I. Optimal consumption with non-time-separable utility

Consider the following non-time-separable preference function similar to Weil:

\[(1) \quad V(w) = \text{Max} \, \left[ \frac{c^{\alpha} + (\beta \, E \, V(w'))^{\sigma/\alpha}}{c} \right]^{\alpha/\sigma}\]

where \(\sigma < 1\), \(\alpha < 1\) and \(0 < \beta < 1\); where \(w\) and \(w'\) denote current period and next period's wealth; where \(c\) denotes current period consumption; and where \(E\) denotes expectations as of the current period.

The household's dynamic budget constraint is given by

\[(2) \quad w' = R' \, (w - c)\]

where \(R'\) is a stochastic gross return on the household's assets, realized next period and unknown as of the current period. Assume that \(R'\) is an i.i.d. random variable with mean \(R\). Note that there is no labor or transfer income in this model.

PART ONE (75 points): Solve for the value function and optimal decision rule, using the guess and verify method. Specifically, first guess that the value function takes the form:

\[(3) \quad V(w) = A \, w^\alpha, \quad \text{where } A \text{ is some positive constant.}\]

(a) Use this guess to solve the RHS of (1). Show that optimal consumption is a linear function of current wealth. Letting \(\mu\) denote the optimal marginal propensity to consume out of wealth, write \(\mu\) as a function of \(A\) and model parameters. Prove that \(\mu\) is between 0 and 1. For a given \(A\), analyze how changes in the mean interest rate \(R\) affect \(\mu\)—for what parameter values is the interest elasticity of current saving positive, zero or negative?

(b) Using your solution to part (a), verify that your guess (3) is correct and write down an expression for \(A\) in terms of \(\mu\) and model parameters. (In principle you can solve the two equations in two unknowns to express \(A\) and \(\mu\) in terms of the model's underlying parameters, but you do not need to do this).

PART TWO (25 points): Define the equity premium puzzle and the risk free rate puzzle. Discuss whether preferences such as (1) can help resolve each of these puzzles. No math is needed for this part—your answer will be graded according to your grasp of the concepts and intuition (although if you think equations can help you present your intuition, you can use them).
II. Stochastic Growth with Capital Utilization

Consider the following problem. Households maximize the usual time-separable discounted expected utility function, where period utility is logarithmic in the difference between consumption $C$ and the disutility of hours worked $L$. The horizon is infinite.

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \ln(C_t - BL_t)$$

where $B > 0$ and $0 < \beta < 1$. Households own one asset, physical capital, and have access to a production technology for making output from capital and labor, as follows:

$$Y_t = A_t L_t^{\alpha} (u_t K_t)^{1-\alpha}$$

where $0 < \alpha < 1$; $Y_t$ is output at time $t$; $K_t$ is capital at time $t$, which is predetermined as of $t$; $L_t$ is labor input at time $t$, which households choose each period; $A_t$ is a stochastic exogenous technology shock, realized at time $t$ and known to households as of period $t$; and $u_t$ is the utilization of capital, which households choose each period. Capital is more productive if households choose to utilize it more intensively. Intuitively, $K$ could be the number of machines that the household owns, while $u$ would represent how many hours per week the household uses each machine.

Output can be used either for consumption or for investment in capital:

$$Y_t = C_t + I_t$$

Finally, the household's capital accumulation is as follows:

$$K_{t+1} = K_t + I_t - DK_t u_t^{1+\phi}$$

where $D > 0$ and $\phi > 0$ are parameters governing the depreciation of capital. The more heavily capital is utilized, the more rapidly it depreciates.

(a) Identify state and choice variables and write down the Bellman Equation at $t$.

(b) Use the first order conditions from the household's choice problem to find closed-form decision rules expressing both optimal labor supply $L_t$ and optimal utilization $u_t$ in terms of the model's state variables and parameters. Determine the sign of the contemporaneous impact of each state variable on $L$ and $u$, and explain the intuition for the sign of each of these impacts. Let $W_t$ denote the marginal product of labor at time $t$, and rewrite the labor supply equation in terms of $W_t$.

(c) Derive the Euler Equation linking optimal consumption in periods $t$ and $t+1$ to state variables and model parameters. Let $R_{t+1}$ denote the marginal product of capital at $t+1$; rewrite the Euler equation using $R_{t+1}$, and determine the sign of the impact of changes in state variables at $t+1$ on $R_{t+1}$.