \left[ a \left( y_2 - y_1 - \frac{a}{2} \sigma^2 \right) \right] \left[ (y_2 - y_1 - \frac{a}{2} \sigma^2) - \frac{a}{2} \sigma^2 \right]$$

$$\frac{dR}{da} = \exp \left[ a \left( y_2 - y_1 - \frac{a}{2} \sigma^2 \right) \right] \left[ (y_2 - y_1) - a \sigma^2 \right]$$

Given that $\exp \left[ a \left( y_2 - y_1 - \frac{a}{2} \sigma^2 \right) \right] > 0$, the sign of $\frac{dR}{da}$ depends on the sign of $\left[ (y_2 - y_1) - a \sigma^2 \right]$. The sign of $\left[ (y_2 - y_1) - a \sigma^2 \right]$ is unambiguous when $y_2 < y_1$ but ambiguous if $y_2 > y_1$.  

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