DIRECTIONS: answer both questions. Use a separate booklet for each answer. Show all your work; partial credit will be given. You have two hours. Good luck!

I. Consumption, Unemployment and Search

Assume that household preferences and the dynamic budget constraint are given by:

\[
\begin{align*}
\text{(*)} & \quad \max E_0 \sum_{t=0}^{\infty} \beta^t [u(c_t) - C(s_t)] \\
A_{t+1} &= (1+r)(A_t + y_t - c_t),
\end{align*}
\]

where \(u(.)\) is an increasing and concave utility from consumption; \(C\) is a utility cost of searching for a job, discussed further below; \(A\) refers to financial wealth, \(y\) is flow income, and \(c\) is consumption; \(r > 0\) is the return on a risk-free asset, the only form of financial wealth; and \(0 < \beta < 1\).

In any period, households can either be unemployed or employed. Employed households receive \(y_t = w\), where \(w\) is the wage, which is time-invariant. Unemployed households receive a time-invariant unemployment benefit \(y_t = b < w\).

Unemployed households can expend effort to search for a job. Assume that an unemployed household at time \(t\) can choose the probability \(s_t\) that they find a job that starts in period \(t+1\). To clarify the timing: if you are unemployed at time \(t\), your \(y_t\) will equal \(b\) no matter what \(s_t\) you choose. However, if you find a job in period \(t\), then your \(y_t\) will equal \(w\) starting in period \(t+1\).

Search effort is costly to the household. The utility cost of searching is given by \(C(s_t)\), which is an increasing and convex function: \(C'(s) > 0\) and \(C''(s) > 0\). To insure an interior solution for \(s\), we can assume that \(C(0) = 0\); \(C'(0) = 0\) and that \(C'(s)\) approaches \(\infty\) in the limit as \(s\) approaches 1.

For simplicity, assume that jobs last forever: once a household is employed, it will stay employed and will earn \(w\) for ever. Therefore, employed households never have to search, and they will have \(C(s) = C(0) = 0\) each period. For employed households, we can simplify preferences by omitting the \(C(s)\) term from (*) above.

(a) Clearly, one state variable of the household is whether or not it is employed or unemployed. Identify the minimum set of additional state and control variables necessary to characterize this problem.
(b) Write down Bellman Equations defining $V_{ut}$ and $V_{et}$, the value functions of a household that is unemployed or employed at the start of period $t$. Write $V_{ut}$ and $V_{et}$ as explicit functions of all other state variables besides whether the household is employed or unemployed. Also, write these equations so that uncertainty is accounted for explicitly; in other words, don't use $E_t$ to capture expectations, but write out any expected values explicitly in terms of underlying future outcomes and their conditional probabilities. [Hint: $V_{et+1}$ should appear somewhere in the Bellman Equation for $V_{ut}$].

(c) Write down first order conditions and envelope conditions for all state and control variables from the Bellman Equations for $V_{ut}$ and $V_{et}$. Derive an Euler Equation linking $c_{et}$ and $c_{et+1}$, the consumption of an employed worker in $t$ and $t+1$ Derive an Euler Equation linking $c_{ut}$, $c_{ut+1}$, $c_{et+1}$, and $s_t$, where $c_{ut}$ refers to $c$ of an unemployed worker at $t$.

(d) Using your results from (c), establish the following result: for an unemployed agent, optimal search effort $s_t$ is a decreasing function of financial wealth, $A_t$. Provide an intuition for this result. In establishing this result, you may use the following (without having to derive them): (1) given $A_t$, $c_{et} > c_{ut}$ for all $t$; (2) for any agent, optimal financial wealth next period, $A_{t+1}$, is increasing in financial wealth this period: $d(A_{t+1})/d(A_t) > 0$.

(e) Using your results from (c), establish the following result: for an agent that is unemployed in periods $t$ and $t+1$, consumption is decreasing over time: $c_{ut} > c_{ut+1}$. Provide an intuition for this result. In establishing this result, you may use the results (1) and (2) from part (d), and you may also assume that $\beta(1+r) = 1$.

(f) Derive a closed form solution for the optimal level of $c_{et}$.

II. Equilibrium Interest Rates in a Heterogeneous Agent Endowment Economy

This problem asks you to solve for equilibrium interest rates in an endowment economy with incomplete markets in which agents face both idiosyncratic risk (as in Huggett (1993)) and aggregate risk. To make the problem tractable, I assume CARA (exponential) utility and lognormal shocks, as in your recent problem set. I remind you of the following:

DEFINITION: a random variable $X$ is lognormally distributed if $\log(X)$ is normal.

FACT: if $\log(X)$ is a normal random variable with mean $\mu$ and variance $\sigma^2$, then

$$E(X) = \exp(\mu + (1/2) * \sigma^2), \text{ where } \exp(.) \text{ is the exponential function.}$$

Assume that there are a large number of agents in the economy, indexed by $i$. There are two periods. Each agent has identical preferences and dynamic budget constraint:

$$\max U(c_{i1}) + \beta E_1 U(c_{i2}), \text{ s.t. } c_{i2} = (1+r)(y_{i1} - c_{i1}) + y_{i2}$$
where $c$ and $y$ refer to consumption and endowments, and $r$ is the rate of return on risk-free bonds. The interest rate $r$ will be determined in equilibrium. Endowments are perishable, so bonds are the only way that households can transfer consumption over time; there are no other securities markets to insure against risk. Assume that period utility is CARA (exponential):

$$U(c_{it}) = (-1/a) \exp(-ac_{it}),$$

where $a > 0$ is the coefficient of absolute risk aversion (and $1/a$ is the intertemporal elasticity of substitution).

First period endowments are given by $y_{i1} = y_1 + u_{i1}$, where $y_1$ is the aggregate mean endowment in period 1, common to all $i$, and where $u_{i1}$ is a mean-zero idiosyncratic shock, i.i.d. across agents. Second period endowments are given by $y_{i2} = y_2 + u_{i2}$, where $y_2$ is aggregate mean endowment in period 2, known to all agents in period 1, and $u_{i2}$ is a mean-zero idiosyncratic variable, i.i.d. across agents. Specifically, assume that $u_{i2}$ is normally distributed with mean zero and variance $\sigma_2^2$, so that $y_{i2}$ is normally distributed with mean $y_2$ and variance $\sigma_2^2$ for all households. Note that $y_2$, aggregate mean endowment in period 2, need not equal $y_1$; in other words, there may be an aggregate shock to income in period one relative to period 2. Also note that I did not specify the distribution of $u_{i1}$; the distribution of $u_{i1}$ is irrelevant, so long as it is mean zero and i.i.d. across agents.

Since endowments are perishable, in the aggregate it must be true that all endowment is consumed in both period one and period two. Letting $c_1$ and $c_2$ denote average consumption across agents in periods 1 and 2, we have $c_1 = y_1$ and $c_2 = y_2$. This assumption implies that bonds are in zero net supply in the first period (note that there will be no bond market in the second period, since the economy ends after the second period). However, since there are idiosyncratic income shocks in period one, there will be nontrivial trade in the bond market; households with good income shocks in period 1 will buy bonds in period one from people with bad shocks.

(a) Solve for optimal individual consumption in period one ($c_{i1}$) as a function of individual endowment $y_{i1}$, the equilibrium interest rate $r$, and other model parameters.

(b) Find a closed-form expression for the equilibrium interest rate that clears the bond market in period 1. [Hint: by Walras' Law, the bond market will clear as long as the goods market clears, that is, as long as $c_1 = y_1$. How do changes in $\beta$, $\sigma_2^2$, $y_1$ and $y_2$ affect $r$? Provide an intuition for the sign of each effect.

(c) Show that the sign of the impact of the utility parameter $a$ on $r$ is unambiguous when $y_2 \leq y_1$, but ambiguous when $y_2 > y_1$. Describe intuitively the two competing forces affecting the sign of this impact when $y_2 > y_1$, and explain why the sign is unambiguous when $y_2 \leq y_1$. [Hint: remember that the parameter $a$ governs both risk aversion and the intertemporal elasticity of substitution].