

Online Appendix for “Regularized Partial Least Squares Estimation of an Approximate Envelope Model for Supervised Dimension Reduction in Time Series Forecasting”¹

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Abstract

This online appendix is organized into five parts. Appendix A provides the proofs of Theorems 1-2, 4, 5(b), 6(b), and 7 as well as Example 1 and some further remarks. Appendix B presents Lemmas B-1 to B-5 as well as the proofs of these lemmas. Appendix C gives Lemmas C-1 to C-9 as well as their proofs. Appendix D provides the proof of Proposition 1. In addition, Appendix D presents Lemmas D-1 to D-11 as well as their proofs. Finally, Appendix E gives some key lemmas that are used in the proofs of Theorems 4 and 7. More specifically, Appendix E presents Lemmas E-1 to E-16 as well as the proofs of these lemmas. Moreover, Remark E-17 is also included at the end of Appendix E.

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JEL classification: C32, C33, C38, C52, C53, C55, C58.

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Appendix A: Proofs of Theorems 1-2, 4, 5(b), 6(b), and 7

Proof of Theorem 1:

To proceed, note first that, given expression $X_t = \mu_X + \Gamma(F_t - \mu_F) + u_t$, we can write

$$(X_T - \mu_X)' \beta = (F_T - \mu_F)' \Gamma' \beta + u_T' \beta. \quad (1)$$

Now, consider first the term $u_T' \beta$. Note that, by condition (iii) of this theorem,

$$\beta = \frac{L_1 \varphi}{\sqrt{\lambda_1}}, \text{ where } L_1 = \frac{\Gamma}{\sqrt{\lambda_1}}$$

from which it follows that

$$\begin{aligned} & E \left[(u_T' \beta)^2 \right] \\ = & E \left[\left(\frac{u_T' L_1 \varphi}{\sqrt{\lambda_1}} \right)^2 \right] \\ = & \frac{\varphi'}{\sqrt{\lambda_1}} \frac{\Gamma'}{\sqrt{\lambda_1}} E[u_T u_T'] \frac{\Gamma}{\sqrt{\lambda_1}} \frac{\varphi}{\sqrt{\lambda_1}} \\ = & \frac{\varphi'}{\sqrt{\lambda_1}} \frac{\Gamma'}{\sqrt{\lambda_1}} \Sigma_{u,T} \frac{\Gamma}{\sqrt{\lambda_1}} \frac{\varphi}{\sqrt{\lambda_1}} \\ \leq & \lambda_{\max}(\Sigma_{u,T}) \frac{\varphi'}{\sqrt{\lambda_1}} \frac{\Gamma' \Gamma}{\lambda_1} \frac{\varphi}{\sqrt{\lambda_1}} \\ = & \lambda_{\max}(\Sigma_{u,T}) \frac{\|\varphi\|_2^2}{\lambda_1} \left(\text{since } \frac{\Gamma' \Gamma}{\lambda_1} = I_K \text{ and } \varphi' \varphi = \|\varphi\|_2^2 \right) \\ \leq & \frac{C^* K \bar{c}^2}{\lambda_1} \left(\text{since, by assumptions (i) and (ii) of this theorem, there exist constants} \right. \\ & \left. C^* \text{ and } \bar{c} \text{ such that } C^* \geq \sup_t \lambda_{\max}(\Sigma_{u,t}) \text{ and } K \bar{c}^2 \geq K \left(\max_{1 \leq g \leq K} |\varphi_{\cdot g}| \right)^2 \geq \|\varphi\|_2^2 \right) \\ = & O \left(\frac{1}{\lambda_1} \right) \end{aligned}$$

so that, upon application of the Markov's inequality, we deduce that

$$|u_T' \beta| = \left| \frac{u_T' L_1 \varphi}{\sqrt{\lambda_1}} \right| = O_p \left(\frac{1}{\sqrt{\lambda_1}} \right). \quad (2)$$

Next, consider the term $(F_T - \mu_F)' \Gamma' \beta$. For this term, note that

$$\begin{aligned}
& (F_T - \mu_F)' \Gamma' \beta \\
= & (F_T - \mu_F)' \Gamma' \frac{\Gamma}{\sqrt{\underline{\lambda}_1}} \frac{\varphi}{\sqrt{\underline{\lambda}_1}} \\
& \left(\text{since } \beta = \frac{L_1 \varphi}{\sqrt{\underline{\lambda}_1}} \text{ with } L_1 = \frac{\Gamma}{\sqrt{\underline{\lambda}_1}} \text{ by condition (iii) of this theorem} \right) \\
= & (F_T - \mu_F)' \frac{\Gamma' \Gamma}{\underline{\lambda}_1} \varphi \\
= & (F_T - \mu_F)' \varphi \left(\text{since } \frac{\Gamma' \Gamma}{\underline{\lambda}_1} = I_K \text{ by expression (7) of the main paper} \right). \tag{3}
\end{aligned}$$

It follows by expressions (2) and (3) that

$$\begin{aligned}
(X_T - \mu_X)' \beta &= (F_T - \mu_F)' \Gamma' \beta + u_T' \beta \\
&= (F_T - \mu_F)' \varphi + O_p \left(\frac{1}{\sqrt{\underline{\lambda}_1}} \right) \\
&= (F_T - \mu_F)' \varphi + O_p \left(\frac{1}{\sqrt{\lambda_1}} \right), \tag{4}
\end{aligned}$$

where the last equality above follows from part (a) of Lemma D-3, which shows that there exists an absolute constant $c^* > 1$ such that²

$$0 < \frac{1}{c^*} < \frac{\lambda_1}{\underline{\lambda}_1} \leq c^* < \infty \text{ for all } N \text{ and } T \text{ sufficiently large. } \square$$

Proof of Theorem 2:

²Part (a) of Lemma D-3 and its proof is given in Appendix D below.

To show part (a), note that, in this case $\bar{\Sigma}_u = 0$; and, for any integer $k \geq 0$, we have

$$\begin{aligned}
\frac{\bar{\sigma}'_{Xy} \bar{\Sigma}_X^k \bar{\sigma}_{Xy}}{\lambda_1^{k+1}} &= \frac{\beta' \bar{\Sigma}_X^{k+2} \beta}{\lambda_1^{k+1}} \quad (\text{since } \bar{\sigma}_{Xy} = \bar{\Sigma}_X \beta \text{ in the many-predictors regression case}) \\
&= \frac{\beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{k+2} \beta}{\lambda_1^{k+1}} \quad (\text{by Assumption 4-2 given that } \bar{\Sigma}_u = 0) \\
&= \left(\frac{\phi}{\sqrt{\underline{\Delta}_{1,1}}} \right)' \frac{L'_1 [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{k+2} L_1}{\lambda_1^{k+1}} \frac{\phi}{\sqrt{\underline{\Delta}_{1,1}}} \\
&\quad \left(\text{since } \beta = \frac{L_1 \phi}{\sqrt{\underline{\Delta}_{1,1}}} \text{ by Assumption 4-4(a)} \right) \\
&= \left(\frac{\lambda_1}{\underline{\Delta}_{1,1}} \right) \frac{\phi' L'_1 (L_1 \underline{\Delta}_1^{k+2} L'_1 + L_2 \underline{\Delta}_2^{k+2} L'_2) L_1 \phi}{\lambda_1^{k+2}} \\
&\quad \left(\text{since, } \bar{\Sigma}_X^{k+2} = [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{k+2} = L_1 \underline{\Delta}_1^{k+2} L'_1 + L_2 \underline{\Delta}_2^{k+2} L'_2 \right) \\
&= \left(\frac{\lambda_1}{\underline{\Delta}_{1,1}} \right) \frac{\phi' \underline{\Delta}_1^{k+2} \phi}{\lambda_1^{k+2}} \\
&= \left(\frac{\lambda_1}{\underline{\Delta}_{1,1}} \right) \frac{1}{\lambda_1^{k+2}} \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^{k+2} \phi_g^2 \tag{5}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\bar{\Sigma}_X^k \bar{\sigma}_{Xy}}{\lambda_1^{k+1}} &= \frac{\bar{\Sigma}_X^{k+1} \beta}{\lambda_1^{k+1}} \quad (\text{since } \bar{\sigma}_{Xy} = \bar{\Sigma}_X \beta) \\
&= \frac{\bar{\Sigma}_X^{k+1}}{\lambda_1^{k+1}} \frac{L_1 \phi}{\sqrt{\underline{\Delta}_{1,1}}} \quad \left(\text{since } \beta = \frac{L_1 \phi}{\sqrt{\underline{\Delta}_{1,1}}} \text{ by Assumption 4-4 (a)} \right) \\
&= \sqrt{\frac{\lambda_1}{\underline{\Delta}_{1,1}}} \frac{[L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{k+1} L_1 \phi}{\lambda_1^{k+\frac{3}{2}}} \quad (\text{by Assumption 4-2 given that } \bar{\Sigma}_u = 0 \text{ in this case}) \\
&= \sqrt{\frac{\lambda_1}{\underline{\Delta}_{1,1}}} \frac{(L_1 \underline{\Delta}_1^{k+1} L'_1 + L_2 \underline{\Delta}_2^{k+1} L'_2) L_1 \phi}{\lambda_1^{k+\frac{3}{2}}} \\
&\quad \left(\text{since } [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{k+1} = L_1 \underline{\Delta}_1^{k+1} L'_1 + L_2 \underline{\Delta}_2^{k+1} L'_2 \right) \\
&= \sqrt{\frac{\lambda_1}{\underline{\Delta}_{1,1}}} L_1 \frac{\underline{\Delta}_1^{k+1}}{\lambda_1^{k+\frac{3}{2}}} \phi
\end{aligned}$$

Now, write

$$\begin{aligned}\bar{\beta} &= G (G' \bar{\Sigma}_X G)^{-1} G' \bar{\sigma}_{Xy} \\ &= \frac{G D_{\lambda, \kappa}^{-1}}{\sqrt{\lambda_1}} \left[\frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right]^{-1} \frac{D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}}.\end{aligned}$$

where

$$D_{\lambda, \kappa} = \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{\kappa - \frac{1}{2}} \end{pmatrix} \text{ and } G = \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{pmatrix}$$

Next, making use of expression (5), we obtain

$$\begin{aligned}
& \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
&= \left\{ \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_{Xy} \bar{\Sigma}_X \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \end{pmatrix} \frac{\bar{\Sigma}_X}{\lambda_1} \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{pmatrix} \right. \\
&\quad \times \left. \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \right\} \\
&= \begin{pmatrix} \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} / \lambda_1^2 & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} / \lambda_1^3 & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^\kappa \bar{\sigma}_{Xy} / \lambda_1^{\kappa+1} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} / \lambda_1^3 & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^3 \bar{\sigma}_{Xy} / \lambda_1^4 & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa+1} \bar{\sigma}_{Xy} / \lambda_1^{\kappa+2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^\kappa \bar{\sigma}_{Xy} / \lambda_1^{\kappa+1} & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa+1} \bar{\sigma}_{Xy} / \lambda_1^{\kappa+2} & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{2\kappa-1} \bar{\sigma}_{Xy} / \lambda_1^{2\kappa} \end{pmatrix} \\
&= \left(\frac{\lambda_1}{\lambda_{1,1}} \right) \begin{pmatrix} \sum_{g=1}^\kappa \lambda_{1,g}^3 \phi_g^2 / \lambda_1^3 & \sum_{g=1}^\kappa \lambda_{1,g}^4 \phi_g^2 / \lambda_1^4 & \cdots & \sum_{g=1}^\kappa \lambda_{1,g}^{\kappa+2} \phi_g^2 / \lambda_1^{\kappa+2} \\ \sum_{g=1}^\kappa \lambda_{1,g}^4 \phi_g^2 / \lambda_1^4 & \sum_{g=1}^\kappa \lambda_{1,g}^5 \phi_g^2 / \lambda_1^5 & \cdots & \sum_{g=1}^\kappa \lambda_{1,g}^{\kappa+3} \phi_g^2 / \lambda_1^{\kappa+3} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{g=1}^\kappa \lambda_{1,g}^{\kappa+2} \phi_g^2 / \lambda_1^{\kappa+2} & \sum_{g=1}^\kappa \lambda_{1,g}^{\kappa+3} \phi_g^2 / \lambda_1^{\kappa+3} & \cdots & \sum_{g=1}^\kappa \lambda_{1,g}^{2\kappa+1} \phi_g^2 / \lambda_1^{2\kappa+1} \end{pmatrix} \\
&\quad \left(\text{since } \frac{\bar{\sigma}'_{Xy} \bar{\Sigma}_X^k \bar{\sigma}_{Xy}}{\lambda_1^{k+1}} = \left(\frac{\lambda_1}{\lambda_{1,1}} \right) \frac{1}{\lambda_1^{k+2}} \sum_{g=1}^\kappa \lambda_{1,g}^{k+2} \phi_g^2 \right)
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \right. \\
&\quad \times \left(\frac{\lambda_1}{\lambda_{1,1}} \right) \begin{pmatrix} \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^3 \phi_g^2 / \lambda_1^3 & \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^4 \phi_g^2 / \lambda_1^3 & \cdots & \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^{\kappa+2} \phi_g^2 / \lambda_1^3 \\ \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^4 \phi_g^2 / \lambda_1^3 & \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^5 \phi_g^2 / \lambda_1^3 & \cdots & \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^{\kappa+3} \phi_g^2 / \lambda_1^3 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^{\kappa+2} \phi_g^2 / \lambda_1^3 & \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^{\kappa+3} \phi_g^2 / \lambda_1^3 & \cdots & \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^{2\kappa+1} \phi_g^2 / \lambda_1^3 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \Bigg\} \\
&= \left(\frac{\lambda_1}{\lambda_{1,1}} \right) \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \begin{pmatrix} \underline{\lambda}_{1,1}^3 \phi_1^2 / \lambda_1^3 & \underline{\lambda}_{1,2}^3 \phi_2^2 / \lambda_1^3 & \cdots & \underline{\lambda}_{1,\kappa}^3 \phi_{\kappa}^2 / \lambda_1^3 \\ \underline{\lambda}_{1,1}^4 \phi_1^2 / \lambda_1^3 & \underline{\lambda}_{1,2}^4 \phi_2^2 / \lambda_1^3 & \cdots & \underline{\lambda}_{1,\kappa}^4 \phi_{\kappa}^2 / \lambda_1^3 \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\lambda}_{1,1}^{\kappa+2} \phi_1^2 / \lambda_1^3 & \underline{\lambda}_{1,2}^{\kappa+2} \phi_2^2 / \lambda_1^3 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa+2} \phi_{\kappa}^2 / \lambda_1^3 \end{pmatrix} \right. \\
&\quad \times \begin{pmatrix} 1 & \underline{\lambda}_{1,1} & \underline{\lambda}_{1,1}^2 & \cdots & \underline{\lambda}_{1,1}^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2} & \underline{\lambda}_{1,2}^2 & \cdots & \underline{\lambda}_{1,2}^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underline{\lambda}_{1,\kappa} & \underline{\lambda}_{1,\kappa}^2 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \Bigg\} \\
&= \left(\frac{\lambda_1}{\lambda_{1,1}} \right) \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \underline{\lambda}_{1,1} & \underline{\lambda}_{1,2} & \cdots & \underline{\lambda}_{1,\kappa} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\lambda}_{1,1}^{\kappa-1} & \underline{\lambda}_{1,2}^{\kappa-1} & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} \end{pmatrix} \right. \\
&\quad \times \begin{pmatrix} \underline{\lambda}_{1,1}^3 \phi_1^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \phi_2^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa}^3 \phi_{\kappa}^2 / \lambda_1^3 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & \underline{\lambda}_{1,1} & \underline{\lambda}_{1,1}^2 & \cdots & \underline{\lambda}_{1,1}^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2} & \underline{\lambda}_{1,2}^2 & \cdots & \underline{\lambda}_{1,2}^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underline{\lambda}_{1,\kappa} & \underline{\lambda}_{1,\kappa}^2 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \Bigg\}
\end{aligned}$$

Moreover, note that

$$\begin{aligned}
& \begin{pmatrix} 1 & \underline{\lambda}_{1,1} & \underline{\lambda}_{1,1}^2 & \cdots & \underline{\lambda}_{1,1}^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2} & \underline{\lambda}_{1,2}^2 & \cdots & \underline{\lambda}_{1,2}^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\lambda}_{1,\kappa} & \underline{\lambda}_{1,\kappa}^2 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1^{-2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \\
&= \begin{pmatrix} 1 & \underline{\lambda}_{1,1}/\lambda_1 & \underline{\lambda}_{1,1}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,1}^{\kappa-1}/\lambda_1^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2}/\lambda_1 & \underline{\lambda}_{1,2}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,2}^{\kappa-1}/\lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\lambda}_{1,\kappa}/\lambda_1 & \underline{\lambda}_{1,\kappa}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1}/\lambda_1^{\kappa-1} \end{pmatrix}
\end{aligned}$$

from which it follows that

$$\begin{aligned}
& \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
&= \left(\frac{\lambda_1}{\underline{\lambda}_{1,1}} \right) \left\{ \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \underline{\lambda}_{1,1}/\lambda_1 & \underline{\lambda}_{1,2}/\lambda_1 & \cdots & \underline{\lambda}_{1,\kappa}/\lambda_1 \\ \underline{\lambda}_{1,1}^2/\lambda_1^2 & \underline{\lambda}_{1,2}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,\kappa}^2/\lambda_1^2 \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\lambda}_{1,1}^{\kappa-1}/\lambda_1^{\kappa-1} & \underline{\lambda}_{1,2}^{\kappa-1}/\lambda_1^{\kappa-1} & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1}/\lambda_1^{\kappa-1} \end{pmatrix} \right. \\
&\quad \times \begin{pmatrix} \underline{\lambda}_{1,1}^3 \phi_1^2/\lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \phi_2^2/\lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa}^3 \phi_\kappa^2/\lambda_1^3 \end{pmatrix} \begin{pmatrix} 1 & \underline{\lambda}_{1,1}/\lambda_1 & \underline{\lambda}_{1,1}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,1}^{\kappa-1}/\lambda_1^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2}/\lambda_1 & \underline{\lambda}_{1,2}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,2}^{\kappa-1}/\lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\lambda}_{1,\kappa}/\lambda_1 & \underline{\lambda}_{1,\kappa}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1}/\lambda_1^{\kappa-1} \end{pmatrix} \left. \right\} \\
&= \left(\frac{\lambda_1}{\underline{\lambda}_{1,1}} \right) \underline{V}' D_{\phi,\lambda} \underline{V},
\end{aligned}$$

where

$$\begin{aligned}
D_{\phi,\lambda} &= \begin{pmatrix} \underline{\lambda}_{1,1}^3 \phi_1^2/\lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \phi_2^2/\lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa}^3 \phi_\kappa^2/\lambda_1^3 \end{pmatrix}, \\
\underline{V} &= \begin{pmatrix} 1 & \underline{\lambda}_{1,1}/\lambda_1 & \underline{\lambda}_{1,1}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,1}^{\kappa-1}/\lambda_1^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2}/\lambda_1 & \underline{\lambda}_{1,2}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,2}^{\kappa-1}/\lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\lambda}_{1,\kappa}/\lambda_1 & \underline{\lambda}_{1,\kappa}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1}/\lambda_1^{\kappa-1} \end{pmatrix}.
\end{aligned}$$

Next, note that

$$\begin{aligned}
\frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} &= \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_{Xy} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \end{pmatrix} \frac{\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \\
&= \begin{pmatrix} \bar{\sigma}'_{Xy} \bar{\sigma}_{Xy} / \lambda_1 \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} / \lambda_1^2 \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} / \lambda_1^\kappa \end{pmatrix} \\
&= \left(\frac{\lambda_1}{\lambda_{1,1}} \right) \begin{pmatrix} \sum_{g=1}^\kappa \lambda_{1,g}^2 \phi_g^2 / \lambda_1^2 \\ \sum_{g=1}^\kappa \lambda_{1,g}^3 \phi_g^2 / \lambda_1^3 \\ \vdots \\ \sum_{g=1}^\kappa \lambda_{1,g}^{\kappa+1} \phi_g^2 / \lambda_1^{\kappa+1} \end{pmatrix} \\
&\quad \left(\text{since } \frac{\bar{\sigma}'_{Xy} \bar{\Sigma}_X^k \bar{\sigma}_{Xy}}{\lambda_1^{k+1}} = \left(\frac{\lambda_1}{\lambda_{1,1}} \right) \frac{1}{\lambda_1^{k+2}} \sum_{g=1}^\kappa \lambda_{1,g}^{k+2} \phi_g^2 \text{ for integer } k \geq 0 \right) \\
&= \left(\frac{\lambda_1}{\lambda_{1,1}} \right) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \begin{pmatrix} \sum_{g=1}^\kappa \lambda_{1,g}^2 \phi_g^2 / \lambda_1^2 \\ \sum_{g=1}^\kappa \lambda_{1,g}^3 \phi_g^2 / \lambda_1^3 \\ \vdots \\ \sum_{g=1}^\kappa \lambda_{1,g}^{\kappa+1} \phi_g^2 / \lambda_1^{\kappa+1} \end{pmatrix} \\
&= \left(\frac{\lambda_1}{\lambda_{1,1}} \right) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,\kappa} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1,1}^{\kappa-1} & \lambda_{1,2}^{\kappa-1} & \cdots & \lambda_{1,\kappa}^{\kappa-1} \end{pmatrix} \begin{pmatrix} \lambda_{1,1}^2 \phi_1^2 / \lambda_1^2 \\ \lambda_{1,2}^2 \phi_2^2 / \lambda_1^2 \\ \vdots \\ \lambda_{1,\kappa}^2 \phi_\kappa^2 / \lambda_1^2 \end{pmatrix} \\
&= \left(\frac{\lambda_1}{\lambda_{1,1}} \right) \underline{V}' \underline{w}_\phi,
\end{aligned}$$

where

$$\begin{aligned}
\underline{V}' &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \underline{\lambda}_{1,1} & \underline{\lambda}_{1,2} & \cdots & \underline{\lambda}_{1,\kappa} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\lambda}_{1,1}^{\kappa-1} & \underline{\lambda}_{1,2}^{\kappa-1} & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \underline{\lambda}_{1,1}/\lambda_1 & \underline{\lambda}_{1,2}/\lambda_1 & \cdots & \underline{\lambda}_{1,\kappa}/\lambda_1 \\ \underline{\lambda}_{1,1}^2/\lambda_1^2 & \underline{\lambda}_{1,2}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,\kappa}^2/\lambda_1^2 \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\lambda}_{1,1}^{\kappa-1}/\lambda_1^{\kappa-1} & \underline{\lambda}_{1,2}^{\kappa-1}/\lambda_1^{\kappa-1} & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1}/\lambda_1^{\kappa-1} \end{pmatrix} \\
\underline{w}_\phi &= \begin{pmatrix} \underline{\lambda}_{1,1}^2 \phi_1^2 / \lambda_1^2 \\ \underline{\lambda}_{1,2}^2 \phi_2^2 / \lambda_1^2 \\ \vdots \\ \underline{\lambda}_{1,\kappa}^2 \phi_\kappa^2 / \lambda_1^2 \end{pmatrix}.
\end{aligned}$$

In addition, note that

$$\begin{aligned}
& \frac{GD_{\lambda,\kappa}^{-1}}{\sqrt{\lambda_1}} \\
&= \frac{1}{\sqrt{\lambda_1}} \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{pmatrix} \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \\
&= \begin{pmatrix} \bar{\sigma}_{Xy}/\lambda_1 & \bar{\Sigma}_X \bar{\sigma}_{Xy}/\lambda_1^2 & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy}/\lambda_1^\kappa \end{pmatrix} \\
&= \sqrt{\frac{\lambda_1}{\lambda_{1,1}}} L_1 \begin{pmatrix} \underline{\Delta}_1 \phi / \lambda_1^{\frac{3}{2}} & \underline{\Delta}_1^2 \phi / \lambda_1^{\frac{5}{2}} & \cdots & \underline{\Delta}_1^\kappa \phi / \lambda_1^{\kappa+\frac{1}{2}} \end{pmatrix} \left(\text{since } \frac{\bar{\Sigma}_X^k \bar{\sigma}_{Xy}}{\lambda_1^{k+1}} = \sqrt{\frac{\lambda_1}{\lambda_{1,1}}} L_1 \frac{\underline{\Delta}_1^{k+1}}{\lambda_1^{k+\frac{3}{2}}} \phi \right) \\
&= \sqrt{\frac{\lambda_1}{\lambda_{1,1}}} L_1 \begin{pmatrix} \underline{\Delta}_{1,1} \phi_1 / \lambda_1^{\frac{3}{2}} & \underline{\Delta}_{1,1}^2 \phi_1 / \lambda_1^{\frac{5}{2}} & \cdots & \underline{\Delta}_{1,1}^\kappa \phi_1 / \lambda_1^{\kappa+\frac{1}{2}} \\ \underline{\Delta}_{1,2} \phi_2 / \lambda_1^{\frac{3}{2}} & \underline{\Delta}_{1,2}^2 \phi_2 / \lambda_1^{\frac{5}{2}} & \cdots & \underline{\Delta}_{1,2}^\kappa \phi_2 / \lambda_1^{\kappa+\frac{1}{2}} \\ \vdots & \vdots & & \vdots \\ \underline{\Delta}_{1,\kappa} \phi_\kappa / \lambda_1^{\frac{3}{2}} & \underline{\Delta}_{1,\kappa}^2 \phi_\kappa / \lambda_1^{\frac{5}{2}} & \cdots & \underline{\Delta}_{1,\kappa}^\kappa \phi_\kappa / \lambda_1^{\kappa+\frac{1}{2}} \end{pmatrix} \\
&= \sqrt{\frac{\lambda_1}{\lambda_{1,1}}} L_1 \left\{ \begin{pmatrix} \underline{\Delta}_{1,1} \phi_1 / \lambda_1^{\frac{3}{2}} & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2} \phi_2 / \lambda_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,\kappa} \phi_\kappa / \lambda_1^{\frac{3}{2}} \end{pmatrix} \right. \\
&\quad \times \left. \begin{pmatrix} 1 & \underline{\Delta}_{1,1}/\lambda_1 & \underline{\Delta}_{1,1}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,1}^{\kappa-1}/\lambda_1^{\kappa-1} \\ 1 & \underline{\Delta}_{1,2}/\lambda_1 & \underline{\Delta}_{1,2}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^{\kappa-1}/\lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\Delta}_{1,\kappa}/\lambda_1 & \underline{\Delta}_{1,\kappa}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,\kappa}^{\kappa-1}/\lambda_1^{\kappa-1} \end{pmatrix} \right\} \\
&= \sqrt{\frac{\lambda_1}{\lambda_{1,1}}} L_1 \left(\frac{1}{\sqrt{\lambda_1}} D_{\phi,\lambda}^* \right) \underline{V}
\end{aligned}$$

where

$$D_{\phi,\lambda}^* = \begin{pmatrix} \underline{\Delta}_{1,1} \phi_1 / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2} \phi_2 / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,\kappa} \phi_\kappa / \lambda_1 \end{pmatrix}.$$

Making use of these calculations, we have

$$\begin{aligned}
\bar{\beta} &= \frac{GD_{\lambda,\kappa}^{-1}}{\sqrt{\lambda_1}} \left[\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \\
&= \sqrt{\frac{\lambda_1}{\underline{\lambda}_{1,1}}} L_1 \left(\frac{1}{\sqrt{\lambda_1}} D_{\phi,\lambda}^* \right) \underline{V} \left(\left(\frac{\lambda_1}{\underline{\lambda}_{1,1}} \right) \underline{V}' D_{\phi,\lambda} \underline{V} \right)^{-1} \left(\frac{\lambda_1}{\underline{\lambda}_{1,1}} \right) \underline{V}' \underline{w}_\phi \\
&= L_1 \left(\frac{1}{\sqrt{\underline{\lambda}_{1,1}}} D_{\phi,\lambda}^* \right) \underline{V} \underline{V}^{-1} D_{\phi,\lambda}^{-1} (\underline{V}')^{-1} \left(\frac{\lambda_1}{\underline{\lambda}_{1,1}} \right)^{-1} \left(\frac{\lambda_1}{\underline{\lambda}_{1,1}} \right) \underline{V}' \underline{w}_\phi \\
&= L_1 \left(\frac{1}{\sqrt{\underline{\lambda}_{1,1}}} D_{\phi,\lambda}^* \right) D_{\phi,\lambda}^{-1} \underline{w}_\phi
\end{aligned}$$

Moreover, note that

$$\begin{aligned}
&\left(\frac{1}{\sqrt{\underline{\lambda}_{1,1}}} D_{\phi,\lambda}^* \right) D_{\phi,\lambda}^{-1} \underline{w}_\phi \\
&= \frac{1}{\sqrt{\underline{\lambda}_{1,1}}} \begin{pmatrix} \underline{\lambda}_{1,1} \phi_1 / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2} \phi_2 / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa} \phi_\kappa / \lambda_1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} \underline{\lambda}_{1,1}^3 \phi_1^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \phi_2^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa}^3 \phi_\kappa^2 / \lambda_1^3 \end{pmatrix}^{-1} \begin{pmatrix} \underline{\lambda}_{1,1}^2 \phi_1^2 / \lambda_1^2 \\ \underline{\lambda}_{1,2}^2 \phi_2^2 / \lambda_1^2 \\ \vdots \\ \underline{\lambda}_{1,\kappa}^2 \phi_\kappa^2 / \lambda_1^2 \end{pmatrix} \\
&= \frac{1}{\sqrt{\underline{\lambda}_{1,1}}} \begin{pmatrix} \underline{\lambda}_{1,1} \phi_1 / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2} \phi_2 / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa} \phi_\kappa / \lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1 / \underline{\lambda}_{1,1} \\ \lambda_1 / \underline{\lambda}_{1,2} \\ \vdots \\ \lambda_1 / \underline{\lambda}_{1,\kappa} \end{pmatrix} \\
&= \begin{pmatrix} \phi_1 / \sqrt{\underline{\lambda}_{1,1}} \\ \phi_2 / \sqrt{\underline{\lambda}_{1,1}} \\ \vdots \\ \phi_\kappa / \sqrt{\underline{\lambda}_{1,1}} \end{pmatrix} \\
&= \phi / \sqrt{\underline{\lambda}_{1,1}}.
\end{aligned}$$

It follows that, in the case where $\overline{\Sigma}_u = 0$,

$$\overline{\beta} = L_1 \left(\frac{1}{\sqrt{\Delta_{1,1}}} D_{\phi,\lambda}^* \right) D_{\phi,\lambda}^{-1} \underline{w}_\phi = L_1 \left(\frac{\phi}{\sqrt{\Delta_{1,1}}} \right) = \beta.$$

Next, to show part (b), note first that

$$\begin{aligned}
& \sqrt{\underline{\lambda}_{1,1}} (\bar{\beta} - \beta) \\
&= \sqrt{\underline{\lambda}_{1,1}} \left[G (G' \bar{\Sigma}_X G)^{-1} G' \bar{\sigma}_{Xy} - L_1 \left(\frac{\phi}{\sqrt{\underline{\lambda}_{1,1}}} \right) \right] \\
&= \sqrt{\underline{\lambda}_{1,1}} \frac{GD_{\lambda,\kappa}^{-1}}{\sqrt{\lambda_1}} \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - L_1 \phi \\
&= \left\{ \left(\sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - L_1 D_{\phi,\lambda}^* \underline{V} + L_1 D_{\phi,\lambda}^* \underline{V} \right) \right. \\
&\quad \times \left(\left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} + (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right) \\
&\quad \times \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\underline{\lambda}_{1,1}}} - \underline{V}' \underline{w}_\phi + \underline{V}' \underline{w}_\phi \right] \Big\} - L_1 \phi \\
&= \left(\sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - L_1 D_{\phi,\lambda}^* \underline{V} \right) \left(\left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right) \\
&\quad \times \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\phi \right] \\
&\quad + \left(\sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - L_1 D_{\phi,\lambda}^* \underline{V} \right) \left(\left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right) \underline{V}' \underline{w}_\phi \\
&\quad + \left(\sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - L_1 D_{\phi,\lambda}^* \underline{V} \right) (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\phi \right] \\
&\quad + \left(\sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - L_1 D_{\phi,\lambda}^* \underline{V} \right) (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \underline{V}' \underline{w}_\phi \\
&\quad + L_1 D_{\phi,\lambda}^* \underline{V} (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\phi \right] \\
&\quad + L_1 D_{\phi,\lambda}^* \underline{V} \left(\left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right) \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\phi \right] \\
&\quad + L_1 D_{\phi,\lambda}^* \underline{V} \left(\left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right) \underline{V}' \underline{w}_\phi \\
&\quad + L_1 D_{\phi,\lambda}^* \underline{V} (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \underline{V}' \underline{w}_\phi - L_1 \phi. \quad 14
\end{aligned} \tag{6}$$

Next, note that

$$\begin{aligned}
& L_1 D_{\phi, \lambda}^* \underline{V} (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} \underline{V}' \underline{w}_\phi - L_1 \phi \\
= & L_1 D_{\phi, \lambda}^* \underline{V} \underline{V}^{-1} D_{\phi, \lambda}^{-1} (\underline{V}')^{-1} \underline{V}' \underline{w}_\phi - L_1 \phi \\
= & L_1 D_{\phi, \lambda}^* D_{\phi, \lambda}^{-1} \underline{w}_\phi - L_1 \phi \\
= & L_1 \left\{ \begin{pmatrix} \underline{\lambda}_{1,1} \phi_1 / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2} \phi_2 / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa} \phi_\kappa / \lambda_1 \end{pmatrix} \right. \\
& \times \begin{pmatrix} \underline{\lambda}_{1,1}^3 \phi_1^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \phi_2^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa}^3 \phi_\kappa^2 / \lambda_1^3 \end{pmatrix}^{-1} \begin{pmatrix} \underline{\lambda}_{1,1}^2 \phi_1^2 / \lambda_1^2 \\ \underline{\lambda}_{1,2}^2 \phi_2^2 / \lambda_1^2 \\ \vdots \\ \underline{\lambda}_{1,\kappa}^2 \phi_\kappa^2 / \lambda_1^2 \end{pmatrix} \Bigg\} \\
& - L_1 \phi \\
= & L_1 \begin{pmatrix} \underline{\lambda}_{1,1} \phi_1 / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2} \phi_2 / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa} \phi_\kappa / \lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1 / \underline{\lambda}_{1,1} \\ \lambda_1 / \underline{\lambda}_{1,2} \\ \vdots \\ \lambda_1 / \underline{\lambda}_{1,\kappa} \end{pmatrix} - L_1 \phi \\
= & L_1 \phi - L_1 \phi \left(\text{since } \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_\kappa \end{pmatrix} \right) \\
= & 0.
\end{aligned}$$

Hence, in light of expression (6), we can write

$$\begin{aligned}
& \left\| \sqrt{\underline{\lambda}_{1,1}} (\bar{\beta} - \beta) \right\|_2 \\
\leq & \left\| \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - L_1 D_{\phi,\lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\|_2 \\
& \times \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\phi \right\|_2 \\
& + \left\| \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - L_1 D_{\phi,\lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \underline{V}' \underline{w}_\phi \right\|_2 \\
& + \left\| \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - L_1 D_{\phi,\lambda}^* \underline{V} \right\|_2 \left\| (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\phi \right\|_2 \\
& + \left\| \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - L_1 D_{\phi,\lambda}^* \underline{V} \right\|_2 \left\| (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \underline{V}' \underline{w}_\phi \right\|_2 \\
& + \left\| L_1 D_{\phi,\lambda}^* \underline{V} \right\|_2 \left\| (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_{\phi 2} \right\|_2 \\
& + \left\| L_1 D_{\phi,\lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\phi \right\|_2 \\
& + \left\| L_1 D_{\phi,\lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \underline{V}' \underline{w}_\phi \right\|_2 \tag{7}
\end{aligned}$$

We shall analyze each term on the majorant side of expression (7) in turn. Consider initially the first term on the majorant side of expression (7). In this case, we can apply parts (b),

(c), and (d) of Lemma B-5, given in Appendix B below, to obtain

$$\begin{aligned}
& \left\| \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - L_1 D_{\phi,\lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\|_2 \\
& \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\phi \right\|_2 \\
& = O\left(\frac{1}{\lambda_1}\right) O\left(\frac{1}{\lambda_1}\right) O\left(\frac{1}{\lambda_1}\right) \\
& = O\left(\frac{1}{\lambda_1^3}\right), \tag{8}
\end{aligned}$$

To analyze the second term on the majorant side of expression (7), we first note that

$$\begin{aligned}
\underline{V}' \underline{w}_\phi &= \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \underline{\lambda}_{1,1}/\lambda_1 & \underline{\lambda}_{1,2}/\lambda_1 & \cdots & \underline{\lambda}_{1,\kappa}/\lambda_1 \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\lambda}_{1,1}^{\kappa-1}/\lambda_1^{\kappa-1} & \underline{\lambda}_{1,2}^{\kappa-1}/\lambda_1^{\kappa-1} & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1}/\lambda_1^{\kappa-1} \end{pmatrix} \begin{pmatrix} \underline{\lambda}_{1,1}^2 \phi_1^2 / \lambda_1^2 \\ \underline{\lambda}_{1,2}^2 \phi_2^2 / \lambda_1^2 \\ \vdots \\ \underline{\lambda}_{1,\kappa}^2 \phi_\kappa^2 / \lambda_1^2 \end{pmatrix} \\
&= \begin{pmatrix} \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^2 \phi_g^2 / \lambda_1^2 \\ \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^3 \phi_g^2 / \lambda_1^3 \\ \vdots \\ \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^{\kappa+1} \phi_g^2 / \lambda_1^{\kappa+1} \end{pmatrix}
\end{aligned}$$

so that

$$\begin{aligned}
& \| \underline{V}' \underline{w}_\phi \|_2 \\
&= \sqrt{\sum_{h=1}^{\kappa} \left(\sum_{g=1}^{\kappa} \frac{\lambda_{1,g}^{h+1} \phi_g^2}{\lambda_1^{h+1}} \right)^2} \\
&\leq \sqrt{\sum_{h=1}^{\kappa} \kappa \sum_{g=1}^{\kappa} \frac{\lambda_{1,g}^{2(h+1)} \phi_g^4}{\lambda_1^{2(h+1)}}} \left(\text{using the inequality } \left| \sum_{i=1}^m a_i \right|^2 \leq m \sum_{i=1}^m |a_i|^2 \right) \\
&\leq \kappa \left(\max_{1 \leq g \leq \kappa} \phi_g^2 \right) \sqrt{\sum_{h=1}^{\kappa} \left(\frac{\lambda_{1,1}^{2(h+1)}}{\lambda_1^{2(h+1)}} \right)} \\
&\leq \kappa \left(\max_{1 \leq g \leq \kappa} \phi_g^2 \right) \sum_{h=1}^{\kappa} \left(\frac{\lambda_{1,1}^{(h+1)}}{\lambda_1^{(h+1)}} \right) \left(\text{by the inequality } \left| \sum_{i=1}^m a_i \right|^r \leq \sum_{i=1}^m |a_i|^r \text{ for } 0 < r \leq 1 \right) \\
&\leq \kappa^2 \left(\max_{1 \leq g \leq \kappa} \phi_g^2 \right) (c^*)^{\kappa+1} \left(\text{applying Lemma D-3(a) which implies that there exists constant } c^* > 1 \right. \\
&\quad \left. \text{such that } 0 < \frac{1}{c^*} \leq \frac{\lambda_{1,1}}{\lambda_1} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large} \right) \\
&\leq \kappa^2 \bar{c}^2 (c^*)^{\kappa+1} < \infty \quad (\text{by Assumptions 4-4}) \tag{9}
\end{aligned}$$

Making use of this result along with parts (c) and (d) of Lemma B-5 given in Appendix B below, we see that

$$\begin{aligned}
& \left\| \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda, \kappa}^{-1} - L_1 D_{\phi, \lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} \right\|_2 \| \underline{V}' \underline{w}_\phi \|_2 \\
&= O\left(\frac{1}{\lambda_1}\right) O\left(\frac{1}{\lambda_1}\right) O(1) \\
&= O\left(\frac{1}{\lambda_1^2}\right). \tag{10}
\end{aligned}$$

Next, to analyze the third term on the majorant side of expression (7), note first that

$$\begin{aligned}
\left\| (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} \right\|_2 &\leq \sqrt{\lambda_{\max} \left\{ \left[(\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} \right]' (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} \right\}} \\
&= \sqrt{\lambda_{\max} \left\{ ([\underline{V}' D_{\phi, \lambda} \underline{V}]')^{-1} (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} \right\}} \\
&= \sqrt{\lambda_{\max} \left\{ (\underline{V}' D_{\phi, \lambda} \underline{V})^{-2} \right\}} \quad (\text{since } \underline{V}' D_{\phi, \lambda} \underline{V} \text{ is symmetric}) \\
&= \sqrt{\frac{1}{(\lambda_{\min} \{ \underline{V}' D_{\phi, \lambda} \underline{V} \})^2}} \\
&= \frac{1}{\lambda_{\min} \{ \underline{V}' D_{\phi, \lambda} \underline{V} \}} \\
&\leq \frac{1}{C_{\#}} < \infty \quad (\text{for some constant } C_{\#} > 0 \text{ by Lemma B-3 (c)}) \quad (11)
\end{aligned}$$

Making use of this result along with parts (b) and (c) of Lemma B-5 given in Appendix B of this Online Appendix, we obtain

$$\begin{aligned}
&\left\| \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda, \kappa}^{-1} - L_1 D_{\phi, \lambda}^* \underline{V} \right\|_2 \left\| (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} \right\|_2 \left\| \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_{\phi} \right\|_2 \\
&= O\left(\frac{1}{\lambda_1}\right) O(1) O\left(\frac{1}{\lambda_1}\right) \\
&= O\left(\frac{1}{\lambda_1^2}\right). \quad (12)
\end{aligned}$$

For the fourth term on the majorant side of expression (7), we have, by the results given in expressions (9) and (11) as well as part (c) of Lemma B-5,

$$\begin{aligned}
&\left\| \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda, \kappa}^{-1} - L_1 D_{\phi, \lambda}^* \underline{V} \right\|_2 \left\| (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} \right\|_2 \left\| \underline{V}' \underline{w}_{\phi} \right\|_2 \\
&= O\left(\frac{1}{\lambda_1}\right) O(1) O(1) \\
&= O\left(\frac{1}{\lambda_1}\right) \quad (13)
\end{aligned}$$

To analyze the fifth term on the majorant side of expression (7), note first that

$$\begin{aligned}
& D_{\phi, \lambda}^* \underline{V} \\
&= \begin{pmatrix} \underline{\lambda}_{1,1} \phi_1 / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2} \phi_2 / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa} \phi_\kappa / \lambda_1 \end{pmatrix} \begin{pmatrix} 1 & \underline{\lambda}_{1,1} / \lambda_1 & \underline{\lambda}_{1,1}^2 / \lambda_1^2 & \cdots & \underline{\lambda}_{1,1}^{\kappa-1} / \lambda_1^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2} / \lambda_1 & \underline{\lambda}_{1,2}^2 / \lambda_1^2 & \cdots & \underline{\lambda}_{1,2}^{\kappa-1} / \lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underline{\lambda}_{1,\kappa} / \lambda_1 & \underline{\lambda}_{1,\kappa}^2 / \lambda_1^2 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} / \lambda_1^{\kappa-1} \end{pmatrix} \\
&= \begin{pmatrix} \underline{\lambda}_{1,1} \phi_1 / \lambda_1 & \underline{\lambda}_{1,1}^2 \phi_1 / \lambda_1^2 & \underline{\lambda}_{1,1}^3 \phi_1 / \lambda_1^3 & \cdots & \underline{\lambda}_{1,1}^\kappa \phi_1 / \lambda_1^\kappa \\ \underline{\lambda}_{1,2} \phi_2 / \lambda_1 & \underline{\lambda}_{1,2}^2 \phi_2 / \lambda_1^2 & \underline{\lambda}_{1,2}^3 \phi_2 / \lambda_1^3 & \cdots & \underline{\lambda}_{1,2}^\kappa \phi_2 / \lambda_1^\kappa \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{\lambda}_{1,\kappa} \phi_\kappa / \lambda_1 & \underline{\lambda}_{1,\kappa}^2 \phi_\kappa / \lambda_1^2 & \underline{\lambda}_{1,\kappa}^3 \phi_\kappa / \lambda_1^3 & \cdots & \underline{\lambda}_{1,\kappa}^\kappa \phi_\kappa / \lambda_1^\kappa \end{pmatrix}
\end{aligned}$$

from which it follows, by straightforward calculation, that

$$tr \left\{ \underline{V}' (D_{\phi, \lambda}^*)' D_{\phi, \lambda}^* \underline{V} \right\} = \sum_{h=1}^{\kappa} \sum_{g=1}^{\kappa} \frac{\underline{\lambda}_{1,g}^{2h} \phi_g^2}{\lambda_1^{2h}}$$

Making use of this calculation, we see that

$$\begin{aligned}
& \|L_1 D_{\phi,\lambda}^* \underline{V}\|_2 \\
&= \|L_1\|_2 \|D_{\phi,\lambda}^* \underline{V}\|_2 \\
&\leq \|L_1\|_2 \|D_{\phi,\lambda}^* \underline{V}\|_F \\
&\leq \sqrt{\lambda_{\max}(L_1' L_1)} \sqrt{\text{tr} \left\{ \underline{V}' (D_{\phi,\lambda}^*)' D_{\phi,\lambda}^* \underline{V} \right\}} \\
&= \sqrt{\lambda_{\max}(I_\kappa)} \sqrt{\sum_{h=1}^{\kappa} \sum_{g=1}^{\kappa} \frac{\underline{\lambda}_{1,g}^{2h} \phi_g^2}{\lambda_1^{2h}}} \\
&= \sqrt{\sum_{h=1}^{\kappa} \sum_{g=1}^{\kappa} \frac{\underline{\lambda}_{1,g}^{2h} \phi_g^2}{\lambda_1^{2h}}} \\
&\leq \left(\max_{1 \leq g \leq \kappa} |\phi_g| \right) \sqrt{\sum_{h=1}^{\kappa} \sum_{g=1}^{\kappa} \frac{\underline{\lambda}_{1,g}^{2h}}{\lambda_1^{2h}}} \\
&\leq \sqrt{\kappa} \left(\max_{1 \leq g \leq \kappa} |\phi_g| \right) \sqrt{\sum_{h=1}^{\kappa} \frac{\underline{\lambda}_{1,1}^{2h}}{\lambda_1^{2h}}} \\
&\leq \sqrt{\kappa} \left(\max_{1 \leq g \leq \kappa} |\phi_g| \right) \sum_{h=1}^{\kappa} \frac{\underline{\lambda}_{1,1}^h}{\lambda_1^h} \left(\text{applying the inequality } \left| \sum_{i=1}^m a_i \right|^r \leq \sum_{i=1}^m |a_i|^r \text{ for } 0 < r \leq 1 \right) \\
&\leq \kappa^{\frac{3}{2}} \left(\max_{1 \leq g \leq \kappa} |\phi_g| \right) (c^*)^\kappa \quad (\text{applying Lemma D-3(a) which implies that there exists constant } \\
&\quad c^* > 1 \text{ such that } 0 < \frac{1}{c^*} \leq \frac{\underline{\lambda}_{1,1}}{\lambda_1} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large}) \\
&\leq \kappa^{\frac{3}{2}} \bar{c} (c^*)^\kappa \quad \left(\text{for some constant } \bar{c} \geq \max_{1 \leq g \leq \kappa} |\phi_g| \text{ by Assumption 4-4} \right) \tag{14}
\end{aligned}$$

Using this result along with expression (11) and part (b) of Lemma B-5, given in Appendix B of this Online Appendix, we see that

$$\begin{aligned}
& \|L_1 D_{\phi,\lambda}^* \underline{V}\|_2 \left\| (V' D_{\phi,\lambda} V)^{-1} \right\|_2 \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' w_\phi \right\|_2 \\
&= O(1) O(1) O\left(\frac{1}{\lambda_1}\right) \\
&= O\left(\frac{1}{\lambda_1}\right). \tag{15}
\end{aligned}$$

Turning our attention to the sixth term, note that, by expression (14) and by the results given in parts (b) and (d) of Lemma B-5, we have

$$\begin{aligned}
& \left\| L_1 D_{\phi, \lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} \right\|_2 \\
& \times \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\phi \right\|_2 \\
& = O(1) O\left(\frac{1}{\lambda_1}\right) O\left(\frac{1}{\lambda_1}\right) \\
& = O\left(\frac{1}{\lambda_1^2}\right)
\end{aligned} \tag{16}$$

Now, for the last term on the majorant side of expression (7), we can apply part (d) of Lemma B-5 and the results given in expressions (9) and (14) to obtain

$$\begin{aligned}
& \left\| L_1 D_{\phi, \lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} \right\|_2 \left\| \underline{V}' \underline{w}_\phi \right\|_2 \\
& = O(1) O\left(\frac{1}{\lambda_1}\right) O(1) \\
& = O\left(\frac{1}{\lambda_1}\right).
\end{aligned} \tag{17}$$

Combining the results given in expressions (8), (10), (12)-(13), and (15)-(17); we obtain

$$\begin{aligned}
& \left\| \sqrt{\underline{\lambda}_{1,1}} (\bar{\beta} - \beta) \right\|_2 \\
& = O\left(\frac{1}{\lambda_1^3}\right) + O\left(\frac{1}{\lambda_1^2}\right) + O\left(\frac{1}{\lambda_1^2}\right) + O\left(\frac{1}{\lambda_1}\right) + O\left(\frac{1}{\lambda_1}\right) + O\left(\frac{1}{\lambda_1^2}\right) + O\left(\frac{1}{\lambda_1}\right) \\
& = O\left(\frac{1}{\lambda_1}\right),
\end{aligned}$$

which, in turn, implies that

$$\left\| \bar{\beta} - \beta \right\|_2 = O\left(\frac{1}{\lambda_1 \sqrt{\underline{\lambda}_{1,1}}}\right) = O\left(\frac{1}{\lambda_1^{\frac{3}{2}}}\right),$$

since, by part (a) of Lemma D-3 given in Appendix D of this Online Appendix, there exists

an absolute constant $c^* > 1$ such that

$$0 < \frac{1}{c^*} < \frac{\lambda_1}{\Delta_{1,1}} \leq c^* < \infty \text{ for all } N \text{ and } T \text{ sufficiently large. } \square$$

Proof of Theorem 4:

To show part (a), first write

$$[X_T - \mu_X]' \bar{\beta} - (F_{1,T} - \mu_{F,1})' \varphi_1 = [X_T - \mu_X]' (\bar{\beta} - \beta) + [X_T - \mu_X]' \beta - (F_{1,T} - \mu_{F,1})' \varphi_1$$

where, in this case,

$$\begin{aligned} \beta &= \frac{\Gamma}{\sqrt{\Delta_{1,1}}} \left(\frac{\varphi}{\sqrt{\Delta_{1,1}}} \right) = \frac{1}{\Delta_{1,1}} \begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \\ &= \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \left(\frac{\varphi_1}{\sqrt{\Delta_{1,1}}} \right) + \frac{\Gamma_2}{\sqrt{\Delta_{1,1}}} \left(\frac{\varphi_2}{\sqrt{\Delta_{1,1}}} \right) \\ &= \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \left(\frac{\varphi_1}{\sqrt{\Delta_{1,1}}} \right) \quad (\text{given that } \varphi_2 = 0 \text{ under Assumption S1-1}) \end{aligned}$$

and

$$\bar{\beta} = G (G' \bar{\Sigma}_X G)^{-1} G' \bar{\sigma}_{Xy} \text{ with } G = \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{pmatrix} \text{ for } 1 \leq \kappa < K$$

By applying the same argument as that given in the proof of Theorem 3, we can show that

$$[X_T - \mu_X]' \beta - (F_{1,T} - \mu_{F,1})' \varphi_1 = O_p \left(\frac{1}{\sqrt{\Delta_{1,1}}} \right) = O_p \left(\frac{1}{\sqrt{\lambda_1}} \right) \quad (18)$$

where $\lambda_1 = \lambda_{\max}(\bar{\Sigma}_X)$ and where the second equality above follows from Lemma D-3 part (a) given in Appendix D of this Online Appendix.

Next, note that, since $X_t - \mu_X = \Gamma (F_t - \mu_F) + u_t$, we can write

$$\begin{aligned}
& |[X_T - \mu_X]' (\bar{\beta} - \beta)| \\
& \leq |(F_T - \mu_F)' \Gamma' (\bar{\beta} - \beta)| + |u_T' (\bar{\beta} - \beta)| \quad (\text{by triangle inequality}) \\
& \leq \sqrt{(F_T - \mu_F)' \Gamma' \Gamma (F_T - \mu_F)} \sqrt{(\bar{\beta} - \beta)' (\bar{\beta} - \beta)} + |u_T' (\bar{\beta} - \beta)| \\
& = \sqrt{\underline{\lambda}_{1,1}} \sqrt{(F_T - \mu_F)' \frac{\Gamma' \Gamma}{\underline{\lambda}_{1,1}} (F_T - \mu_F)} \|(\bar{\beta} - \beta)\|_2 + |u_T' (\bar{\beta} - \beta)| \\
& = \sqrt{\underline{\lambda}_{1,1}} \sqrt{(F_T - \mu_F)' (F_T - \mu_F)} \|(\bar{\beta} - \beta)\|_2 + |u_T' (\bar{\beta} - \beta)| \\
& \quad \left(\text{since } \frac{\Gamma' \Gamma}{\underline{\lambda}_{1,1}} = I_K \text{ by Assumption (i)} \right)
\end{aligned} \tag{19}$$

Note also that, for all T ,

$$\begin{aligned}
E \left\{ \sqrt{(F_T - \mu_F)' (F_T - \mu_F)} \right\} & \leq \sqrt{E \left\{ (F_T - \mu_F)' (F_T - \mu_F) \right\}} \quad (\text{by Jensen's inequality}) \\
& = \sqrt{\sum_{k=1}^K E \left\{ (F_{k,T} - \mu_{k,F})^2 \right\}} \\
& \leq \sqrt{\sum_{k=1}^K \left(E \left\{ (F_{k,T} - \mu_{k,F})^4 \right\} \right)^{\frac{1}{2}}} \quad (\text{by Liapunov's inequality}) \\
& \leq \sqrt{K} [C^\dagger]^{\frac{1}{4}} \quad (\text{by assumption (ii)}) \\
& \leq C_1 \quad \left(\text{for some constant } C_1 > 0 \text{ such that } C_1 \geq \sqrt{K} [C^\dagger]^{\frac{1}{4}} \right) \\
& < \infty
\end{aligned}$$

It then follows that for any $\epsilon > 0$, we can set $C_{1,\epsilon} = C_1/\epsilon$; and, by Markov's inequality, we have

$$\begin{aligned}
P \left(\left| \sqrt{(F_T - \mu_F)' (F_T - \mu_F)} \right| \geq C_{1,\epsilon} \right) & = P \left(\left| \sqrt{(F_T - \mu_F)' (F_T - \mu_F)} \right| \geq \frac{C_1}{\epsilon} \right) \\
& \leq \frac{\epsilon}{C_1} E \left(\sqrt{(F_T - \mu_F)' (F_T - \mu_F)} \right) \\
& \leq \frac{\epsilon}{C_1} C_1 \\
& = \epsilon
\end{aligned}$$

from which we further deduce that

$$\sqrt{(F_T - \mu_F)' (F_T - \mu_F)} = O_p(1)$$

Moreover,

$$\begin{aligned} E |u'_T (\bar{\beta} - \beta)| &\leq \sqrt{(\bar{\beta} - \beta)' E [u_T u'_T] (\bar{\beta} - \beta)} \\ &\quad \text{(by Liapunov's inequality)} \\ &= \sqrt{(\bar{\beta} - \beta)' \Sigma_{u,T} (\bar{\beta} - \beta)} \\ &\leq \sqrt{\lambda_{\max}(\Sigma_{u,T})} \|\bar{\beta} - \beta\|_2 \\ &\leq \sqrt{C^\dagger} \|\bar{\beta} - \beta\|_2 \quad \text{(by assumption (ii))} \\ &= C_2 \|\bar{\beta} - \beta\|_2 \quad \left(\text{setting } C_2 = \sqrt{C^\dagger}\right) \end{aligned}$$

It follows that for any $\epsilon > 0$, we can set $C_{2,\epsilon} = C_2/\epsilon$; and, by Markov's inequality, we have

$$\begin{aligned} P \left(\|\bar{\beta} - \beta\|_2^{-1} |u'_T (\bar{\beta} - \beta)| \geq C_{2,\epsilon} \right) &= P \left(\|\bar{\beta} - \beta\|_2^{-1} |u'_T (\bar{\beta} - \beta)| \geq \frac{C_2}{\epsilon} \right) \\ &\leq \frac{\epsilon}{C_2} \|\bar{\beta} - \beta\|_2^{-1} E (|u'_T (\bar{\beta} - \beta)|) \\ &\leq \frac{\epsilon}{C_2} \|\bar{\beta} - \beta\|_2^{-1} C_2 \|\bar{\beta} - \beta\|_2 \\ &= \epsilon \end{aligned}$$

so that

$$\|\bar{\beta} - \beta\|_2^{-1} |u'_T (\bar{\beta} - \beta)| = O_p(1) \quad \text{or} \quad |u'_T (\bar{\beta} - \beta)| = O_p(\|\bar{\beta} - \beta\|_2) \quad (20)$$

Applying the result given in part (a) of Lemma E-1, where it is shown that $\|\bar{\beta} - \beta\|_2 = O\left(\lambda_1^{-\frac{3}{2}}\right)$ under Scenario 1³; we have

$$|u'_T (\bar{\beta} - \beta)| = O_p(\|\bar{\beta} - \beta\|_2) = O_p\left(\frac{1}{\lambda_1^{\frac{3}{2}}}\right)$$

³Lemma E-1 and its proof can be found in Appendix E of this Online Appendix.

Putting things together, we then obtain

$$\begin{aligned}
& |[X_T - \mu_X]' (\bar{\beta} - \beta)| \\
& \leq \sqrt{\underline{\lambda}_{1,1}} \sqrt{(F_T - \mu_F)' (F_T - \mu_F)} \|(\bar{\beta} - \beta)\|_2 + |u_T' (\bar{\beta} - \beta)| \\
& \quad (\text{by the inequality given in expression (19)}) \\
& = O\left(\sqrt{\underline{\lambda}_{1,1}}\right) O_p(1) O\left(\lambda_1^{-\frac{3}{2}}\right) + O\left(\lambda_1^{-\frac{3}{2}}\right) \quad (\text{by part (a) of Lemma E-1}) \\
& = O\left(\sqrt{\lambda_1}\right) O_p(1) O\left(\lambda_1^{-\frac{3}{2}}\right) + O_p\left(\lambda_1^{-\frac{3}{2}}\right) \quad (\text{since } \underline{\lambda}_{1,1} \asymp \lambda_1 \text{ by Lemma D-3 (a)}) \\
& = O_p\left(\frac{1}{\lambda_1}\right) + O_p\left(\lambda_1^{-\frac{3}{2}}\right) \\
& = O_p\left(\frac{1}{\lambda_1}\right). \tag{21}
\end{aligned}$$

Now, combining the results given in expressions (18) and (21), we then obtain

$$\begin{aligned}
[X_T - \mu_X]' \bar{\beta} - (F_{1,T} - \mu_{F,1})' \varphi_1 &= [X_T - \mu_X]' (\bar{\beta} - \beta) + [X_T - \mu_X]' \beta - (F_{1,T} - \mu_{F,1})' \varphi_1 \\
&= O_p\left(\frac{1}{\lambda_1}\right) + O_p\left(\frac{1}{\sqrt{\lambda_1}}\right) \\
&= O_p\left(\frac{1}{\sqrt{\lambda_1}}\right) \tag{22}
\end{aligned}$$

The result for part (b) can be shown in a way similar to the proof of part (a). Here, since $\kappa = K$ under Scenario 2, we can write

$$[X_T - \mu_X]' \bar{\beta} - (F_T - \mu_F)' \varphi = [X_T - \mu_X]' (\bar{\beta} - \beta) + [X_T - \mu_X]' \beta - (F_T - \mu_F)' \varphi$$

where

$$\beta = \frac{\Gamma}{\sqrt{\underline{\lambda}_{1,1}}} \left(\frac{\varphi}{\sqrt{\underline{\lambda}_{1,1}}} \right).$$

and

$$\bar{\beta} = G (G' \bar{\Sigma}_X G)^{-1} G' \bar{\sigma}_{Xy} \text{ with } G = \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{K-1} \bar{\sigma}_{Xy} \end{pmatrix}.$$

Now, in this case, by applying the same argument as that given in the proof of Theorem 1, we can show that

$$[X_T - \mu_X]' \beta - (F_T - \mu_F)' \varphi = O_p\left(\frac{1}{\sqrt{\underline{\lambda}_{1,1}}}\right) = O_p\left(\frac{1}{\sqrt{\lambda_1}}\right) \tag{23}$$

where $\lambda_1 = \lambda_{\max}(\bar{\Sigma}_X)$ and where again the second equality above follows from part (a) of Lemma D-3 given in Appendix D.

Moreover, following the same argument as that given in the proof of part (a), we can again show that

$$\begin{aligned}
& |[X_T - \mu_X]' (\bar{\beta} - \beta)| \\
& \leq |(F_T - \mu_F)' \Gamma' (\bar{\beta} - \beta)| + |u_T' (\bar{\beta} - \beta)| \quad (\text{by triangle inequality}) \\
& \leq \sqrt{(F_T - \mu_F)' \Gamma' \Gamma (F_T - \mu_F)} \sqrt{(\bar{\beta} - \beta)' (\bar{\beta} - \beta)} + |u_T' (\bar{\beta} - \beta)| \\
& = \sqrt{\underline{\lambda}_{1,1}} \sqrt{(F_T - \mu_F)' \frac{\Gamma' \Gamma}{\underline{\lambda}_{1,1}} (F_T - \mu_F)} \|(\bar{\beta} - \beta)\|_2 + |u_T' (\bar{\beta} - \beta)| \\
& = \sqrt{\underline{\lambda}_{1,1}} \sqrt{(F_T - \mu_F)' (F_T - \mu_F)} \|(\bar{\beta} - \beta)\|_2 + |u_T' (\bar{\beta} - \beta)| \tag{24}
\end{aligned}$$

where

$$\sqrt{(F_T - \mu_F)' (F_T - \mu_F)} = O_p(1).$$

In this case, applying part (b) of Lemma E-1, given in Appendix E of this Online Appendix, we also get

$$|u_T' (\bar{\beta} - \beta)| = O_p(\|(\bar{\beta} - \beta)\|_2) = O_p\left(\frac{1}{\lambda_1^{\frac{3}{2}}}\right)$$

Putting things together, we have

$$\begin{aligned}
& |[X_T - \mu_X]' (\bar{\beta} - \beta)| \\
& \leq \sqrt{\underline{\lambda}_{1,1}} \sqrt{(F_T - \mu_F)' (F_T - \mu_F)} \|(\bar{\beta} - \beta)\|_2 + |u_T' (\bar{\beta} - \beta)| \\
& \quad (\text{by the inequality given in expression (24)}) \\
& = O\left(\sqrt{\underline{\lambda}_{1,1}}\right) O_p(1) O\left(\lambda_1^{-\frac{3}{2}}\right) + O\left(\lambda_1^{-\frac{3}{2}}\right) \quad (\text{by part (b) of Lemma E-1}) \\
& = O\left(\sqrt{\lambda_1}\right) O_p(1) O\left(\lambda_1^{-\frac{3}{2}}\right) + O_p\left(\lambda_1^{-\frac{3}{2}}\right) \quad (\text{since } \underline{\lambda}_{1,1} \asymp \lambda_1 \text{ by Lemma D-3 part (a)}) \\
& = O_p\left(\frac{1}{\lambda_1}\right) + O_p\left(\lambda_1^{-\frac{3}{2}}\right) \\
& = O_p\left(\frac{1}{\lambda_1}\right) \tag{25}
\end{aligned}$$

Finally, combining the results given in expressions (23) and (25), we then obtain

$$\begin{aligned}
[X_T - \mu_X]' \bar{\beta} - (F_T - \mu_F)' \varphi &= [X_T - \mu_X]' (\bar{\beta} - \beta) + [X_T - \mu_X]' \beta - (F_T - \mu_F)' \varphi \\
&= O_p \left(\frac{1}{\lambda_1} \right) + O_p \left(\frac{1}{\sqrt{\lambda_1}} \right) \\
&= O_p \left(\frac{1}{\sqrt{\lambda_1}} \right). \quad \square
\end{aligned} \tag{26}$$

Proof of Part (b) of Theorem 5:

Part (b) of Theorem 5 can be shown in a manner similar to the proof of part (a) given in the Supplemental Appendix, although with some modifications. For completeness, we will provide the details of the proof of part (b) here. To proceed, write

$$\hat{\beta}_{\text{grpls}} - \beta = \hat{\beta}_{\text{grpls}} - \bar{\beta} + \bar{\beta} - \beta \tag{27}$$

where

$$\begin{aligned}
\hat{\beta}_{\text{grpls}} &= \hat{G}_S \hat{D}_{\lambda, \kappa}^{-1} \left[\hat{D}_{\lambda, \kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S \hat{D}_{\lambda, \kappa}^{-1} \right]^+ \hat{D}_{\lambda, \kappa}^{-1} \hat{G}'_S \hat{\sigma}_{Xy}^S, \quad \bar{\beta} = G (G' \bar{\Sigma}_X G)^{-1} G' \bar{\sigma}_{Xy}, \\
\beta &= \frac{L_1 \phi}{\sqrt{\lambda_{1,1}}}, \text{ and } \hat{D}_{\lambda, \kappa} = \begin{pmatrix} \hat{\lambda}_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \hat{\lambda}_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \hat{\lambda}_1^{\kappa - \frac{1}{2}} \end{pmatrix}.
\end{aligned}$$

Consider first the term $\hat{\beta}_{\text{grpls}} - \bar{\beta}$. Let

$$D_{\lambda, \kappa} = \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{\kappa - \frac{1}{2}} \end{pmatrix},$$

and note that, by straightforward calculations,

$$\begin{aligned}
& \widehat{\beta}_{\text{grpls}} - \overline{\beta} \\
&= \widehat{G}_S \widehat{D}_{\lambda, \kappa}^{-1} \left[\widehat{D}_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda, \kappa}^{-1} \right]^+ \widehat{D}_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\sigma}_{Xy}^S - G (G' \overline{\Sigma}_X G)^{-1} G' \overline{\sigma}_{Xy} \\
&= \widehat{G}_S \widehat{D}_{\lambda, \kappa}^{-1} \left[\widehat{D}_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda, \kappa}^{-1} \right]^+ \widehat{D}_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\sigma}_{Xy}^S - G D_{\lambda, \kappa}^{-1} (D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1})^{-1} D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy} \\
&= \left[(\widehat{G}_S - G) \widehat{D}_{\lambda, \kappa}^{-1} + G D_{\lambda, \kappa}^{-1} (D_{\lambda, \kappa} \widehat{D}_{\lambda, \kappa}^{-1} - I_\kappa) + G D_{\lambda, \kappa}^{-1} \right] \\
&\quad \times \left\{ \left(\widehat{D}_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda, \kappa}^{-1} \right)^+ - (D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1})^{-1} + (D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1})^{-1} \right\} \\
&\quad \times \left[\widehat{D}_{\lambda, \kappa}^{-1} (\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}) + (\widehat{D}_{\lambda, \kappa}^{-1} D_{\lambda, \kappa} - I_\kappa) D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy} + D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy} \right] \\
&\quad - G D_{\lambda, \kappa}^{-1} (D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1})^{-1} D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy} \\
&= \mathcal{T}\mathcal{T}_1 + \mathcal{T}\mathcal{T}_2 + \mathcal{T}\mathcal{T}_3 + \mathcal{T}\mathcal{T}_4 + \mathcal{T}\mathcal{T}_5 + \mathcal{T}\mathcal{T}_6 + \mathcal{T}\mathcal{T}_7 + \mathcal{T}\mathcal{T}_8 + \mathcal{T}\mathcal{T}_9 + \mathcal{T}\mathcal{T}_{10} + \mathcal{T}\mathcal{T}_{11} \\
&\quad + \mathcal{T}\mathcal{T}_{12} + \mathcal{T}\mathcal{T}_{13} + \mathcal{T}\mathcal{T}_{14} + \mathcal{T}\mathcal{T}_{15} + \mathcal{T}\mathcal{T}_{16} + \mathcal{T}\mathcal{T}_{17} + \mathcal{T}\mathcal{T}_{18} - \mathcal{T}\mathcal{T}_{18}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{T}\mathcal{T}_1 &= \frac{1}{\sqrt{\lambda_1}} (\widehat{G}_S - G) \widehat{D}_{\lambda, \kappa}^{-1} \left\{ \left(\frac{\widehat{D}_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \\
&\quad \times \widehat{D}_{\lambda, \kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \\
\mathcal{T}\mathcal{T}_2 &= \frac{1}{\sqrt{\lambda_1}} (\widehat{G}_S - G) \widehat{D}_{\lambda, \kappa}^{-1} \left\{ \left(\frac{\widehat{D}_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \\
&\quad \times (\widehat{D}_{\lambda, \kappa}^{-1} D_{\lambda, \kappa} - I_\kappa) \frac{D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \\
\mathcal{T}\mathcal{T}_3 &= \frac{1}{\sqrt{\lambda_1}} (\widehat{G}_S - G) \widehat{D}_{\lambda, \kappa}^{-1} \left\{ \left(\frac{\widehat{D}_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \frac{D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \\
\mathcal{T}\mathcal{T}_4 &= \frac{1}{\sqrt{\lambda_1}} (\widehat{G}_S - G) \widehat{D}_{\lambda, \kappa}^{-1} \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \widehat{D}_{\lambda, \kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \\
\mathcal{T}\mathcal{T}_5 &= \frac{1}{\sqrt{\lambda_1}} (\widehat{G}_S - G) \widehat{D}_{\lambda, \kappa}^{-1} \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} (\widehat{D}_{\lambda, \kappa}^{-1} D_{\lambda, \kappa} - I_\kappa) \frac{D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}}
\end{aligned}$$

$$\begin{aligned}
\mathcal{TT}_6 &= \frac{1}{\sqrt{\lambda_1}} \left(\hat{G}_S - G \right) \hat{D}_{\lambda,\kappa}^{-1} \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \\
\mathcal{TT}_7 &= \frac{1}{\sqrt{\lambda_1}} G D_{\lambda,\kappa}^{-1} \left(D_{\lambda,\kappa} \hat{D}_{\lambda,\kappa}^{-1} - I_\kappa \right) \left(\left\{ \left(\frac{\hat{D}_{\lambda,\kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S \hat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \right. \\
&\quad \left. \times \hat{D}_{\lambda,\kappa}^{-1} \left[\frac{\hat{G}'_S \hat{\sigma}_{Xy}^S - G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \right) \\
\mathcal{TT}_8 &= \frac{1}{\sqrt{\lambda_1}} G D_{\lambda,\kappa}^{-1} \left(D_{\lambda,\kappa} \hat{D}_{\lambda,\kappa}^{-1} - I_\kappa \right) \left(\left\{ \left(\frac{\hat{D}_{\lambda,\kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S \hat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \right. \\
&\quad \left. \times \left(\hat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right) \\
\mathcal{TT}_9 &= \frac{1}{\sqrt{\lambda_1}} G D_{\lambda,\kappa}^{-1} \left(D_{\lambda,\kappa} \hat{D}_{\lambda,\kappa}^{-1} - I_\kappa \right) \left\{ \left(\frac{\hat{D}_{\lambda,\kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S \hat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \\
&\quad \times \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \\
\mathcal{TT}_{10} &= \frac{1}{\sqrt{\lambda_1}} G D_{\lambda,\kappa}^{-1} \left(D_{\lambda,\kappa} \hat{D}_{\lambda,\kappa}^{-1} - I_\kappa \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \hat{D}_{\lambda,\kappa}^{-1} \left[\frac{\hat{G}'_S \hat{\sigma}_{Xy}^S - G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \\
\mathcal{TT}_{11} &= \frac{1}{\sqrt{\lambda_1}} G D_{\lambda,\kappa}^{-1} \left(D_{\lambda,\kappa} \hat{D}_{\lambda,\kappa}^{-1} - I_\kappa \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \left(\hat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \\
\mathcal{TT}_{12} &= \frac{1}{\sqrt{\lambda_1}} G D_{\lambda,\kappa}^{-1} \left(D_{\lambda,\kappa} \hat{D}_{\lambda,\kappa}^{-1} - I_\kappa \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \\
\mathcal{TT}_{13} &= \frac{1}{\sqrt{\lambda_1}} G D_{\lambda,\kappa}^{-1} \left\{ \left(\frac{\hat{D}_{\lambda,\kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S \hat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \hat{D}_{\lambda,\kappa}^{-1} \left[\frac{\hat{G}'_S \hat{\sigma}_{Xy}^S - G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right]
\end{aligned}$$

$$\begin{aligned}
\mathcal{T}\mathcal{T}_{14} &= \frac{1}{\sqrt{\lambda_1}} G D_{\lambda,\kappa}^{-1} \left\{ \left(\frac{\hat{D}_{\lambda,\kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S \hat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \\
&\quad \times \left(\hat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \\
\mathcal{T}\mathcal{T}_{15} &= \frac{1}{\sqrt{\lambda_1}} G D_{\lambda,\kappa}^{-1} \left\{ \left(\frac{\hat{D}_{\lambda,\kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S \hat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \\
\mathcal{T}\mathcal{T}_{16} &= \frac{1}{\sqrt{\lambda_1}} G D_{\lambda,\kappa}^{-1} \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \hat{D}_{\lambda,\kappa}^{-1} \left[\frac{\hat{G}'_S \hat{\sigma}_{Xy}^S - G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \\
\mathcal{T}\mathcal{T}_{17} &= \frac{1}{\sqrt{\lambda_1}} G D_{\lambda,\kappa}^{-1} \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \left(\hat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}}.
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{T}\mathcal{T}_{18} &= G D_{\lambda,\kappa}^{-1} \left(D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1} \right)^{-1} D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy} \\
&= \frac{1}{\sqrt{\lambda_1}} G D_{\lambda,\kappa}^{-1} \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}}.
\end{aligned}$$

Since the last two terms cancel each other out, we further obtain

$$\begin{aligned}
\hat{\beta}_{\text{grpls}} - \bar{\beta} &= \mathcal{T}\mathcal{T}_1 + \mathcal{T}\mathcal{T}_2 + \mathcal{T}\mathcal{T}_3 + \mathcal{T}\mathcal{T}_4 + \mathcal{T}\mathcal{T}_5 + \mathcal{T}\mathcal{T}_6 + \mathcal{T}\mathcal{T}_7 + \mathcal{T}\mathcal{T}_8 + \mathcal{T}\mathcal{T}_9 \\
&\quad + \mathcal{T}\mathcal{T}_{10} + \mathcal{T}\mathcal{T}_{11} + \mathcal{T}\mathcal{T}_{12} + \mathcal{T}\mathcal{T}_{13} + \mathcal{T}\mathcal{T}_{14} + \mathcal{T}\mathcal{T}_{15} + \mathcal{T}\mathcal{T}_{16} + \mathcal{T}\mathcal{T}_{17}
\end{aligned}$$

Analyzing each of the terms $\mathcal{T}\mathcal{T}_1$ to $\mathcal{T}\mathcal{T}_{17}$ individually, we see first of all that, for $\mathcal{T}_1$, we can apply parts (c), (d), and (f) of Lemma B-4, given in Appendix B below, to obtain

$$\begin{aligned}
\|\mathcal{T}\mathcal{T}_1\|_2 &\leq \frac{1}{\sqrt{\lambda_1}} \left\{ \left\| \left(\hat{G}_S - G \right) \hat{D}_{\lambda,\kappa}^{-1} \right\|_2 \left\| \left(\frac{\hat{D}_{\lambda,\kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S \hat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \right. \\
&\quad \left. \times \left\| \hat{D}_{\lambda,\kappa}^{-1} \left[\frac{\hat{G}'_S \hat{\sigma}_{Xy}^S - G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \right\|_2 \right\} \\
&= O\left(\frac{1}{\sqrt{\lambda_1}}\right) O_p\left(\max\left\{\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) \\
&\quad \times O_p\left(\max\left\{\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) \\
&\quad \times O_p\left(\max\left\{\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) \\
&= O_p\left(\max\left\{\frac{c_0^3(N)}{\lambda_1^{\frac{7}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{3}{2}(1-q)}, \frac{[a(N)]^{\frac{3}{2}}}{\lambda_1^2} \left[\frac{\ln N}{T}\right]^{\frac{3}{2}(1-\frac{q}{2})}, \right. \right. \\
&\quad \left. \left. \frac{c_0(N) a(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T}\right]^{(1-\frac{q}{2})}, \frac{c_0^2(N) \sqrt{a(N)}}{\lambda_1^3} \left[\frac{\ln N}{T}\right]^{1-q} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) \\
&= o_p\left(\max\left\{\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right)
\end{aligned}$$

where the last equality above follows from parts (a), (b), (c), and (j) of Lemma D-10, given in Appendix D below. Next, for $\mathcal{T}\mathcal{T}_2$, we apply parts (b), (c), and (f) of Lemma B-4 given

in Appendix B and part (d) of Lemma D-9 given in Appendix D to get

$$\begin{aligned}
\|\mathcal{T}\mathcal{T}_2\|_2 &\leq \frac{1}{\sqrt{\lambda_1}} \left\{ \left\| \left(\widehat{G}_S - G \right) \widehat{D}_{\lambda,\kappa}^{-1} \right\|_2 \left\| \left(\frac{\widehat{D}_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \right. \\
&\quad \left. \times \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \left\| \frac{D_{\lambda,\kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \right\} \\
&= O\left(\frac{1}{\sqrt{\lambda_1}}\right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
&\quad \times O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
&\quad \times O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) O(1) \\
&= O_p \left(\max \left\{ \frac{c_0^3(N)}{\lambda_1^{\frac{7}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{3}{2}(1-q)}, \frac{c_0(N) a(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})}, \right. \right. \\
&\quad \left. \left. \frac{c_0^2(N) \sqrt{a(N)}}{\lambda_1^3} \left[\frac{\ln N}{T} \right]^{(1-q)} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
&= o_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)
\end{aligned}$$

where the last equality above follows from parts (a), (c), and (j) of Lemma D-10. Similar to

the case of $\mathcal{T}\mathcal{T}_2$, for $\mathcal{T}\mathcal{T}_3$, we can apply parts (b), (c), and (f) of Lemma B-4 to get

$$\begin{aligned}
\|\mathcal{T}\mathcal{T}_3\|_2 &\leq \frac{1}{\sqrt{\lambda_1}} \left\{ \left\| (\hat{G}_S - G) \hat{D}_{\lambda, \kappa}^{-1} \right\|_2 \left\| \left(\frac{\hat{D}_{\lambda, \kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S \hat{D}_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \right. \\
&\quad \left. \times \left\| \frac{D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \right\} \\
&= O\left(\frac{1}{\sqrt{\lambda_1}}\right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
&\quad \times O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) O(1) \\
&= O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{1-q}, \frac{a(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})}, \right. \right. \\
&\quad \left. \left. \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
&= o_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)
\end{aligned}$$

where the last equality above follows from parts (d), (e), and (f) of Lemma D-10. Now, applying parts (c) and (d) of Lemma B-4 as well as expression (66), given in the proof of

part (f) of Lemma B-4, we obtain for $\mathcal{TT}_4$

$$\begin{aligned}
& \|\mathcal{TT}_4\|_2 \\
& \leq \frac{1}{\sqrt{\lambda_1}} \left\| (\widehat{G}_S - G) \widehat{D}_{\lambda, \kappa}^{-1} \right\|_2 \left\| \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \left\| \widehat{D}_{\lambda, \kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \right\|_2 \\
& = O\left(\frac{1}{\sqrt{\lambda_1}}\right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& \quad \times O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& = O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{(1-q)}, \frac{a(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})}, \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& = o_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right),
\end{aligned}$$

where the last equality above follows from parts (d), (e), and (f) of Lemma D-10. For $\mathcal{TT}_5$, we apply part (d) of Lemma D-9 and parts (b) and (c) of Lemma B-4 as well as expression (66), given in the proof of part (f) of Lemma B-4 to obtain

$$\begin{aligned}
\|\mathcal{TT}_5\|_2 & \leq \frac{1}{\sqrt{\lambda_1}} \left\| (\widehat{G}_S - G) \widehat{D}_{\lambda, \kappa}^{-1} \right\|_2 \left\| \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \left\| \widehat{D}_{\lambda, \kappa}^{-1} D_{\lambda, \kappa} - I_\kappa \right\|_2 \left\| \frac{D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\
& = O\left(\frac{1}{\sqrt{\lambda_1}}\right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& \quad \times O(1) O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) O(1) \\
& = O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{1-q}, \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& = o_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right),
\end{aligned}$$

where the last equality follows from parts (d) and (e) of Lemma D-10. For $\mathcal{TT}_6$, we apply parts (b) and (c) of Lemma B-4 as well as expression (66) in the proof of Lemma B-4 part

(f) to obtain

$$\begin{aligned}
\|\mathcal{T}\mathcal{T}_6\|_2 &\leq \frac{1}{\sqrt{\lambda_1}} \left\| \left(\widehat{G}_S - G \right) \widehat{D}_{\lambda,\kappa}^{-1} \right\|_2 \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \left\| \frac{D_{\lambda,\kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\
&= O\left(\frac{1}{\sqrt{\lambda_1}}\right) O_p\left(\max\left\{\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}}\right\}\right) O(1) O(1) \\
&= O_p\left(\max\left\{\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right).
\end{aligned}$$

Next, for $\mathcal{T}\mathcal{T}_7$, we apply part (d) of Lemma D-9 as well as parts (a), (d), and (f) of Lemma B-4 to obtain

$$\begin{aligned}
&\|\mathcal{T}\mathcal{T}_7\|_2 \\
&\leq \frac{1}{\sqrt{\lambda_1}} \left\{ \left\| G D_{\lambda,\kappa}^{-1} \right\|_2 \left\| D_{\lambda,\kappa} \widehat{D}_{\lambda,\kappa}^{-1} - I_\kappa \right\|_2 \left\| \left(\frac{\widehat{D}_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \right. \\
&\quad \left. \times \left\| \widehat{D}_{\lambda,\kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \right\|_2 \right\} \\
&= O\left(\frac{1}{\sqrt{\lambda_1}}\right) O(1) O_p\left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right) \\
&\quad \times O_p\left(\max\left\{\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1} \left(\frac{\ln N}{T}\right)^{\frac{1}{2}(1-\frac{q}{2})}}\right\}\right) \\
&\quad \times O_p\left(\max\left\{\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}}\right\}\right) \\
&= O_p\left(\max\left\{\frac{c_0^3(N)}{\lambda_1^{\frac{7}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{3}{2}(1-q)}, \frac{c_0(N) a(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T}\right]^{(1-\frac{q}{2})}, \right. \right. \\
&\quad \left. \left. \frac{c_0^2(N) \sqrt{a(N)}}{\lambda_1^3} \left[\frac{\ln N}{T}\right]^{1-q} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) \\
&= o_p\left(\max\left\{\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right),
\end{aligned}$$

where the last equality above follows from parts (a), (c), and (j) of Lemma D-10 and from

the fact that $1/\lambda_1 = o(1)$. Consider now $\mathcal{T}\mathcal{T}_8$. Making use part (d) of Lemma D-9 as well as parts (a), (b), and (f) of Lemma B-4, we obtain

$$\begin{aligned}
\|\mathcal{T}\mathcal{T}_8\|_2 &\leq \frac{1}{\sqrt{\lambda_1}} \|GD_{\Delta,\kappa}^{-1}\|_2 \|\widehat{D}_{\lambda,\kappa}^{-1}D_{\lambda,\kappa} - I_\kappa\|_2 \left\| \left(\frac{\widehat{D}_{\lambda,\kappa}^{-1}\widehat{G}'_S\widehat{\Sigma}_X^S\widehat{G}_S\widehat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1}G'\overline{\Sigma}_XGD_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \\
&\quad \times \|\widehat{D}_{\lambda,\kappa}^{-1}D_{\lambda,\kappa} - I_\kappa\|_2 \left\| \frac{D_{\lambda,\kappa}^{-1}G'\overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\
&= O\left(\frac{1}{\sqrt{\lambda_1}}\right) O(1) O_p\left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right) \\
&\quad \times O_p\left(\max\left\{\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T}\right)^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) \\
&\quad \times O_p\left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right) O(1) \\
&= O_p\left(\max\left\{\frac{c_0^3(N)}{\lambda_1^{\frac{7}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{3}{2}(1-q)}, \frac{c_0^2(N)\sqrt{a(N)}}{\lambda_1^3} \left[\frac{\ln N}{T}\right]^{(1-q)} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) \\
&= o_p\left(\max\left\{\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right),
\end{aligned}$$

where the last equality follows from parts (a) and (j) of Lemma D-10. In addition, for $\mathcal{T}\mathcal{T}_9$, we can apply part (d) of Lemma D-9 as well as parts (a), (b), and (f) of Lemma B-4 to

obtain

$$\begin{aligned}
\|\mathcal{T}\mathcal{T}_9\|_2 &\leq \frac{1}{\sqrt{\lambda_1}} \|GD_{\lambda,\kappa}^{-1}\|_2 \|\widehat{D}_{\lambda,\kappa}^{-1}D_{\lambda,\kappa} - I_\kappa\|_2 \left\| \left(\frac{\widehat{D}_{\lambda,\kappa}^{-1}\widehat{G}'_S\widehat{\Sigma}_X^S\widehat{G}_S\widehat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1}G'\overline{\Sigma}_X GD_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \\
&\quad \times \left\| \frac{D_{\lambda,\kappa}^{-1}G'\overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\
&= O\left(\frac{1}{\sqrt{\lambda_1}}\right) O(1) O_p\left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right) \\
&\quad \times O_p\left(\max\left\{\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T}\right)^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) O(1) \\
&= O_p\left(\max\left\{\frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T}\right]^{1-q}, \frac{c_0(N)\sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) \\
&= o_p\left(\max\left\{\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right),
\end{aligned}$$

where the last equality follows from parts (d) and (e) of Lemma D-10. Now, for $\mathcal{T}\mathcal{T}_{10}$, we can apply part (d) of Lemma D-9 and parts (a) and (d) of Lemma B-4 as well as expression (66) in the proof of Lemma B-4 part (f) to obtain

$$\begin{aligned}
\|\mathcal{T}\mathcal{T}_{10}\|_2 &\leq \frac{1}{\sqrt{\lambda_1}} \|GD_{\lambda,\kappa}^{-1}\|_2 \|\widehat{D}_{\lambda,\kappa}^{-1}D_{\lambda,\kappa} - I_\kappa\|_2 \left\| \left(\frac{D_{\lambda,\kappa}^{-1}G'\overline{\Sigma}_X GD_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \\
&\quad \times \left\| \widehat{D}_{\lambda,\kappa}^{-1} \left[\frac{\widehat{G}'_S\widehat{\sigma}_{Xy}^S - G'\overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \right\|_2 \\
&= O\left(\frac{1}{\sqrt{\lambda_1}}\right) O(1) O_p\left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right) O(1) \\
&\quad \times O_p\left(\max\left\{\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) \\
&= O_p\left(\max\left\{\frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T}\right]^{1-q}, \frac{c_0(N)\sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) \\
&= o_p\left(\max\left\{\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right),
\end{aligned}$$

where the last equality above follows from parts (d) and (e) of Lemma D-10. Consider $\mathcal{TT}_{11}$. Making use of part (d) of Lemma D-9 and parts (a) and (b) of Lemma B-4 as well as expression (66) in the proof of Lemma B-4 part (f), we obtain

$$\begin{aligned}
\|\mathcal{TT}_{11}\|_2 &\leq \frac{1}{\sqrt{\lambda_1}} \|GD_{\lambda,\kappa}^{-1}\|_2 \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \\
&\quad \times \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \left\| \frac{D_{\lambda,\kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\
&= O\left(\frac{1}{\sqrt{\lambda_1}}\right) O(1) O_p\left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right) O(1) \\
&\quad \times O_p\left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right) O(1) \\
&= O_p\left(\frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T}\right]^{1-q}\right) \\
&= O_p\left(\max\left\{\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right),
\end{aligned}$$

where the last equality follows from part (d) of Lemma D-10. Similarly, for $\mathcal{TT}_{12}$, we can apply part (d) of Lemma D-9 and parts (a) and (b) of Lemma B-4 as well as expression (66) in the proof of Lemma B-4 part (f) to obtain

$$\begin{aligned}
\|\mathcal{TT}_{12}\|_2 &\leq \frac{1}{\sqrt{\lambda_1}} \|GD_{\lambda,\kappa}^{-1}\|_2 \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \\
&\quad \times \left\| \frac{D_{\lambda,\kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\
&= O\left(\frac{1}{\sqrt{\lambda_1}}\right) O(1) O_p\left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right) O(1) O(1) \\
&= O_p\left(\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right).
\end{aligned}$$

Now, consider $\mathcal{TT}_{13}$. Here, we can apply parts (a), (d), and (f) of Lemma B-4 to obtain

$$\begin{aligned}
& \|\mathcal{TT}_{13}\|_2 \\
& \leq \frac{1}{\sqrt{\lambda_1}} \|GD_{\lambda,\kappa}^{-1}\|_2 \left\| \left(\frac{\widehat{D}_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \left\| \widehat{D}_{\lambda,\kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \right\|_2 \\
& = O\left(\frac{1}{\sqrt{\lambda_1}}\right) O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& \quad \times O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& = O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{1-q}, \frac{a(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})}, \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& = o_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right),
\end{aligned}$$

where the last equality follows from parts (d), (e), and (f) of Lemma D-10. Next, consider $\mathcal{TT}_{14}$. In this case, we can apply part (d) of Lemma D-9 as well as parts (a), (b), and (f) of Lemma B-4 to obtain

$$\begin{aligned}
& \|\mathcal{TT}_{14}\|_2 \\
& \leq \frac{1}{\sqrt{\lambda_1}} \left\{ \|GD_{\lambda,\kappa}^{-1}\|_2 \left\| \left(\frac{\widehat{D}_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \right. \\
& \quad \left. \times \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \left\| \frac{D_{\lambda,\kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \right\} \\
& = O\left(\frac{1}{\sqrt{\lambda_1}}\right) O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& \quad \times O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) O(1) \\
& = O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{1-q}, \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& = o_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right),
\end{aligned}$$

where the last equality follows from parts (d) and (e) of Lemma D-10. In a similar way for the term $\mathcal{T}\mathcal{T}_{15}$, we can apply parts (a), (b), and (f) of Lemma B-4 to obtain

$$\begin{aligned}
& \|\mathcal{T}\mathcal{T}_{15}\|_2 \\
& \leq \frac{1}{\sqrt{\lambda_1}} \|GD_{\lambda,\kappa}^{-1}\|_2 \left\| \left(\frac{\widehat{D}_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \left\| \frac{D_{\lambda,\kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\
& = O\left(\frac{1}{\sqrt{\lambda_1}}\right) O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) O(1) \\
& \quad \text{(given that, by Lemma D-3(a), there exists absolute constant } c^* > 1 \text{ such that} \\
& \quad 0 < \frac{1}{c^*} < \frac{\lambda_1}{\Delta_{1,1}} \leq c^* < \infty \text{ for all } N \text{ and } T \text{ sufficiently large)} \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)
\end{aligned}$$

Turning our attention to $\mathcal{T}\mathcal{T}_{16}$, note that, for this term, we can apply parts (a) and (d) of Lemma B-4 as well as expression (66) in the proof of Lemma B-4 part (f) to obtain

$$\begin{aligned}
& \|\mathcal{T}\mathcal{T}_{16}\|_2 \\
& \leq \frac{1}{\sqrt{\lambda_1}} \|GD_{\lambda,\kappa}^{-1}\|_2 \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \left\| \widehat{D}_{\lambda,\kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \right\|_2 \\
& = O\left(\frac{1}{\sqrt{\lambda_1}}\right) O(1) O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)
\end{aligned}$$

Finally, for $\mathcal{T}\mathcal{T}_{17}$, we can apply part (d) of Lemma D-9 and parts (a) and (b) of Lemma B-4

as well as expression (66) in the proof of part (f) of Lemma B-4 to get

$$\begin{aligned}
\|\mathcal{T}\mathcal{T}_{17}\|_2 &\leq \frac{1}{\sqrt{\lambda_1}} \|GD_{\lambda,\kappa}^{-1}\|_2 \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \left\| \hat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \left\| \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\
&= O\left(\frac{1}{\sqrt{\lambda_1}}\right) O(1) O(1) O_p\left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right) O(1) \\
&= O_p\left(\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right)
\end{aligned}$$

Combining the results given by the above calculations, we see that

$$\begin{aligned}
&\left\| \hat{\beta}_{\text{grpls}} - \bar{\beta} \right\|_2 \\
&\leq \|\mathcal{T}\mathcal{T}_1\|_2 + \|\mathcal{T}\mathcal{T}_2\|_2 + \|\mathcal{T}\mathcal{T}_3\|_2 + \|\mathcal{T}\mathcal{T}_4\|_2 + \|\mathcal{T}\mathcal{T}_5\|_2 + \|\mathcal{T}\mathcal{T}_6\|_2 + \|\mathcal{T}\mathcal{T}_7\|_2 + \|\mathcal{T}\mathcal{T}_8\|_2 \\
&\quad + \|\mathcal{T}\mathcal{T}_9\|_2 + \|\mathcal{T}\mathcal{T}_{10}\|_2 + \|\mathcal{T}\mathcal{T}_{11}\|_2 + \|\mathcal{T}\mathcal{T}_{12}\|_2 + \|\mathcal{T}\mathcal{T}_{13}\|_2 + \|\mathcal{T}\mathcal{T}_{14}\|_2 + \|\mathcal{T}\mathcal{T}_{15}\|_2 \\
&\quad + \|\mathcal{T}\mathcal{T}_{16}\|_2 + \|\mathcal{T}\mathcal{T}_{17}\|_2 \\
&= (\|\mathcal{T}\mathcal{T}_6\|_2 + \|\mathcal{T}\mathcal{T}_{15}\|_2 + \|\mathcal{T}\mathcal{T}_{16}\|_2) + (\|\mathcal{T}\mathcal{T}_{12}\|_2 + \|\mathcal{T}\mathcal{T}_{17}\|_2) \\
&\quad + (\|\mathcal{T}\mathcal{T}_1\|_2 + \|\mathcal{T}\mathcal{T}_2\|_2 + \|\mathcal{T}\mathcal{T}_3\|_2 + \|\mathcal{T}\mathcal{T}_4\|_2 + \|\mathcal{T}\mathcal{T}_5\|_2 + \|\mathcal{T}\mathcal{T}_7\|_2 + \|\mathcal{T}\mathcal{T}_8\|_2 \\
&\quad \quad \quad \|\mathcal{T}\mathcal{T}_9\|_2 + \|\mathcal{T}\mathcal{T}_{10}\|_2 + \|\mathcal{T}\mathcal{T}_{11}\|_2 + \|\mathcal{T}\mathcal{T}_{13}\|_2 + \|\mathcal{T}\mathcal{T}_{14}\|_2) \\
&= O_p\left(\max\left\{\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) \\
&\quad + O_p\left(\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right) \\
&\quad + o_p\left(\max\left\{\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) \\
&= O_p\left(\max\left\{\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right\}\right) \\
&= o_p(1)
\end{aligned}$$

where the last equality above follows from the rate conditions given in Assumption 6-3, i.e.,

$$\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}} = o(1) \text{ and } \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})} = o(1), \text{ as } N, T, \lambda_1 \rightarrow \infty.$$

Next, note that, for the second term $\bar{\beta} - \beta$ on the right-hand side of expression (27), part

(b) of Theorem 2 shows that

$$\|\bar{\beta} - \beta\|_2 = O\left(\frac{1}{\lambda_1^{\frac{3}{2}}}\right) = o(1).$$

where the second equality above follows from the fact that $\lambda_1 \rightarrow \infty$ as $N, T \rightarrow \infty$ since $\lambda_1 \asymp \underline{\lambda}_{1,1}$ by Lemma D-3(a) and $\underline{\lambda}_{1,1} \rightarrow \infty$ by part (a) of Assumption 4-2. It follows immediately by the triangle inequality that

$$\begin{aligned} & \left\| \hat{\beta}_{\text{grpls}} - \beta \right\|_2 \\ & \leq \left\| \hat{\beta}_{\text{grpls}} - \bar{\beta} \right\|_2 + \|\bar{\beta} - \beta\|_2 \\ & = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) + O\left(\frac{1}{\lambda_1^{\frac{3}{2}}}\right) \\ & = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right) \\ & = o_p(1). \quad \square \end{aligned}$$

Proof of Part (b) of Theorem 6:

To show part (b) of Theorem 6, note that, by the results given in Theorem 5, we have that

$$\left\| \hat{\beta}_{\text{grpls}} - \beta \right\|_2 = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right)$$

and

$$\left\| \hat{\beta}_{\text{drpls}} - \beta \right\|_2 = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right)$$

so that $\hat{\beta}_{\text{grpls}}$ and $\hat{\beta}_{\text{drpls}}$ have the same rate of convergence. Hence, it is easily seen that by following the same argument as that given in the Supplemental Appendix for the proof of part (a) of Theorem 6, we can show that

$$\begin{aligned} & \hat{y}_{T+h}^{\text{grpls}} - [\alpha + (X_T - \mu_X)' \beta] \\ & = O_p \left(\max \left\{ \frac{c_0(N) \sqrt{N}}{\lambda_1^{\frac{3}{2}}} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \frac{\sqrt{a(N)N}}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{\sqrt{N}}{\lambda_1^{\frac{3}{2}}} \right\} \right) + O_p \left(\frac{1}{\sqrt{T}} \right), \end{aligned}$$

which is the required result for part (b) of Theorem 6, $\square$

Proof of Theorem 7:

To show part (a), note first that, under Scenario 1 and in the case where the point forecast is based on $\widehat{\beta}_{\text{drpls}}$, we can write the deviation of the point forecast from the conditional mean function of the forecast equation as follows:

$$\begin{aligned}
& \widehat{y}_{T+h}^{\text{drpls}} - \left(\alpha + [F_{1,T} - \mu_{F,1}]' \varphi_1 \right) \\
&= \bar{y}_{T_h} + [X_T - \bar{X}_{T_h}]' \widehat{\beta}_{\text{drpls}} - \left(\alpha + [X_T - \mu_X]' \beta \right) + \left(\alpha + [X_T - \mu_X]' \beta \right) \\
&\quad - \left(\alpha + [F_{1,T} - \mu_{F,1}]' \varphi_1 \right) \\
&= \alpha + (\bar{X}_{T_h} - \mu_X)' \beta - \bar{u}_{T_h}' \beta + \bar{\varepsilon}_{T_h} + [X_T - \bar{X}_{T_h}]' \widehat{\beta}_{\text{drpls}} - \left(\alpha + [X_T - \mu_X]' \beta \right) \\
&\quad + \left([X_T - \mu_X]' \beta - [F_{1,T} - \mu_{F,1}]' \varphi_1 \right) \\
&= [X_T - \mu_X]' \left(\widehat{\beta}_{\text{drpls}} - \beta \right) - (\bar{X}_{T_h} - \mu_X)' \left(\widehat{\beta}_{\text{drpls}} - \beta \right) - \bar{u}_{T_h}' \beta + \bar{\varepsilon}_{T_h} \\
&\quad + \left([X_T - \mu_X]' \beta - [F_{1,T} - \mu_{F,1}]' \varphi_1 \right), \tag{28}
\end{aligned}$$

where, in this case, $\beta = \Gamma_{\lambda,1} (\varphi_1 / \sqrt{\underline{\lambda}_{1,1}})$ with $\Gamma_{\lambda,1} = \Gamma_1 / \sqrt{\underline{\lambda}_{1,1}}$ and where

$$\bar{X}_{T_h} = \frac{1}{T_h} \sum_{t=1}^{T_h} X_t, \quad \bar{u}_{T_h} = \frac{1}{T_h} \sum_{t=1}^{T_h} u_t, \quad \text{and} \quad \bar{\varepsilon}_{T_h} = \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h}, \quad \text{with } T_h = T - h$$

Consider the first term on the right-hand side of expression (28). To proceed, note first that

$$\begin{aligned}
& \left| [X_T - \mu_X]' \left(\widehat{\beta}_{\text{drpls}} - \beta \right) \right| \\
&\leq \|X_T - \mu_X\|_2 \left\| \widehat{\beta}_{\text{drpls}} - \beta \right\|_2 \quad (\text{by the Cauchy-Schwarz inequality}) \\
&= \sqrt{[X_T - \mu_X]' [X_T - \mu_X]} \sqrt{\left(\widehat{\beta}_{\text{drpls}} - \beta \right)' \left(\widehat{\beta}_{\text{drpls}} - \beta \right)} \\
&= \sqrt{\text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \}} \left\| \widehat{\beta}_{\text{drpls}} - \beta \right\|_2
\end{aligned}$$

Applying Assumption (ii), we have, for all N and T ,

$$\begin{aligned}
& E \left(N^{-1} \text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \} \right) \\
&= \frac{1}{N} \text{tr} \{ E \left([X_T - \mu_X] [X_T - \mu_X]' \right) \} \\
&= \frac{1}{N} \text{tr} \{ E \left([\Gamma (F_T - \mu_F) + u_T] [\Gamma (F_T - \mu_F) + u_T]' \right) \} \\
&= \frac{1}{N} \text{tr} \{ \Gamma E \left[(F_T - \mu_F) (F_T - \mu_F)' \right] \Gamma' + \Gamma E \left[(F_T - \mu_F) u_T' \right] + E \left[u_T (F_T - \mu_F)' \right] \Gamma' + E \left[u_T u_T' \right] \} \\
&= \frac{1}{N} \text{tr} \{ \Gamma E \left[(F_T - \mu_F) (F_T - \mu_F)' \right] \Gamma' + \Sigma_{u,T} \} \\
&\quad \left(\text{since } E \left((F_T - \mu_F) u_T' \right) = 0 \text{ and } E \left(u_T (F_T - \mu_F)' \right) = 0 \text{ by Assumption (ii)} \right) \\
&= \frac{\underline{\lambda}_{1,1} E \left[(F_T - \mu_F)' (\Gamma' \Gamma / \underline{\lambda}_{1,1}) (F_T - \mu_F) \right]}{N} + \frac{1}{N} \text{tr} \{ \Sigma_{u,T} \} \\
&= \frac{\underline{\lambda}_{1,1} E \left[(F_T - \mu_F)' (F_T - \mu_F) \right]}{N} + \frac{1}{N} \text{tr} \{ \Sigma_{u,T} \} \quad \left(\text{since } \frac{\Gamma' \Gamma}{\underline{\lambda}_{1,1}} = I_\kappa \right) \\
&\leq \frac{\underline{\lambda}_{1,1}}{N} K \sqrt{C^\dagger} + \frac{1}{N} \text{tr} \{ \Sigma_{u,T} \} \quad \left(\text{since, by Assumption (ii) and Liapunov's inequality, we have} \right. \\
&\quad \left. E \left\{ (F_{T,g} - \mu_{F,g})^2 \right\} \leq \left[E \left\{ (F_{T,g} - \mu_{F,g})^4 \right\} \right]^{\frac{1}{2}} \leq \sqrt{C^\dagger} \text{ for all } g \in \{1, \dots, K\} \right) \\
&\leq \frac{\underline{\lambda}_{1,1}}{N} K \sqrt{C^\dagger} + \frac{N}{N} \lambda_{\max} (\Sigma_{u,T}) \\
&\leq C^* K \sqrt{C^\dagger} + C^\dagger \\
&\quad \left(\text{since } \underline{\lambda}_{1,1} \text{ cannot have order of magnitude greater than } N, \text{ there exists a constant } C^* > 1 \text{ such} \right. \\
&\quad \left. \text{that } \frac{\underline{\lambda}_{1,1}}{N} \leq C^* \text{ for all } N, T \text{ sufficiently large. Also, by Assumption (ii), } \sup_t \lambda_{\max} (\Sigma_{u,t}) \leq C^\dagger \right) \\
&\leq C < \infty
\end{aligned}$$

for some constant $C \geq C^* K \sqrt{C^\dagger} + C^\dagger$. It then follows that for any $\epsilon > 0$, we can set $C_\epsilon = C/\epsilon$; and, by Markov's inequality, we have, for all N and T sufficiently large,

$$\begin{aligned}
P \left(\left| \frac{\text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \}}{N} \right| \geq C_\epsilon \right) &= P \left(\left| \frac{\text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \}}{N} \right| \geq \frac{C}{\epsilon} \right) \\
&\leq \frac{\epsilon}{C} E \left(N^{-1} \text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \} \right) \\
&\leq \frac{\epsilon}{C} C \\
&= \epsilon
\end{aligned}$$

so that

$$\text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \} = O_p(N)$$

from which we further deduce that

$$\sqrt{\text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]'\}} = O_p(\sqrt{N}).$$

Now, to analyze the asymptotic behavior of $\|\widehat{\beta}_{\text{drpls}} - \beta\|_2$, first write

$$\widehat{\beta}_{\text{drpls}} - \beta = \widehat{\beta}_{\text{drpls}} - \bar{\beta} + \bar{\beta} - \beta$$

so that, by the triangle inequality,

$$\|\widehat{\beta}_{\text{drpls}} - \beta\|_2 \leq \|\widehat{\beta}_{\text{drpls}} - \bar{\beta}\|_2 + \|\bar{\beta} - \beta\|_2 \quad (29)$$

We can then make use of argument very similar to what we have given previously in the proof of part (a) of Theorem 5 to show that, for the first term on the right-hand side of expression (29) above, we have

$$\|\widehat{\beta}_{\text{drpls}} - \bar{\beta}\|_2 = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right) \quad (30)$$

To avoid too much duplication, we will only give a sketch of the argument here, although, in the supporting lemmas given in Appendix E, we do provide details in areas where there are sufficient difference between the proof needed here and the proof given earlier for the many-predictors regression case. In particular, as shown in Appendix E, given the diffusion index model setup, i.e.,

$$\begin{aligned} y_{t+h} &= \alpha + (F_t - \mu_F)' \varphi + \varepsilon_{t+h}, \\ X_t &= \mu_X + \Gamma (F_t - \mu_F) + u_t, \end{aligned}$$

the population covariance vector $\bar{\sigma}_{Xy}$ under Scenario 1 has the representation

$$\begin{aligned} \bar{\sigma}_{Xy} &= \frac{1}{T_h} \sum_{t=1}^{T_h} E \{ (X_t - \mu_X) (y_{t+h} - \alpha) \} \\ &= \Gamma_1 \frac{\underline{\Lambda}_1}{\underline{\lambda}_{1,1}} \varphi_1 = \Gamma_{\lambda,1} \frac{\underline{\Lambda}_1}{\sqrt{\underline{\lambda}_{1,1}}} \varphi_1 \left(\text{where } \Gamma_{\lambda,1} = \frac{\Gamma_1}{\sqrt{\underline{\lambda}_{1,1}}} \right), \end{aligned} \quad (31)$$

which is a bit different from its representation in the case where the forecast equation is a many-predictors regression:

$$y_{t+h} = \alpha + (X_t - \mu_X)' \beta + \varepsilon_{t+h}$$

with $E\{(X_t - \mu_X)\varepsilon_{t+h}\} = 0$ for all t^4 . Hence, a key part of our proof for the diffusion index case given here is to show a uniform convergence result

$$\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| = O_p \left(\sqrt{\frac{\ln N}{T}} \right) \quad (32)$$

for this case, where $\bar{\sigma}_{Xy,i}$ here denotes the i^{th} element of the (population) covariance vector $\bar{\sigma}_{Xy}$ given in expression (31) above and where $\hat{\sigma}_{Xy,i}$ is the i^{th} element of the usual sample estimator

$$\hat{\sigma}_{Xy} = \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \bar{X}_{i,T_h}] y_{t+h}.$$

A proof of the result given in expression (32) under Scenario 1 can be found in part (a) of Lemma E-14 in Appendix E. In addition, in Lemma E-10 of Appendix E, we also prove the uniform convergence result

$$\max_{1 \leq i,j \leq N} |\hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)}| = O_p \left(\sqrt{\frac{\ln N}{T}} \right) \quad (33)$$

under the assumptions of this theorem which impose more primitive conditions for the particular case where the underlying data generating mechanism is a diffusion index model. Since the generalized thresholding estimators $\hat{\sigma}_{Xy}^S$ and $\hat{\Sigma}_X^S$ are obtained by performing element-by-element transformation of the usual (unregularized) sample estimators $\hat{\sigma}_{Xy}$ and $\hat{\Sigma}_X$, given the uniform convergence results stated in expressions (32) and (33), we can then use the same argument as that given in the proofs of Lemma C-6 and Lemma C-9 to show that

$$\left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2 = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) = o_p(1) \text{ for } 0 \leq q < 1 \quad (34)$$

and

$$\left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 = O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right) \text{ for } 0 \leq q < 1. \quad (35)$$

The convergence results for the generalized thresholding estimators $\hat{\Sigma}_X^S$ and $\hat{\sigma}_{Xy}^S$ given in expressions (34) and (35) can then be applied to obtain results similar to that stated in Lemma B-3 and Lemma B-4, but for the diffusion index case. The method of proof is essentially the same as that which has previously been used to establish comparable results for the many-predictors regression case as given in these lemmas. We do make some minor modifications in the proof of parts (a) and (c) of Lemma B-3 and of parts (a) and (b) of

⁴An explicit derivation of the representation for $\bar{\sigma}_{Xy}$ given in expression (31) can be found at the proof of part (a) of Lemma E-1 in Appendix E.

Lemma B-4 to obtain analogous results for the diffusion index case. For the case where $1 \leq \kappa < K$ (i.e., Scenario 1), the diffusion index counterparts of the results given in parts (a) and (c) of Lemma B-3 are given in part (a) of Lemma E-15 and in part (b) of Lemma E-2, respectively. On the other hand, the diffusion index counterparts of the results given in parts (a) and (b) of Lemma B-4 are given in parts (a) and (b) of Lemma E-16, respectively. Outside of these modifications, we use essentially the same argument as that given in the proof of part (a) of Theorem 5⁵ to obtain the asymptotic result stated in expression (30) above.

Consider next $\|\bar{\beta} - \beta\|_2$, the second term on the right-hand side of expression (29) above. To analyze the asymptotic behavior of $\|\bar{\beta} - \beta\|_2$ in the diffusion index case under Scenario 1, we need to apply part (a) of Lemma E-1 to obtain

$$\|\bar{\beta} - \beta\|_2 = O\left(\frac{1}{\lambda_1^{\frac{3}{2}}}\right), \text{ where } \lambda_1 = \lambda_{\max}(\bar{\Sigma}_X). \quad (36)$$

It follows from expressions (29), (30), and (36) that

$$\begin{aligned} & \left\| \hat{\beta}_{\text{drpls}} - \beta \right\|_2 \\ & \leq \left\| \hat{\beta}_{\text{drpls}} - \bar{\beta} \right\|_2 + \|\bar{\beta} - \beta\|_2 \\ & = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right) + O\left(\frac{1}{\lambda_1^{\frac{3}{2}}}\right) \\ & = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right). \end{aligned} \quad (37)$$

Making use of the asymptotic result for $\left\| \hat{\beta}_{\text{drpls}} - \beta \right\|_2$ given in expression (37) above, we then obtain

$$\begin{aligned} & \left| [X_T - \mu_X]' (\hat{\beta}_{\text{drpls}} - \beta) \right| \\ & \leq \sqrt{\text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \}} \left\| \hat{\beta}_{\text{drpls}} - \beta \right\|_2 \\ & = O_p(\sqrt{N}) \left[O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right) \right] \\ & = O_p \left(\max \left\{ \sqrt{\frac{N}{\lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{N}{\lambda_1}} \frac{\sqrt{a(N)}}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{\sqrt{N}}{\lambda_1^{\frac{3}{2}}} \right\} \right) \end{aligned}$$

⁵The proof of part (a) of Theorem 5 is given in the Supplemental Appendix.

from which it follows that

$$\begin{aligned} & [X_T - \mu_X]' (\hat{\beta}_{\text{drpls}} - \beta) \\ &= O_p \left(\max \left\{ \sqrt{\frac{N}{\lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{N}{\lambda_1}} \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{\sqrt{N}}{\lambda_1^{\frac{3}{2}}} \right\} \right). \end{aligned} \quad (38)$$

Consider now the second term on the right-hand side of expression (28), i.e., the term $(\bar{X}_{T_h} - \mu_X)' (\hat{\beta}_{\text{drpls}} - \beta)$. To analyze this term, we follow an approach similar to that used in the proof of Theorem 6. To proceed, we first define

$$L_1 = \Gamma_1 / \sqrt{\lambda_{1,1}} \text{ and } L_2 = \begin{pmatrix} \Gamma_2 / \sqrt{\lambda_{1,1}} & L_{22} \end{pmatrix}$$

Here, L_{22} denotes an $N \times (N - K)$ semi-orthogonal matrix whose columns are the orthonormal eigenvectors associated with the 0 eigenvalue of $\Sigma_{\text{com}} = \Gamma \bar{\Sigma}_F \Gamma'$, which in this case has algebraic multiplicity equal to $N - K$. By the orthonormality of eigenvectors, it follows that $L_{22}' L_{22} = I_{N-K}$, $L_{22}' \Gamma_1 / \sqrt{\lambda_{1,1}} = 0$, and $L_{22}' \Gamma_2 / \sqrt{\lambda_{1,1}} = 0$, so that $L = \begin{pmatrix} L_1 & L_2 \end{pmatrix}$ is an $N \times N$ orthogonal matrix. Given these notations, we can write

$$\begin{aligned} & (\bar{X}_{T_h} - \mu_X)' (\hat{\beta}_{\text{drpls}} - \beta) \\ &= \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' (\hat{\beta}_{\text{drpls}} - \beta) \\ &= \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' \begin{pmatrix} L_1 & L_2 \end{pmatrix} \begin{pmatrix} L_1' \\ L_2' \end{pmatrix} (\hat{\beta}_{\text{drpls}} - \beta) \\ &= \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_1 L_1' (\hat{\beta}_{\text{drpls}} - \beta) + \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_2 L_2' (\hat{\beta}_{\text{drpls}} - \beta) \end{aligned} \quad (39)$$

To analyze the first term on the right-hand side of expression (39) above, note that

$$\begin{aligned}
& \left| \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_1 L_1' \left(\widehat{\beta}_{\text{drpls}} - \beta \right) \right| \\
& \leq \left\| L_1' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right) \right\|_2 \left\| L_1' \left(\widehat{\beta}_{\text{drpls}} - \beta \right) \right\|_2 \\
& = \sqrt{\left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right)' L_1 L_1' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right)} \sqrt{\left(\widehat{\beta}_{\text{drpls}} - \beta \right)' L_1 L_1' \left(\widehat{\beta}_{\text{drpls}} - \beta \right)} \\
& = \sqrt{tr \left\{ L_1' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right)' L_1 \right\}} \sqrt{\left(\widehat{\beta}_{\text{drpls}} - \beta \right)' L_1 L_1' \left(\widehat{\beta}_{\text{drpls}} - \beta \right)}
\end{aligned}$$

Now,

$$\begin{aligned}
& tr \left\{ L_1' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X] \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [X_s - \mu_X] \right)' L_1 \right\} \\
& = tr \left\{ \frac{\Gamma_1'}{\sqrt{\Delta_{1,1}}} \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [\Gamma(F_t - \mu_F) + u_t] \right) \frac{1}{T_h} \sum_{s=1}^{T_h} [(F_s - \mu_F)' \Gamma' + u_s'] \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \\
& = tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1'}{\sqrt{\Delta_{1,1}}} \Gamma(F_t - \mu_F) \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& \quad + tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1'}{\sqrt{\Delta_{1,1}}} \Gamma(F_t - \mu_F) \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u_s' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& \quad + tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' u_t}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& \quad + tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' u_t}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u_s' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \tag{40}
\end{aligned}$$

Moreover, let the $\kappa \times 1$ vector $e_{i,\kappa}$ denote an elementary vector whose i^{th} component equals

1 and all other components are 0, and note that

$$\begin{aligned}
& \left| tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} \Gamma(F_t - \mu_F) \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u'_s \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \right| \\
& \leq \sum_{i=1}^{\kappa} \left| e'_{i,\kappa} \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} \Gamma(F_t - \mu_F) \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u'_s \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] e_{i,\kappa} \right| \\
& \leq \sum_{i=1}^{\kappa} \left\{ \frac{1}{2} e'_{i,\kappa} \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} \Gamma(F_t - \mu_F) \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right] e_{i,\kappa} \right. \\
& \quad \left. + \frac{1}{2} e'_{i,\kappa} \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u'_s \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] e_{i,\kappa} \right\} \\
& \quad \left(\text{using the inequality } |xy| \leq \frac{1}{2}x^2 + \frac{1}{2}y^2 \right) \\
& = \frac{1}{2} tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} \Gamma(F_t - \mu_F) \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& \quad + \frac{1}{2} tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u'_s \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \tag{41}
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
& \left| tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \right| \\
& \leq \frac{1}{2} tr \left\{ \frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} \Gamma(F_t - \mu_F) \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \\
& \quad + \frac{1}{2} tr \left\{ \frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u'_s \Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \tag{42}
\end{aligned}$$

Applying expressions (41) and (42) to expression (40), we then have

$$\begin{aligned}
& tr \left\{ L_1' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right)' L_1 \right\} \\
= & tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' \Gamma (F_t - \mu_F)}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{(F_s - \mu_F)' \Gamma' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& + tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' u_t}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u_s' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& + tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' \Gamma (F_t - \mu_F)}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u_s' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& + tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' u_t}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{(F_s - \mu_F)' \Gamma' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
\leq & tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' \Gamma (F_t - \mu_F)}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{(F_s - \mu_F)' \Gamma' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& + tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' u_t}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u_s' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& + \left| tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' \Gamma (F_t - \mu_F)}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u_s' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \right| \\
& + \left| tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' u_t}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{(F_s - \mu_F)' \Gamma' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \right| \\
\leq & tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' \Gamma (F_t - \mu_F)}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{(F_s - \mu_F)' \Gamma' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& + tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' u_t}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u_s' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& + \frac{1}{2} tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' \Gamma (F_t - \mu_F)}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{(F_s - \mu_F)' \Gamma' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& + \frac{1}{2} tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' u_t}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u_s' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& + \frac{1}{2} tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' \Gamma (F_t - \mu_F)}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{(F_s - \mu_F)' \Gamma' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
& + \frac{1}{2} tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma_1' u_t}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u_s' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\}
\end{aligned}$$

$$\begin{aligned}
&= 2tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma'_1 \Gamma (F_t - \mu_F)}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{(F_s - \mu_F)' \Gamma' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
&\quad + 2tr \left\{ \left[\frac{1}{T_h} \sum_{t=1}^{T_h} \frac{\Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} \frac{u'_s \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
&= 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} tr \left\{ \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} \Gamma (F_t - \mu_F) (F_s - \mu_F)' \Gamma' \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \\
&\quad + 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} tr \left\{ \frac{\Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \frac{u'_s \Gamma_1}{\sqrt{\Delta_{1,1}}} \right\}. \tag{43}
\end{aligned}$$

Consider the first term on the right-hand side of expression (43). Taking the expectation of this term and dividing it by $\lambda_1 = \lambda_{\max}(\bar{\Sigma}_X)$, we get

$$\begin{aligned}
0 &\leq \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} tr \left\{ \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} \Gamma E[(F_t - \mu_F)(F_s - \mu_F)'] \Gamma' \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \\
&= \frac{2\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} tr \left\{ \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} \begin{pmatrix} \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} & \frac{\Gamma_2}{\sqrt{\Delta_{1,1}}} \end{pmatrix} E[(F_t - \mu_F)(F_s - \mu_F)'] \begin{pmatrix} \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} \\ \frac{\Gamma'_2}{\sqrt{\Delta_{1,1}}} \end{pmatrix} \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \\
&= 2 \frac{\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} tr \left\{ \begin{pmatrix} I_\kappa & 0 \end{pmatrix} E[(F_t - \mu_F)(F_s - \mu_F)'] \begin{pmatrix} I_\kappa \\ 0 \end{pmatrix} \right\} \\
&= 2 \frac{\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} tr \left\{ E[(F_{1,t} - \mu_{F,1})(F_{1,s} - \mu_{F,1})'] \right\} \\
&= 2 \frac{\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} E[(F_{1,s} - \mu_{F,1})'(F_{1,t} - \mu_{F,1})] \\
&\leq 2 \frac{\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \sum_{g=1}^{\kappa} E[(F_{1,s,\cdot,g} - \mu_{F,1,\cdot,g})(F_{1,t,\cdot,g} - \mu_{F,1,\cdot,g})] \\
&\quad \text{(where } F_{1,s,\cdot,g} \text{ and } F_{1,t,\cdot,g} \text{ are, respectively, the } g^{th} \text{ element of } F_{1,s} \\
&\quad \text{and of } F_{1,t} \text{ and where } \mu_{F,1,\cdot,g} = E[F_{1,t,\cdot,g}] = E[F_{1,s,\cdot,g}]) \\
&= \frac{2\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{g=1}^{\kappa} E[(F_{1,t,\cdot,g} - \mu_{F,1,\cdot,g})^2] \\
&\quad + \frac{4\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} |E[(F_{1,t,\cdot,g} - \mu_{F,1,\cdot,g})(F_{1,t+m,\cdot,g} - \mu_{F,1,\cdot,g})]|
\end{aligned}$$

Applying Corollary 14.3 of Davidson (1994), we further obtain

$$\begin{aligned}
0 &\leq \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \left\{ \frac{\Gamma'_1 \Gamma}{\sqrt{\Delta_{1,1}}} E \left[(F_t - \mu_F) (F_s - \mu_F)' \right] \frac{\Gamma \Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \\
&\leq \frac{2\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{g=1}^{\kappa} E \left[(F_{1,t,g} - \mu_{F,1,g})^2 \right] \\
&\quad + \frac{4\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} \left| E \left[(F_{1,t,g} - \mu_{F,1,g}) (F_{1,t+m,g} - \mu_{F,1,g}) \right] \right| \\
&\leq \frac{2\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{g=1}^{\kappa} E \left[(F_{1,t,g} - \mu_{F,1,g})^2 \right] + \\
&\quad \frac{4\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) |\alpha(m)|^{1-\frac{1}{2}-\frac{1}{4}} \sqrt{E |F_{1,t,g} - \mu_{F,1,g}|^2} \left[E |F_{1,t+m,g} - \mu_{F,1,g}|^4 \right]^{\frac{1}{4}} \\
&\quad \text{(by Corollary 14.3 of Davidson (1994))} \\
&\leq \frac{2\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{g=1}^{\kappa} \sqrt{C^\dagger} + \frac{4\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) |\alpha(m)|^{\frac{1}{4}} \sqrt{C^\dagger} \\
&\quad \text{(applying Assumption (ii) and Liapunov's inequality)} \\
&\leq \frac{2\lambda_{1,1}}{\lambda_1} \kappa \sqrt{C^\dagger} \left(\frac{1}{T_h} \right) + \frac{8\lambda_{1,1}}{\lambda_1} \kappa \left(\frac{1}{T_h} \right) \sqrt{C^\dagger} \left(2^{\frac{1}{2}} + 1 \right) \sum_{m=1}^T |\alpha(m)|^{\frac{1}{4}} \\
&\leq \frac{2\lambda_{1,1}}{\lambda_1} \frac{\kappa}{T_h} \sqrt{C^\dagger} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \right] \\
&\quad \text{(by Assumption (v) part (b))} \\
&\leq \frac{2\lambda_{1,1}}{\lambda_1} \frac{\kappa}{T_h} \sqrt{C^\dagger} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \left(\text{since } \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \leq C^\# < \infty \right. \\
&\quad \left. \text{for some constant } C^\# > 0 \text{ as shown in Lemma D-11 of Appendix D} \right) \\
&\leq 2c^* \frac{\kappa}{T_h} \sqrt{C^\dagger} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \left(\text{since Lemma D-3(a) implies that there exists} \right. \\
&\quad \left. \text{absolute constant } c^* > 1 \text{ such that } 0 < \frac{1}{c^*} < \frac{\lambda_{1,1}}{\lambda_1} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large} \right) \\
&\leq \frac{C_1^\Delta}{T_h} \left(\text{for some constant } C_1^\Delta \geq 2c^* \kappa \sqrt{C^\dagger} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \right)
\end{aligned}$$

It follows that for any $\epsilon > 0$, we can set $C_\epsilon^{(1)} = C_1^\Delta/\epsilon$; and, by Markov's inequality, we have

$$\begin{aligned}
& P \left(\left| \frac{2T_h}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \text{tr} \left\{ \left[\sum_{t=1}^{T_h} \frac{\Gamma'_1 \Gamma (F_t - \mu_F)}{\sqrt{\Delta_{1,1}}} \right] \left[\sum_{s=1}^{T_h} \frac{(F_s - \mu_F)' \Gamma' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \right| \geq C_\epsilon^{(1)} \right) \\
&= P \left(\left| \frac{2T_h}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \text{tr} \left\{ \left[\sum_{t=1}^{T_h} \frac{\Gamma'_1 \Gamma (F_t - \mu_F)}{\sqrt{\Delta_{1,1}}} \right] \left[\sum_{s=1}^{T_h} \frac{(F_s - \mu_F)' \Gamma' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \right| \geq \frac{C_1^\Delta}{\epsilon} \right) \\
&\leq \frac{\epsilon}{C_1^\Delta} E \left[\frac{2T_h}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \text{tr} \left\{ \left[\sum_{t=1}^{T_h} \frac{\Gamma'_1 \Gamma (F_t - \mu_F)}{\sqrt{\Delta_{1,1}}} \right] \left[\sum_{s=1}^{T_h} \frac{(F_s - \mu_F)' \Gamma' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \right] \\
&= \frac{\epsilon T_h}{C_1^\Delta} E \left[\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \text{tr} \left\{ \left[\sum_{t=1}^{T_h} \frac{\Gamma'_1 \Gamma (F_t - \mu_F)}{\sqrt{\Delta_{1,1}}} \right] \left[\sum_{s=1}^{T_h} \frac{(F_s - \mu_F)' \Gamma' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \right] \\
&\leq \frac{\epsilon T_h}{C_1^\Delta} \frac{C_1^\Delta}{T_h} \\
&= \epsilon
\end{aligned}$$

so that

$$\begin{aligned}
& \frac{2T_h}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \text{tr} \left\{ \left[\sum_{t=1}^{T_h} \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} \Gamma (F_t - \mu_F) \right] \left[\sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right] \right\} \\
&= \frac{2T_h}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \left\{ \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} \Gamma (F_t - \mu_F) (F_s - \mu_F)' \Gamma' \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \\
&= O_p(1)
\end{aligned}$$

from which it further follows that

$$2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \left\{ \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} \Gamma (F_t - \mu_F) (F_s - \mu_F)' \Gamma' \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} = O_p \left(\frac{\lambda_1}{T} \right) \quad (44)$$

Next, consider the second term on the right-hand side of expression (43). To analyze this term, we let $e_{g,\kappa}$ denote a $\kappa \times 1$ elementary vector whose g^{th} component is 1 and all other

components are 0. Taking expectation of this term, we obtain

$$\begin{aligned}
0 &\leq 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \left\{ \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} E[u_t u'_s] \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \\
&= 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \sum_{g=1}^{\kappa} \frac{e'_{g,\kappa} \Gamma'_1}{\sqrt{\Delta_{1,1}}} E[u_t u'_s] \frac{\Gamma_1 e_{g,\kappa}}{\sqrt{\Delta_{1,1}}} \\
&= 2 \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h} \frac{e'_{g,\kappa} \Gamma'_1}{\sqrt{\Delta_{1,1}}} E[u_t u'_t] \frac{\Gamma_1 e_{g,\kappa}}{\sqrt{\Delta_{1,1}}} + 4 \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} E \left\{ \left(\frac{e'_{g,\kappa} \Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \frac{u'_{t+m} \Gamma_1 e_{g,\kappa}}{\sqrt{\Delta_{1,1}}} \right) \right\} \\
&\leq 2 \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h} \frac{e'_{g,\kappa} \Gamma'_1}{\sqrt{\Delta_{1,1}}} E[u_t u'_t] \frac{\Gamma_1 e_{g,\kappa}}{\sqrt{\Delta_{1,1}}} + 4 \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} \left| E \left[\frac{e'_{g,\kappa} \Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \frac{u'_{t+m} \Gamma_1 e_{g,\kappa}}{\sqrt{\Delta_{1,1}}} \right] \right| \\
&\leq 2 \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h} \frac{e'_{g,\kappa} \Gamma'_1}{\sqrt{\Delta_{1,1}}} E[u_t u'_t] \frac{\Gamma_1 e_{g,\kappa}}{\sqrt{\Delta_{1,1}}} \\
&\quad + 4 \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) [\alpha(m)]^{1-\frac{1}{2}-\frac{1}{4}} \sqrt{E \left[\left(\frac{e'_{g,\kappa} \Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \right)^2 \right] \left(E \left[\left(\frac{u'_{t+m} \Gamma_1 e_{g,\kappa}}{\sqrt{\Delta_{1,1}}} \right)^4 \right] \right)^{\frac{1}{4}}} \\
&\quad \text{(by Corollary 14.3 of Davidson (1994))} \\
&= 2 \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h} E \left[\left(\frac{e'_{g,\kappa} \Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \right)^2 \right] \\
&\quad + 4 \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) [\alpha(m)]^{\frac{1}{4}} \sqrt{E \left[\left(\frac{e'_{g,\kappa} \Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \right)^2 \right] \left(E \left[\left(\frac{u'_{t+m} \Gamma_1 e_{g,\kappa}}{\sqrt{\Delta_{1,1}}} \right)^4 \right] \right)^{\frac{1}{4}}} \\
&\leq 2 \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h} E \left[\left(\frac{e'_{g,\kappa} \Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \right)^2 \right] \\
&\quad + 4 \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) [\alpha(m)]^{\frac{1}{4}} \sqrt{E \left[\left(\frac{e'_{g,\kappa} \Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \right)^2 \right] (C^\bullet)^{\frac{1}{4}}} \quad \text{(applying} \\
&\quad \text{Lemma D-6 (b) by setting } \delta = \frac{\Gamma_1 e_{g,\kappa}}{\sqrt{\Delta_{1,1}}}, \text{ noting that } \left\| \frac{\Gamma_1 e_{g,\kappa}}{\sqrt{\Delta_{1,1}}} \right\|_2 = \sqrt{\frac{e'_{g,\kappa} \Gamma'_1 \Gamma_1 e_{g,\kappa}}{\Delta_{1,1}}} = 1 \Big) \\
&\leq 2 \frac{\kappa T_h}{T_h^2} (C^\bullet)^{\frac{1}{2}} + 4 \frac{\kappa (T_h - 1)}{T_h^2} 2 \left(2^{\frac{1}{2}} + 1 \right) (C^\bullet)^{\frac{1}{2}} \sum_{m=1}^T [\alpha(m)]^{\frac{1}{4}} \quad \text{(applying Liapunov's} \\
&\quad \text{inequality to show that } \sqrt{E \left[\left(\frac{e'_{g,\kappa} \Gamma'_1 u_t}{\sqrt{\Delta_{1,1}}} \right)^2 \right]} \leq \left(E \left[\left(\frac{u'_t \Gamma_1 e_{g,\kappa}}{\sqrt{\Delta_{1,1}}} \right)^4 \right] \right)^{\frac{1}{4}} \leq (C^\bullet)^{\frac{1}{4}} \Big)
\end{aligned}$$

$$\begin{aligned}
&\leq 2(C^\bullet)^{\frac{1}{2}} \frac{\kappa}{T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \right] \\
&\quad \text{(by Assumption (v) part (b))} \\
&\leq 2(C^\bullet)^{\frac{1}{2}} \frac{\kappa}{T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \left(\text{since } \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \leq C^\# < \infty \right. \\
&\quad \left. \text{for some constant } C^\# > 0, \text{ as shown in Lemma D-11 of Appendix D) } \right] \\
&\leq \frac{C_2^\Delta}{T_h} \left(\text{for some constant } C_2^\Delta \geq 2\kappa(C^\bullet)^{\frac{1}{2}} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \right)
\end{aligned}$$

It follows that for any $\epsilon > 0$, we can set $C_\epsilon^{(2)} = C_2^\Delta / \epsilon$; and, by Markov's inequality, we have

$$\begin{aligned}
&P \left(\left| T_h \left[2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \left\{ \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} u_t u'_s \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \right] \right| \geq C_\epsilon^{(2)} \right) \\
&= P \left(\left| T_h \left[2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \left\{ \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} u_t u'_s \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \right] \right| \geq \frac{C_2^\Delta}{\epsilon} \right) \\
&\leq \frac{\epsilon}{C_2^\Delta} E \left(T_h \left[2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \left\{ \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} u_t u'_s \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \right] \right) \\
&\leq \frac{\epsilon}{C_2^\Delta} T_h \frac{C_2^\Delta}{T_h} \\
&= \epsilon
\end{aligned}$$

so that

$$T_h \left[2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \left\{ \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} u_t u'_s \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \right] = O_p(1)$$

from which we further deduce that

$$2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \left\{ \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} u_t u'_s \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} = O_p \left(\frac{1}{T} \right) \quad (45)$$

Combining the results given in expressions (44) and (45), we see that

$$\begin{aligned}
& tr \left\{ L_1' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X] \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [X_s - \mu_X] \right)' L_1 \right\} \\
& \leq 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} tr \left\{ \frac{\Gamma_1'}{\sqrt{\Delta_{1,1}}} \Gamma (F_t - \mu_F) (F_s - \mu_F)' \Gamma' \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \\
& \quad + 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} tr \left\{ \frac{\Gamma_1' u_t}{\sqrt{\Delta_{1,1}}} \frac{u_s' \Gamma_1}{\sqrt{\Delta_{1,1}}} \right\} \\
& = O_p \left(\frac{\lambda_1}{T} \right) + O_p \left(\frac{1}{T} \right)
\end{aligned}$$

from which it further follows that

$$\begin{aligned}
\sqrt{tr \left\{ L_1' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X] \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [X_s - \mu_X] \right)' L_1 \right\}} &= O_p \left(\sqrt{\frac{\lambda_1}{T}} \right) + O_p \left(\frac{1}{\sqrt{T}} \right) \\
&= O_p \left(\sqrt{\frac{\lambda_1}{T}} \right).
\end{aligned}$$

since $\lambda_1 \gg 1$. Applying the result given in expression (37) above, we then obtain

$$\begin{aligned}
& \left| \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_1 L_1' \left(\hat{\beta}_{\text{drpls}} - \beta \right) \right| \\
& \leq \sqrt{tr \left\{ L_1' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right)' \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right) L_1 \right\}} \sqrt{(\hat{\beta}_{\text{drpls}} - \beta)' L_1 L_1' (\hat{\beta}_{\text{drpls}} - \beta)} \\
& \leq \sqrt{tr \left\{ L_1' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right)' \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right) L_1 \right\}} \left\| \hat{\beta}_{\text{drpls}} - \beta \right\|_2 \\
& = O_p \left(\sqrt{\frac{\lambda_1}{T}} \right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right) \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1 \sqrt{T}} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1 T}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1 \sqrt{T}} \right\} \right). \tag{46}
\end{aligned}$$

Next, consider the term

$$\left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_2 L_2' (\hat{\beta}_{\text{drpls}} - \beta).$$

By a similar argument, we have

$$\begin{aligned} & \left| \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_2 L_2' (\hat{\beta}_{\text{drpls}} - \beta) \right| \\ & \leq \left\| L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right) \right\|_2 \left\| L_2' (\hat{\beta}_{\text{drpls}} - \beta) \right\|_2 \quad (\text{by the Cauchy-Schwarz inequality}) \\ & = \sqrt{\left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right)' L_2 L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right)} \sqrt{(\hat{\beta}_{\text{drpls}} - \beta)' L_2 L_2' (\hat{\beta}_{\text{drpls}} - \beta)} \\ & = \sqrt{\text{tr} \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right)' L_2 \right\}} \sqrt{(\hat{\beta}_{\text{drpls}} - \beta)' L_2 L_2' (\hat{\beta}_{\text{drpls}} - \beta)} \end{aligned}$$

Next, write

$$\begin{aligned} & \text{tr} \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right)' L_2 \right\} \\ & = \text{tr} \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X] \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [X_s - \mu_X]' \right) L_2 \right\} \\ & = \text{tr} \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [\Gamma (F_t - \mu_F) + u_t] \right) \frac{1}{T_h} \sum_{s=1}^{T_h} [(F_s - \mu_F)' \Gamma' + u_s'] L_2 \right\} \\ & = \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \right) \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right\} \\ & \quad + \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \right) \frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2 \right\} + \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' u_t \right) \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right\} \\ & \quad + \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' u_t \right) \frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2 \right\} \end{aligned} \tag{47}$$

Now, let the $(N - \kappa) \times 1$ vector $e_{i,N-\kappa}$ denote an elementary vector whose i^{th} component

equals 1 and all other components are 0, and note that

$$\begin{aligned}
& \left| \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 \Gamma (F_t - \mu_F) \frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right) \right\} \right| \\
& \leq \sum_{i=1}^{N-\kappa} \left| e'_{i,N-\kappa} \left[\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 \Gamma (F_t - \mu_F) \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right] e_{i,N-\kappa} \right| \\
& \leq \sum_{i=1}^{N-\kappa} \left\{ \frac{1}{2} e'_{i,N-\kappa} \left[\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 \Gamma (F_t - \mu_F) \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right] e_{i,N-\kappa} \right. \\
& \quad \left. + \frac{1}{2} e'_{i,N-\kappa} \left[\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \right] \left[\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right] e_{i,N-\kappa} \right\} \\
& \quad \left(\text{using the inequality } |xy| \leq \frac{1}{2}x^2 + \frac{1}{2}y^2 \right) \\
& = \frac{1}{2} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 \Gamma (F_t - \mu_F) \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right) \right\} \\
& \quad + \frac{1}{2} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right) \right\} \tag{48}
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
& \left| \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right) \right\} \right| \\
& \leq \frac{1}{2} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 \Gamma (F_t - \mu_F) \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right) \right\} \\
& \quad + \frac{1}{2} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right) \right\} \tag{49}
\end{aligned}$$

Applying expressions (48) and (49) to expression (47), we then have

$$\begin{aligned}
& tr \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right)' L_2 \right\} \\
= & tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right) \right\} \\
& + tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2 \right) \right\} + tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' u_t \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right) \right\} \\
& + tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' u_t \frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2 \right) \right\} \\
\leq & tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right) \right\} + tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' u_t \frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2 \right) \right\} \\
& + \left| tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2 \right) \right\} \right| \\
& + \left| tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' u_t \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right) \right\} \right| \\
\leq & tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right) \right\} + tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' u_t \frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2 \right) \right\} \\
& + \frac{1}{2} tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right) \right\} \\
& + \frac{1}{2} tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' u_t \frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2 \right) \right\} \\
& + \frac{1}{2} tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right) \right\} \\
& + \frac{1}{2} tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' u_t \frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2 \right) \right\} \\
= & 2tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \frac{1}{T_h} \sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right) \right\} \\
& + 2tr \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' u_t \frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2 \right) \right\}
\end{aligned}$$

$$\begin{aligned}
&= 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' \Gamma (F_t - \mu_F) (F_s - \mu_F)' \Gamma' L_2 \} \\
&\quad + 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' u_t u_s' L_2 \}
\end{aligned} \tag{50}$$

Consider now the first term on the right-hand side of expression (50). To analyze this term, note first that

$$\begin{aligned}
&L_2' \Gamma E [(F_t - \mu_F) (F_s - \mu_F)'] \Gamma' L_2 \\
&= \lambda_{1,1} \begin{pmatrix} \Gamma_2' / \sqrt{\lambda_{1,1}} \\ L_{22}' \end{pmatrix} \begin{pmatrix} \Gamma_1 / \sqrt{\lambda_{1,1}} & \Gamma_2 / \sqrt{\lambda_{1,1}} \end{pmatrix} \\
&\quad \times E \left[\begin{pmatrix} F_{1,t} - \mu_{F,1} \\ F_{2,t} - \mu_{F,2} \end{pmatrix} (F_{1,s} - \mu_{F,1})' (F_{2,s} - \mu_{F,2})' \right] \begin{pmatrix} \Gamma_1' / \sqrt{\lambda_{1,1}} \\ \Gamma_2' / \sqrt{\lambda_{1,1}} \end{pmatrix} \begin{pmatrix} \Gamma_2 / \sqrt{\lambda_{1,1}} & L_{22} \end{pmatrix} \\
&= \lambda_{1,1} \begin{pmatrix} 0 & I_{K-\kappa} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E \left[(F_{1,t} - \mu_{F,1}) (F_{1,s} - \mu_{F,1})' \right] & E \left[(F_{1,t} - \mu_{F,1}) (F_{2,s} - \mu_{F,2})' \right] \\ E \left[(F_{2,t} - \mu_{F,2}) (F_{1,s} - \mu_{F,1})' \right] & E \left[(F_{2,t} - \mu_{F,2}) (F_{2,s} - \mu_{F,2})' \right] \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 & 0 \\ I_{K-\kappa} & 0 \end{pmatrix} \\
&= \lambda_{1,1} \begin{pmatrix} 0 & I_{K-\kappa} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} E \left[(F_{1,t} - \mu_{F,1}) (F_{2,s} - \mu_{F,2})' \right] & 0 \\ E \left[(F_{2,t} - \mu_{F,2}) (F_{2,s} - \mu_{F,2})' \right] & 0 \end{pmatrix} \\
&= \lambda_{1,1} \begin{pmatrix} E \left[(F_{2,t} - \mu_{F,2}) (F_{2,s} - \mu_{F,2})' \right] & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

Making use of this calculation, we can divide first term on the right-hand side of expression

(50) by $\lambda_1 = \lambda_{\max}(\bar{\Sigma}_X)$ and write

$$\begin{aligned}
0 &\leq \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 E \left(\text{tr} \left\{ \left[\sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \right] \left[\sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right] \right\} \right) \\
&= \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' \Gamma E [(F_t - \mu_F) (F_s - \mu_F)'] \Gamma' L_2 \} \\
&= \frac{2\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \left\{ \begin{pmatrix} E [(F_{2,t} - \mu_{F,2}) (F_{2,s} - \mu_{F,2})'] & 0 \\ 0 & 0 \end{pmatrix} \right\} \\
&= \frac{2\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} E \left[\text{tr} \{ (F_{2,t} - \mu_{F,2}) (F_{2,s} - \mu_{F,2})' \} \right] \\
&= \frac{2\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} E \left[(F_{2,s} - \mu_{F,2})' (F_{2,t} - \mu_{F,2}) \right] \\
&= \frac{2\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \sum_{g=1}^{K-\kappa} E [(F_{2,s,\cdot g} - \mu_{F,2,\cdot g}) (F_{2,t,\cdot g} - \mu_{F,2,\cdot g})] \quad (\text{where } F_{2,s,\cdot g} \text{ and } F_{2,t,\cdot g} \text{ are,} \\
&\quad \text{respectively, the } g^{\text{th}} \text{ element of } F_{2,s} \text{ and } F_{2,t} \text{ and where } \mu_{F,2,\cdot g} = E[F_{2,s,\cdot g}] = E[F_{2,t,\cdot g}]) \\
&\leq \frac{2\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{g=1}^{K-\kappa} E [(F_{2,t,\cdot g} - \mu_{F,2,\cdot g})^2] \\
&\quad + \frac{4\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{K-\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} E [(F_{2,t,\cdot g} - \mu_{F,2,\cdot g}) (F_{2,t+m,\cdot g} - \mu_{F,2,\cdot g})]
\end{aligned}$$

Applying Corollary 14.3 of Davidson (1994), we further obtain

$$\begin{aligned}
0 &\leq \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 E \left(\text{tr} \left\{ \left[\sum_{t=1}^{T_h} L'_2 \Gamma (F_t - \mu_F) \right] \left[\sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right] \right\} \right) \\
&\leq \frac{2\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{g=1}^{K-\kappa} E \left[(F_{2,t,g} - \mu_{F,2,g})^2 \right] \\
&\quad + \frac{4\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{K-\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} |E [(F_{2,t,g} - \mu_{F,2,g}) (F_{2,t+m,g} - \mu_{F,2,g})]| \\
&\leq \frac{2\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{g=1}^{K-\kappa} E \left[(F_{2,t,g} - \mu_{F,2,g})^2 \right] + \\
&\quad \frac{4\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{K-\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) |\alpha(m)|^{1-\frac{1}{2}-\frac{1}{4}} \sqrt{E |F_{2,t,g} - \mu_{F,2,g}|^2} \left[E |F_{2,t+m,g} - \mu_{F,2,g}|^4 \right]^{\frac{1}{4}} \\
&\leq \frac{2\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{g=1}^{K-\kappa} \sqrt{C^\dagger} + \frac{4\lambda_{1,1}}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{K-\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) |\alpha(m)|^{\frac{1}{4}} \sqrt{C^\dagger} \\
&\quad (\text{applying Assumption (ii) and Liapunov's inequality}) \\
&\leq \frac{2\lambda_{1,1}}{\lambda_1} (K - \kappa) \sqrt{C^\dagger} \left(\frac{1}{T_h} \right) + \frac{8\lambda_{1,1}}{\lambda_1} (K - \kappa) \left(\frac{1}{T_h} \right)^2 (T_h - 1) \sqrt{C^\dagger} \left(2^{\frac{1}{2}} + 1 \right) \sum_{m=1}^T |\alpha(m)|^{\frac{1}{4}} \\
&\leq \frac{2\lambda_{1,1}}{\lambda_1} \frac{(K - \kappa)}{T_h} \sqrt{C^\dagger} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \right] \\
&\quad (\text{by Assumption (v) part (b)}) \\
&\leq \frac{2\lambda_{1,1}}{\lambda_1} \frac{(K - \kappa)}{T_h} \sqrt{C^\dagger} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \left(\text{since } \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \leq C^\# < \infty \right. \\
&\quad \left. \text{for some constant } C^\# > 0 \text{ as shown in Lemma D-11 of Appendix D} \right) \\
&\leq 2c^* \frac{(K - \kappa)}{T_h} \sqrt{C^\dagger} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \left(\text{since Lemma D-3(a) implies that there exists} \right. \\
&\quad \left. \text{absolute constant } c^* > 1 \text{ such that } 0 < \frac{1}{c^*} < \frac{\lambda_{1,1}}{\lambda_1} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large} \right) \\
&\leq \frac{C_1^\dagger}{T_h} \left(\text{for some constant } C_1^\dagger \geq 2c^* (K - \kappa) \sqrt{C^\dagger} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \right)
\end{aligned}$$

It follows that for any $\epsilon > 0$, we can set $C_\epsilon^* = C_1^\dagger/\epsilon$; and, by Markov's inequality, we have

$$\begin{aligned}
& P \left(\left| \frac{2T_h}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \text{tr} \left\{ \left[\sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \right] \left[\sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right] \right\} \right| \geq C_\epsilon^* \right) \\
&= P \left(\left| \frac{2T_h}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \text{tr} \left\{ \left[\sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \right] \left[\sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right] \right\} \right| \geq \frac{C_1^\dagger}{\epsilon} \right) \\
&\leq \frac{\epsilon}{C_1^\dagger} E \left[\frac{2T_h}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \text{tr} \left\{ \left[\sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \right] \left[\sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right] \right\} \right] \\
&= \frac{\epsilon T_h}{C_1^\dagger} E \left(\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \text{tr} \left\{ \left[\sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \right] \left[\sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right] \right\} \right) \\
&\leq \frac{\epsilon T_h}{C_1^\dagger} \frac{C_1^\dagger}{T_h} \\
&= \epsilon
\end{aligned}$$

so that

$$\begin{aligned}
& \frac{2T_h}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \text{tr} \left\{ \left[\sum_{t=1}^{T_h} L_2' \Gamma (F_t - \mu_F) \right] \left[\sum_{s=1}^{T_h} (F_s - \mu_F)' \Gamma' L_2 \right] \right\} \\
&= \frac{2T_h}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' \Gamma (F_t - \mu_F) (F_s - \mu_F)' \Gamma' L_2 \} \\
&= O_p(1)
\end{aligned}$$

or

$$2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' \Gamma (F_t - \mu_F) (F_s - \mu_F)' \Gamma' L_2 \} = O_p \left(\frac{\lambda_1}{T} \right) \quad (51)$$

Next, consider the second term on the right-hand side of expression (50). For this term, let

$l_{2,k}$ be the k^{th} column of L_2 , and note that

$$\begin{aligned}
& 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' E [u_t u_s'] L_2 \} \\
&= 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \sum_{k=1}^{N-\kappa} l_{2,k}' E [u_t u_s'] l_{2,k} \\
&= 2 \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h} l_{2,k}' E [u_t u_t'] l_{2,k} + 4 \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} E \{ (l_{2,k}' u_t) (l_{2,k}' u_{t+m}) \} \\
&\leq 2 \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h} l_{2,k}' E [u_t u_t'] l_{2,k} + 4 \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} |E \{ (l_{2,k}' u_t) (l_{2,k}' u_{t+m}) \}| \\
&\leq 2 \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h} E |l_{2,k}' u_t|^2 \\
&\quad + 4 \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) [\alpha(m)]^{1-\frac{1}{2}-\frac{1}{4}} \sqrt{E |l_{2,k}' u_t|^2} \left(E |l_{2,k}' u_{t+m}|^4 \right)^{\frac{1}{4}} \\
&\quad \text{(by Corollary 14.3 of Davidson (1994))} \\
&\leq 2 \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h} E |l_{2,k}' u_t|^2 + 4 \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) [\alpha(m)]^{\frac{1}{4}} \sqrt{E |l_{2,k}' u_t|^2} (C^\bullet)^{\frac{1}{4}} \\
&\quad \left(\text{applying Lemma D-6 part (b) by setting } \delta = l_{2,k}, \text{ noting that } \|l_{2,k}\|_2 = \sqrt{l_{2,k}' l_{2,k}} = 1 \right) \\
&\leq 2 \frac{(N-\kappa) T_h}{T_h^2} (C^\bullet)^{\frac{1}{2}} + 4 \frac{(N-\kappa) (T_h-1)}{T_h^2} 2 \left(2^{\frac{1}{2}} + 1 \right) (C^\bullet)^{\frac{1}{2}} \sum_{m=1}^T [\alpha(m)]^{\frac{1}{4}} \\
&\quad \left(\text{applying Liapunov's inequality to show that } \sqrt{E |l_{2,k}' u_t|^2} \leq \left(E |l_{2,k}' u_t|^4 \right)^{\frac{1}{4}} \leq (C^\bullet)^{\frac{1}{4}} \right) \\
&\leq 2 (C^\bullet)^{\frac{1}{2}} \frac{N-\kappa}{T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \right] \\
&\quad \text{(by Assumption (v) part (b))} \\
&\leq 2 (C^\bullet)^{\frac{1}{2}} \frac{N-\kappa}{T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \left(\text{since } \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \leq C^\# < \infty \text{ for some} \right. \\
&\quad \left. \text{constant } C^\# > 0 \text{ as shown in Lemma D-11 of Appendix D} \right) \\
&< \frac{C_2^\dagger N}{T_h} \left(\text{for some constant } C_2^\dagger \geq 2 (C^\bullet)^{\frac{1}{2}} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \text{ and given that } \kappa \geq 1 \right)
\end{aligned}$$

It follows that for any $\epsilon > 0$, we can set $C_\epsilon^\% = C_2^\dagger/\epsilon$; and, by Markov's inequality, we have

$$\begin{aligned}
& P \left(\left| \frac{T_h}{N} \left[2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' u_t u_s' L_2 \} \right] \right| \geq C_\epsilon^\% \right) \\
&= P \left(\left| \frac{T_h}{N} \left[2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' u_t u_s' L_2 \} \right] \right| \geq \frac{C_2^\dagger}{\epsilon} \right) \\
&\leq \frac{\epsilon}{C_2^\dagger} \frac{T_h}{N} E \left(\left[2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' u_t u_s' L_2 \} \right] \right) \\
&\leq \frac{\epsilon}{C_2^\dagger} \frac{T_h}{N} \frac{C_2^\dagger N}{T_h} \\
&= \epsilon
\end{aligned}$$

so that

$$\frac{T_h}{N} \left[2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' u_t u_s' L_2 \} \right] = O_p(1)$$

from which we further deduce that

$$2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' u_t u_s' L_2 \} = O_p \left(\frac{N}{T} \right) \quad (52)$$

Combining the results given in expressions (51) and (52), we see that

$$\begin{aligned}
& \text{tr} \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X] \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [X_s - \mu_X] \right)' L_2 \right\} \\
&\leq 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' \Gamma (F_t - \mu_F) (F_s - \mu_F)' \Gamma' L_2 \} \\
&\quad + 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' u_t u_s' L_2 \} \\
&= O_p \left(\frac{\lambda_1}{T} \right) + O_p \left(\frac{N}{T} \right),
\end{aligned}$$

from which it further follows that

$$\sqrt{\text{tr} \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X] \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [X_s - \mu_X] \right)' L_2 \right\}} = O_p \left(\sqrt{\frac{\lambda_1}{T}} \right) + O_p \left(\sqrt{\frac{N}{T}} \right).$$

Applying the result given in expression (37), we then obtain

$$\begin{aligned}
& \left| \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_2 L_2' \left(\hat{\beta}_{\text{drpls}} - \beta \right) \right| \\
& \leq \sqrt{tr \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right)' \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right) L_2 \right\}} \sqrt{\left(\hat{\beta}_{\text{drpls}} - \beta \right)' L_2 L_2' \left(\hat{\beta}_{\text{drpls}} - \beta \right)} \\
& \leq \sqrt{tr \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right)' \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right) L_2 \right\}} \left\| \hat{\beta}_{\text{drpls}} - \beta \right\|_2 \\
& = \left[O_p \left(\sqrt{\frac{\lambda_1}{T}} \right) + O_p \left(\sqrt{\frac{N}{T}} \right) \right] \\
& \quad \times \left[O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right) \right] \\
& = O_p \left(\max \left\{ \sqrt{\frac{\lambda_1}{T}} \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{\lambda_1}{T}} \frac{\sqrt{a(N)}}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \sqrt{\frac{\lambda_1}{T}} \right\} \right) \\
& \quad + O_p \left(\max \left\{ \sqrt{\frac{N}{T}} \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{N}{T}} \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \sqrt{\frac{N}{T}} \right\} \right) \\
& = O_p \left(\max \left\{ \sqrt{\frac{N}{T\lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{N}{T\lambda_1}} \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \sqrt{\frac{N}{T\lambda_1^3}} \right\} \right). \quad (53)
\end{aligned}$$

Making use of expressions (39), (46), and (53), we see that

$$\begin{aligned}
& (\bar{X}_{T_h} - \mu_X)' (\hat{\beta}_{\text{drpls}} - \beta) \\
&= \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_1 L_1' (\hat{\beta}_{\text{drpls}} - \beta) + \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_2 L_2' (\hat{\beta}_{\text{drpls}} - \beta) \\
&= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1 \sqrt{T}} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1 T}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1 \sqrt{T}} \right\} \right) \\
&\quad + O_p \left(\max \left\{ \sqrt{\frac{N}{T \lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{N}{T \lambda_1}} \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \sqrt{\frac{N}{T \lambda_1^3}} \right\} \right) \\
&= O_p \left(\max \left\{ \sqrt{\frac{N}{T \lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{N}{T \lambda_1}} \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \sqrt{\frac{N}{T \lambda_1^3}} \right\} \right) \quad (54)
\end{aligned}$$

Consider next the third term on the right-hand side of expression (28), i.e., the term

$-\vec{u}'_{T_h}\beta = -T_h^{-1} \sum_{t=1}^{T_h} u'_t\beta$. For this term, it is easy to see that under Assumption (v)

$$\begin{aligned}
0 &\leq E \left[(\vec{u}'_{T_h}\beta)^2 \right] \\
&= \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} E [\beta' u_t u'_s \beta] \\
&= \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} E [\delta' u_t u'_s \delta] \|\beta\|_2^2 \quad \left(\text{letting } \delta = \frac{\beta}{\|\beta\|_2} \text{ so that } \|\delta\|_2 = 1 \right) \\
&= \frac{\|\beta\|_2^2}{T_h^2} \sum_{t=1}^{T_h} E [\delta' u_t u'_t \delta] + 2 \|\beta\|_2^2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} E [\delta' u_t u'_{t+m} \delta] \\
&\leq \frac{\|\beta\|_2^2}{T_h^2} \sum_{t=1}^{T_h} E [\delta' u_t u'_t \delta] + 2 \|\beta\|_2^2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} |E [\delta' u_t u'_{t+m} \delta]| \\
&\leq \frac{\|\beta\|_2^2}{T_h^2} \sum_{t=1}^{T_h} E [(u'_t \delta)^2] + 2 \|\beta\|_2^2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) [\alpha(m)]^{1-\frac{1}{2}-\frac{1}{4}} \sqrt{E |u'_t \delta|^2} \left(E |u'_{t+m} \delta|^4 \right)^{\frac{1}{4}} \\
&\quad (\text{by Corollary 14.3 of Davidson (1994)}) \\
&\leq \frac{\|\beta\|_2^2}{T_h^2} \sum_{t=1}^{T_h} E |u'_t \delta|^2 + 2 \|\beta\|_2^2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) [\alpha(m)]^{\frac{1}{4}} \sqrt{E |u'_t \delta|^2} (C^\bullet)^{\frac{1}{4}} \\
&\quad (\text{applying the second result given in part (b) of Lemma D-6}) \\
&\leq \frac{\|\beta\|_2^2}{T_h} (C^\bullet)^{\frac{1}{2}} + \frac{2 \|\beta\|_2^2}{T_h} 2 \left(2^{\frac{1}{2}} + 1 \right) (C^\bullet)^{\frac{1}{2}} \sum_{m=1}^T [\alpha(m)]^{\frac{1}{4}} \\
&\quad \left(\text{applying Liapunov's inequality to show that } \sqrt{E |u'_t \delta|^2} \leq \left(E |u'_t \delta|^4 \right)^{\frac{1}{4}} \leq (C^\bullet)^{\frac{1}{4}} \right) \\
&\leq \frac{\|\varphi_1\|_2^2 (C^\bullet)^{\frac{1}{2}}}{\underline{\lambda}_{1,1} T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \right] \quad (\text{by Assumption (v) part (b) and by} \\
&\quad \text{the fact that, under Scenario 1, } \|\beta\|_2^2 = \frac{\varphi'_1}{\sqrt{\underline{\lambda}_{1,1}}} \frac{\Gamma'_1 \Gamma_1}{\underline{\lambda}_{1,1}} \frac{\varphi_1}{\sqrt{\underline{\lambda}_{1,1}}} = \frac{\varphi'_1}{\sqrt{\underline{\lambda}_{1,1}}} \frac{\varphi_1}{\sqrt{\underline{\lambda}_{1,1}}} = \frac{\|\varphi_1\|_2^2}{\underline{\lambda}_{1,1}}) \\
&\leq \frac{\lambda_1}{\underline{\lambda}_{1,1}} \kappa \left(\max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \right)^2 \frac{(C^\bullet)^{\frac{1}{2}}}{\lambda_1 T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \quad \left(\text{since } \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \leq C^\# < \infty \right. \\
&\quad \left. \text{for some constant } C^\# > 0 \text{ as shown in Lemma D-11 of Appendix D} \right) \\
&\leq c^* \kappa \left(\max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \right)^2 \frac{(C^\bullet)^{\frac{1}{2}}}{\lambda_1 T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \quad (\text{by Lemma D-3 part (a) there exists} \\
&\quad \text{absolute constant } c^* > 1 \text{ such that } 0 < \frac{1}{c^*} < \frac{\lambda_1}{\underline{\lambda}_{1,1}} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large})
\end{aligned}$$

$$\begin{aligned}
&\leq c^* \kappa \bar{c}^2 \frac{(C^\bullet)^{\frac{1}{2}}}{\lambda_1 T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \quad (\text{by Assumption S1-1}) \\
&\leq \frac{C^+}{\lambda_1 T_h} \quad \left(\text{for some constant } C^+ \geq c^* \kappa \bar{c}^2 (C^\bullet)^{\frac{1}{2}} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \right)
\end{aligned}$$

It follows that for any $\epsilon > 0$, we can set $C_\epsilon^+ = \sqrt{C^+/\epsilon}$; and, by Markov's inequality, we have

$$\begin{aligned}
P \left(\left| \sqrt{\lambda_1 T_h} (-\bar{u}'_{T_h} \beta) \right| \geq C_\epsilon^+ \right) &= P \left(\left| \lambda_1 T_h (\bar{u}'_{T_h} \beta)^2 \right| \geq (C_\epsilon^+)^2 \right) \\
&\leq \frac{\epsilon}{C^+} \lambda_1 T_h E \left[(\bar{u}'_{T_h} \beta)^2 \right] \\
&\leq \frac{\epsilon}{C^+} \lambda_1 T_h \frac{C^+}{\lambda_1 T_h} \\
&= \epsilon
\end{aligned}$$

so that

$$-\bar{u}'_{T_h} \beta = -\frac{1}{T_h} \sum_{t=1}^{T_h} u'_t \beta = O_p \left(\frac{1}{\sqrt{\lambda_1 T}} \right) \quad (55)$$

Turning our attention now to the fourth term on the right-hand side of expression (28), i.e., the term $\bar{\varepsilon}_{T_h} = T_h^{-1} \sum_{t=1}^{T_h} \varepsilon_{t+h}$. For this term, it is easy to see that, under Assumption

(v),

$$\begin{aligned}
0 &\leq E[\bar{\varepsilon}_{T_h}^2] \\
&= \left(\frac{1}{T_h}\right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} E[\varepsilon_{t+h} \varepsilon_{s+h}] \quad (\text{where } T_h = T - h) \\
&= \frac{1}{T_h^2} \sum_{t=1}^{T_h} E[\varepsilon_{t+h}^2] + 2 \left(\frac{1}{T_h}\right)^2 \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} E\{\varepsilon_{t+h} \varepsilon_{t+h+m}\} \\
&\leq \frac{1}{T_h^2} \sum_{t=1}^{T_h} E[\varepsilon_{t+h}^2] + 2 \left(\frac{1}{T_h}\right)^2 \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} |E[\varepsilon_{t+h} \varepsilon_{t+h+m}]| \\
&\leq \frac{1}{T_h^2} \sum_{t=1}^{T_h} E[\varepsilon_{t+h}^2] + 2 \left(\frac{1}{T_h}\right)^2 \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1\right) [\alpha(m)]^{1-\frac{1}{2}-\frac{1}{4}} \sqrt{E|\varepsilon_{t+h}|^2} (E|\varepsilon_{t+h+m}|^4)^{\frac{1}{4}} \\
&\quad (\text{by Corollary 14.3 of Davidson (1994)}) \\
&\leq \frac{1}{T_h^2} \sum_{t=1}^{T_h} E|\varepsilon_{t+h}|^2 + 2 \left(\frac{1}{T_h}\right)^2 \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1\right) [\alpha(m)]^{\frac{1}{4}} \sqrt{E|\varepsilon_{t+h}|^2} (C^\bullet)^{\frac{1}{4}} \\
&\quad (\text{applying the first result given in part (b) of Lemma D-6}) \\
&\leq \frac{T_h}{T_h^2} (C^\bullet)^{\frac{1}{2}} + 2 \frac{(T_h - 1)}{T_h^2} 2 \left(2^{\frac{1}{2}} + 1\right) (C^\bullet)^{\frac{1}{2}} \sum_{m=1}^T [\alpha(m)]^{\frac{1}{4}} \\
&\quad \left(\text{applying Liapunov's inequality to show that } \sqrt{E|\varepsilon_{t+h}|^2} \leq (E|\varepsilon_{t+h}|^4)^{\frac{1}{4}} \leq (C^\bullet)^{\frac{1}{4}}\right) \\
&\leq \frac{(C^\bullet)^{\frac{1}{2}}}{T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1\right) a^{\frac{1}{4}} \sum_{m=1}^T \exp\left(-\frac{cm^{\gamma_2}}{4}\right)\right] \\
&\quad (\text{by Assumption (v) part (b)}) \\
&\leq \frac{(C^\bullet)^{\frac{1}{2}}}{T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1\right) a^{\frac{1}{4}} C^\# \right] \left(\text{since } \sum_{m=1}^T \exp\left(-\frac{cm^{\gamma_2}}{4}\right) \leq C^\# < \infty \text{ for some}\right. \\
&\quad \left.\text{constant } C^\# > 0 \text{ as shown in Lemma D-11 of Appendix D}\right) \\
&\leq \frac{C^\dagger}{T_h} \left(\text{for some constant } C^\dagger \geq (C^\bullet)^{\frac{1}{2}} \left[1 + 4 \left(2^{\frac{1}{2}} + 1\right) a^{\frac{1}{4}} C^\#\right]\right)
\end{aligned}$$

It follows that for any $\epsilon > 0$, we can set $C_\epsilon^\dagger = \sqrt{C^\dagger/\epsilon}$; and, by Markov's inequality, we have

$$\begin{aligned}
P\left(\left|\sqrt{T_h}\bar{\varepsilon}_{T_h}\right| \geq C_\epsilon^\dagger\right) &= P\left(|T_h\bar{\varepsilon}_{T_h}^2| \geq (C_\epsilon^\dagger)^2\right) \\
&\leq \frac{\epsilon}{C_\epsilon^\dagger} T_h E[\bar{\varepsilon}_{T_h}^2] \\
&\leq \frac{\epsilon}{C_\epsilon^\dagger} T_h \frac{C^\dagger}{T_h} \\
&= \epsilon
\end{aligned}$$

so that

$$\bar{\varepsilon}_{T_h} = \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} = O_p\left(\frac{1}{\sqrt{T}}\right), \quad (56)$$

given that $h \geq 1$ is a fixed finite integer.

Finally, consider the last term on the right-hand side of expression (28), i.e., the term $[X_T - \mu_X]' \beta - [F_{1,T} - \mu_{F,1}]' \varphi_1$. For this term, we apply the result given in Theorem 3 to obtain

$$\begin{aligned}
&[X_T - \mu_X]' \beta - [F_{1,T} - \mu_{F,1}]' \varphi_1 \\
&= O_p\left(\frac{1}{\sqrt{\Delta_{1,1}}}\right) \\
&= O_p\left(\frac{1}{\sqrt{\lambda_1}}\right) \left(\text{given that, by Lemma D-3(a), there exists absolute constant } c^* > 1 \text{ such that}\right. \\
&\quad \left.0 < \frac{1}{c^*} < \frac{\lambda_1}{\Delta_{1,1}} \leq c^* < \infty \text{ for all } N \text{ and } T \text{ sufficiently large so that } \lambda_1 \asymp \Delta_{1,1}\right) \quad (57)
\end{aligned}$$

Putting the results given in expressions (38), (54), (55), (56), and (57) together, we see

that

$$\begin{aligned}
& \widehat{y}_{T+h}^{\text{drpls}} - [F_{1,T} - \mu_{F,1}]' \varphi_1 \\
&= [X_T - \mu_X]' (\widehat{\beta}_{\text{drpls}} - \beta) - (\overline{X}_{T_h} - \mu_X)' (\widehat{\beta}_{\text{drpls}} - \beta) - \overline{u}_{T_h}' \beta + \overline{\varepsilon}_{T_h} \\
&\quad + ([X_T - \mu_X]' \beta - [F_{1,T} - \mu_{F,1}]' \varphi_1) \\
&= O_p \left(\max \left\{ \sqrt{\frac{N}{\lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{N}{\lambda_1}} \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{\sqrt{N}}{\lambda_1^{\frac{3}{2}}} \right\} \right) \\
&\quad + O_p \left(\max \left\{ \sqrt{\frac{N}{T\lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{N}{T\lambda_1}} \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \sqrt{\frac{N}{T\lambda_1^3}} \right\} \right) \\
&\quad + O_p \left(\frac{1}{\sqrt{\lambda_1 T}} \right) + O_p \left(\frac{1}{\sqrt{T}} \right) + O_p \left(\frac{1}{\sqrt{\lambda_1}} \right) \\
&= O_p \left(\max \left\{ \frac{c_0(N) \sqrt{N}}{\lambda_1^{\frac{3}{2}}} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \frac{\sqrt{a(N)N}}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{\sqrt{N}}{\lambda_1^{\frac{3}{2}}} \right\} \right) + O_p \left(\frac{1}{\sqrt{T}} \right) \\
&\quad + O_p \left(\frac{1}{\sqrt{\lambda_1}} \right) \quad \left(\text{since } \frac{1}{\sqrt{\lambda_1 T}} = o \left(\frac{1}{\sqrt{T}} \right) \text{ and } \frac{1}{\sqrt{\lambda_1 T}} = o \left(\frac{1}{\sqrt{\lambda_1}} \right) \right),
\end{aligned}$$

which shows the result for part (a) under Scenario 1.

The result given for Scenario 2 can be shown using essentially the same argument as that given for Scenario 1 with minor modifications in notations and calculations to accommodate the case where $\kappa = K$. In particular, under Scenario 2, we take

$$L_{1, N \times K} = \Gamma / \sqrt{\lambda_{1,1}}$$

since all available factors are now relevant for the purpose of forecasting. Moreover, L_2 now denotes the $N \times (N - K)$ semi-orthogonal matrix whose columns are the orthonormal eigenvectors associated with the 0 eigenvalue of $\Sigma_{\text{com}} = \Gamma \overline{\Sigma}_F \Gamma'$, and $\varphi_1 = (\varphi_{1,1}, \dots, \varphi_{1,K})' = \varphi$ under Scenario 2 since none of the coefficients of the diffusion index forecast equation is equal to zero in this case. In addition, to obtain the result

$$\left\| \widehat{\beta}_{\text{drpls}} - \beta \right\|_2 = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right),$$

we apply part (b) of Lemmas E-1, E-14, and E-15 instead of part (a) of Lemmas E-1, E-14, and E-15 and also apply part (b) of Lemma E-3 instead of part (b) of Lemma E-2.

For part (b), note that, by using essentially the same argument as that given in the proof

of part (b) of Theorem 5, we can show that

$$\left\| \widehat{\beta}_{\text{grpls}} - \beta \right\|_2 = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right)$$

so that, similar to the result we have obtained previously in Theorem 5 for the many-predictors regression model, $\widehat{\beta}_{\text{grpls}}$ and $\widehat{\beta}_{\text{drpls}}$ also have the same rate of convergence here for the diffusion index case. Hence, it is easily seen that by following essentially the same argument as that given in the proof of part (a) above, we can, in addition, show that, under Scenario 1,

$$\begin{aligned} & \widehat{y}_{T+h}^{\text{grpls}} - \left(\alpha + [F_{1,T} - \mu_{F,1}]' \varphi_1 \right) \\ = & O_p \left(\max \left\{ \frac{c_0(N) \sqrt{N}}{\lambda_1^{\frac{3}{2}}} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \frac{\sqrt{a(N)N}}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{\sqrt{N}}{\lambda_1^{\frac{3}{2}}} \right\} \right) \\ & + O_p \left(\frac{1}{\sqrt{T}} \right) + O_p \left(\frac{1}{\sqrt{\lambda_1}} \right) \end{aligned}$$

and that, under Scenario 2,

$$\begin{aligned} & \widehat{y}_{T+h}^{\text{grpls}} - \left(\alpha + [F_T - \mu_F]' \varphi \right) \\ = & O_p \left(\max \left\{ \frac{c_0(N) \sqrt{N}}{\lambda_1^{\frac{3}{2}}} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \frac{\sqrt{a(N)N}}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{\sqrt{N}}{\lambda_1^{\frac{3}{2}}} \right\} \right) \\ & + O_p \left(\frac{1}{\sqrt{T}} \right) + O_p \left(\frac{1}{\sqrt{\lambda_1}} \right). \quad \square \end{aligned}$$

Remark A.1:

It should be noted that, in some special cases, a factor model can have an exact envelope decomposition. For example, if X_t is generated by an exact factor model whose error covariance matrix does not vary with t and is given by $\Sigma_u = E[u_t u_t'] = \sigma_u^2 I_N$. Here, given that $L = \begin{pmatrix} L_1 & L_2 \end{pmatrix}$ is a $N \times N$ orthogonal matrix, we can write:

$$\begin{aligned} \overline{\Sigma}_X &= L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2' + \sigma_u^2 I_N \\ &= L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2' + \sigma_u^2 L_1 L_1' + \sigma_u^2 L_2 L_2' \\ &= L_1 (\underline{\Delta}_1 + \sigma_u^2 I_N) L_1' + L_2 (\underline{\Delta}_2 + \sigma_u^2 I_N) L_2' \\ &= L_1 \Delta_1^* L_1' + L_2 \Delta_2^* L_2', \end{aligned}$$

where $\Delta_1^* = \underline{\Delta}_1 + \sigma_u^2 I_N$ and $\Delta_2^* = \underline{\Delta}_2 + \sigma_u^2 I_N$. In this case it is thus possible for $\overline{\Sigma}_X$ to have an exact envelope representation. However, specification of an exact factor model is often considered too restrictive for applications in economics and finance. For this reason, we have

chosen to consider the approximate envelope setup introduced here. In addition, although we use the factor model as an example in much of our intuitive discussion of the approximate envelope framework, it is worth emphasizing that there is nothing in the specification of the envelope model that requires X_t to be generated by a factor model. The approximate envelope framework is general enough to allow for dimension reduction in certain situations where there is no factor structure.

Remark A.2

Related to the discussion given in Remark 4.1(e) of the main paper, we provide the following example to illustrate how the envelope framework can lead to additional dimension reduction in the case where the leading eigenvalue has algebraic multiplicity greater than 1.

Example 1: Consider again the forecasting equation $y_{t+h} = \alpha + (X_t - \mu_X)' \beta + \varepsilon_{t+h}$, and suppose initially that we specify the covariance matrix decomposition as:

$$\Sigma_X = \underline{L}_1 \underline{\Lambda}_1 \underline{L}_1' + \underline{L}_2 \underline{\Lambda}_2 \underline{L}_2', \quad (58)$$

where $\underline{L}_1 = \begin{pmatrix} \underline{l}_1 & \underline{l}_2 \end{pmatrix}_{N \times 2}$. Here, for simplicity of explanation, we have assumed an exact envelope model so that $\bar{\Sigma}_u = 0$, but it is easy to see that the same story applies to a more general approximate envelope model. Also, again for simplicity of explanation, we have assumed covariance stationarity, so that $\bar{\Sigma}_X = T_*^{-1} \sum_{t=1}^{T_*} \Sigma_X = \Sigma_X$. Now, suppose that $\beta = \underline{L}_1 \underline{\varphi}$, for some 2×1 parameter vector $\underline{\varphi}$. Suppose also that:

$$\underline{\Lambda}_1 = \begin{pmatrix} \underline{\lambda}_1 & 0 \\ 0 & \underline{\lambda}_1 \end{pmatrix} = \underline{\lambda}_1 I_2 \text{ and } \underline{\Lambda}_2 = \begin{pmatrix} \underline{\lambda}_2 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_N \end{pmatrix}.$$

That is, assume that the top eigenvalue has an algebraic multiplicity equal to 2 and the remaining eigenvalues are given as the diagonal elements of the matrix $\underline{\Lambda}_2$, so that $\Sigma_X \underline{L}_1 = \underline{\lambda}_1 \underline{L}_1$, i.e. the columns of the $N \times 2$ matrix $\underline{L}_1$ are the eigenvectors of Σ_X associated with the eigenvalue $\underline{\lambda}_1$. Note, however, that in this case the vector $\ell_1 = \beta / \|\underline{\varphi}\|_2$ is also a normalized eigenvector of Σ_X associated with the eigenvalue $\underline{\lambda}_1$ since:

$$\begin{aligned} \Sigma_X \ell_1 &= \Sigma_X \left(\frac{\beta}{\|\underline{\varphi}\|_2} \right) = \Sigma_X \underline{L}_1 \frac{\underline{\varphi}}{\|\underline{\varphi}\|_2} = \underline{\lambda}_1 \underline{L}_1 \frac{\underline{\varphi}}{\|\underline{\varphi}\|_2} = \underline{\lambda}_1 \left(\frac{\beta}{\|\underline{\varphi}\|_2} \right) = \underline{\lambda}_1 \ell_1, \\ \|\ell_1\|_2^2 &= \frac{\beta' \beta}{\|\underline{\varphi}\|_2^2} = \frac{\underline{\varphi}' \underline{L}_1' \underline{L}_1 \underline{\varphi}}{\|\underline{\varphi}\|_2^2} = \frac{\underline{\varphi}' \underline{\varphi}}{\|\underline{\varphi}\|_2^2} = \frac{\|\underline{\varphi}\|_2^2}{\|\underline{\varphi}\|_2^2} = 1. \end{aligned}$$

In addition, let $\underline{\varphi}^\perp$ be a vector that is orthogonal to $\underline{\varphi}$, i.e. $\underline{\varphi}' \underline{\varphi}^\perp = 0$. Further, define $\ell_2 = \underline{L}_1 \underline{\varphi}^\perp / \|\underline{\varphi}^\perp\|_2$. Now, note that ℓ_2 is also a normalized eigenvector of Σ_X associated with

the eigenvalue $\underline{\lambda}_1$ and is orthogonal to ℓ_1 given that:

$$\begin{aligned}\Sigma_X \ell_2 &= \Sigma_X \underline{L}_1 \frac{\underline{\varphi}^\perp}{\|\underline{\varphi}^\perp\|_2} = \underline{\lambda}_1 \underline{L}_1 \frac{\underline{\varphi}^\perp}{\|\underline{\varphi}^\perp\|_2} = \underline{\lambda}_1 \ell_2, \\ \|\ell_2\|_2^2 &= \frac{\underline{\varphi}^{\perp'} \underline{L}_1' \underline{L}_1 \underline{\varphi}^\perp}{\|\underline{\varphi}^\perp\|_2^2} = \frac{\underline{\varphi}^{\perp'} \underline{\varphi}^\perp}{\|\underline{\varphi}^\perp\|_2^2} = \frac{\|\underline{\varphi}^\perp\|_2^2}{\|\underline{\varphi}^\perp\|_2^2} = 1, \\ \ell_1' \ell_2 &= \frac{\beta' \underline{L}_1 \underline{\varphi}^\perp}{\|\underline{\varphi}\|_2 \|\underline{\varphi}^\perp\|_2} = \frac{\underline{\varphi}' \underline{L}_1' \underline{L}_1 \underline{\varphi}^\perp}{\|\underline{\varphi}\|_2 \|\underline{\varphi}^\perp\|_2} = \frac{\underline{\varphi}' \underline{\varphi}^\perp}{\|\underline{\varphi}\|_2 \|\underline{\varphi}^\perp\|_2} = 0.\end{aligned}$$

It is easy to see that $\underline{L}_1 = \begin{pmatrix} \underline{L}_1 & \underline{L}_2 \end{pmatrix}$ and $\begin{pmatrix} \ell_1 & \ell_2 \end{pmatrix} = \begin{pmatrix} \beta / \|\underline{\varphi}\|_2 & \underline{L}_1 \underline{\varphi}^\perp / \|\underline{\varphi}^\perp\|_2 \end{pmatrix}$ span the same eigenspace associated with the eigenvalue $\underline{\lambda}_1$, so that there exist alternative choices of basis functions for this eigenspace. However, the second pair of eigenvectors provides more precise information about the subspace that actually contains β . In particular, information about β is entirely contained in the span of the eigenvector $\ell_1 = \beta / \|\underline{\varphi}\|_2$, whereas the eigenvector $\ell_2 = \underline{L}_1 \underline{\varphi}^\perp / \|\underline{\varphi}^\perp\|_2$ does not help to span the eigen-subspace containing β , so that the dimension of the envelope in this case should be $\kappa = 1$ and not $\kappa = 2$, and a more meaningful grouping of the eigenvalues/eigenvectors than that given by expression (58) is the decomposition:

$$\Sigma_X = \underline{\lambda}_1 \ell_1 \ell_1' + L_2 \Delta_2 L_2',$$

where $L_2 = \begin{pmatrix} \underline{L}_1 \underline{\varphi}^\perp / \|\underline{\varphi}^\perp\|_2 & \underline{L}_2 \end{pmatrix}$ and $\Delta_2 = \begin{pmatrix} \underline{\lambda}_1 & 0 \\ 0 & \underline{\Lambda}_2 \end{pmatrix}$. Another way to view this is to consider the covariance vector $\sigma_{Xy} = Cov(X_t, y_{t+h})$. Since:

$$\begin{aligned}\ell_1' \underline{L}_1 \underline{\varphi} &= \frac{\beta' \underline{L}_1 \underline{\varphi}}{\|\underline{\varphi}\|_2} = \frac{\underline{\varphi}' \underline{L}_1' \underline{L}_1 \underline{\varphi}}{\|\underline{\varphi}\|_2} = \frac{\underline{\varphi}' \underline{\varphi}}{\|\underline{\varphi}\|_2} = \|\underline{\varphi}\|_2 \text{ and} \\ L_2' \underline{L}_1 \underline{\varphi} &= \begin{pmatrix} \|\underline{\varphi}^\perp\|_2^{-1} \underline{\varphi}^{\perp'} \underline{L}_1' \\ \underline{L}_2' \end{pmatrix} \underline{L}_1 \underline{\varphi} = \begin{pmatrix} \|\underline{\varphi}^\perp\|_2^{-1} \underline{\varphi}^{\perp'} \underline{\varphi} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\end{aligned}$$

it follows that

$$\begin{aligned}\sigma_{Xy} &= E \{ (X_t - \mu_X) (y_{t+h} - \alpha) \} \\ &= E \{ (X_t - \mu_X) (X_t - \mu_X)' \} \beta + E \{ (X_t - \mu_X) \varepsilon_{t+h} \} \\ &= \Sigma_X \beta \quad (\text{since } \Sigma_X = E \{ (X_t - \mu_X) (X_t - \mu_X)' \} \text{ and } E \{ (X_t - \mu_X) \varepsilon_{t+h} \} = 0) \\ &= (\underline{\lambda}_1 \ell_1 \ell_1' + L_2 \Delta_2 L_2') \underline{L}_1 \underline{\varphi} \\ &= \underline{\lambda}_1 \|\underline{\varphi}\|_2 \ell_1.\end{aligned}$$

Thus, the covariance vector $\sigma_{Xy} = Cov(X_t, y_{t+h})$ depends just on ℓ_1 and not on ℓ_2 . Hence, the direction given by ℓ_2 is not correlated with y_{t+h} and will not be useful for predicting the target variable of interest y_{t+h} .

Appendix B: Some Key Support Lemmas and Their Proofs

Lemma B-1:

- (a) Let A be an $m \times \kappa$ matrix of rank κ , let $\tilde{A} = A + E$ be a perturbation of A , let $\|\cdot\|$ be any consistent matrix norm, and let E be decomposed into its components lying in and lying orthogonal to the column space of A :

$$E = E_1 + E_2 = P_A E + (I - P_A) E,$$

where $P_A = AA^+$ with A^+ denoting the Moore-Penrose generalized inverse of A . Moreover, let $\text{cond}(A) = \|A\| \|A^+\|$ denote the condition number of the matrix A . Under these conditions, if

$$\|A^+\| \|E_1\| < 1$$

then

$$\frac{\|(A + E)^+ - A^+\|}{\|A^+\|} \leq \beta_1 + \gamma \left(\frac{\beta_2^2}{1 + \beta_2^2} \right)^{\frac{1}{2}},$$

where

$$\gamma = \left(1 - \frac{\text{cond}(A) \|E_1\|}{\|A\|} \right)^{-1}$$

and

$$\beta_i = \frac{\gamma \text{cond}(A) \|E_i\|}{\|A\|} \text{ for } i = 1, 2.$$

- (b) Suppose that A is a nonsingular $\kappa \times \kappa$ matrix, and let $\tilde{A} = A + E$ be a perturbation of A where $\tilde{A}$ is not necessarily assumed to be nonsingular. In addition, let $\text{cond}(A) = \|A\| \|A^{-1}\| = \|A\| \|A^{-1}\|$ denote the condition number of the matrix A . Now, suppose that

$$\|A^{-1}\| \|E\| < 1$$

then

$$\begin{aligned} \frac{\|(A + E)^+ - A^{-1}\|}{\|A^{-1}\|} &= \frac{\|(A + E)^+ - A^+\|}{\|A^+\|} \\ &\leq \frac{\gamma \text{cond}(A) \|E\|}{\|A\|} \\ &= (1 - \|A^{-1}\| \|E\|)^{-1} \|A^{-1}\| \|E\|, \end{aligned}$$

where

$$\gamma = \left(1 - \frac{\text{cond}(A) \|E\|}{\|A\|} \right)^{-1} = (1 - \|A^{-1}\| \|E\|)^{-1}$$

- (c) Consider the same conditions as in part (b) above, except that $\tilde{A} = A + E$ is explicitly assumed to be nonsingular. Then,

$$\begin{aligned} \frac{\|(A + E)^{-1} - A^{-1}\|}{\|A^{-1}\|} &= \frac{\|(A + E)^+ - A^+\|}{\|A^+\|} \\ &\leq (1 - \|A^{-1}\| \|E\|)^{-1} \|A^{-1}\| \|E\|. \end{aligned}$$

- (d) Let $A \in \mathbb{C}^{p \times p}$ be nonsingular and let $\tilde{A} = A + E$ be a perturbation of A . Let $\|\cdot\|$ be a consistent matrix norm. If $\tilde{A}$ is nonsingular, then

$$\frac{\|\tilde{A}^{-1} - A^{-1}\|}{\|\tilde{A}^{-1}\|} \leq \|A^{-1}E\|$$

Alternatively, if

$$\|A^{-1}E\| < 1,$$

then, $\tilde{A}$ is perforce nonsingular and

$$\|\tilde{A}^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}E\|}.$$

Moreover,

$$\frac{\|\tilde{A}^{-1} - A^{-1}\|}{\|A^{-1}\|} \leq \frac{\|A^{-1}E\|}{1 - \|A^{-1}E\|}.$$

Proof of Lemma B-1:

Note that part (a) is essentially Theorem 4.3 of Stewart (1969). For a proof, please see pages 39-40 of Stewart (1969).

To show part (b), note first that part (b) is a specialization of the result given in part (a) in which A is taken to be a square and nonsingular matrix. In this case, we have $A^+ = A^{-1}$ so that $P_A = AA^{-1} = I$ and $\text{cond}(A) = \|A\| \|A^+\| = \|A\| \|A^{-1}\|$, from which it follows by

direct calculations

$$\begin{aligned}
E_2 &= (I - P_A) E = (I - I) E = 0 \text{ so that } E = E_1, \\
\gamma &= \left(1 - \frac{\text{cond}(A) \|E_1\|}{\|A\|}\right)^{-1} = \left(1 - \frac{\|A\| \|A^{-1}\| \|E\|}{\|A\|}\right)^{-1} = (1 - \|A^{-1}\| \|E\|)^{-1}, \\
\beta_1 &= \frac{\gamma \text{cond}(A) \|E_1\|}{\|A\|} \\
&= \frac{\gamma \text{cond}(A) \|E\|}{\|A\|} \quad (\text{since } E = E_1 \text{ in this case}) \\
&= (1 - \|A^{-1}\| \|E\|)^{-1} \frac{\|A\| \|A^{-1}\| \|E\|}{\|A\|} \\
&\quad \left(\text{since } \gamma = (1 - \|A^{-1}\| \|E\|)^{-1} \text{ and } \text{cond}(A) = \|A\| \|A^{-1}\|\right) \\
&= (1 - \|A^{-1}\| \|E\|)^{-1} \|A^{-1}\| \|E\| \\
\beta_2 &= \frac{\gamma \kappa \|E_2\|}{\|A\|} = 0 \quad (\text{since } E_2 = 0)
\end{aligned}$$

Hence, by the bound given in part (a), we obtain

$$\begin{aligned}
\frac{\|(A + E)^+ - A^{-1}\|}{\|A^{-1}\|} &= \frac{\|(A + E)^+ - A^+\|}{\|A^+\|} \quad (\text{since } A^+ = A^{-1} \text{ given that } A \text{ is nonsingular}) \\
&\leq \beta_1 + \gamma \left(\frac{\beta_2^2}{1 + \beta_2^2}\right)^{\frac{1}{2}} \\
&= \beta_1 \quad (\text{given that } \beta_2 = 0) \\
&= (1 - \|A^{-1}\| \|E\|)^{-1} \|A^{-1}\| \|E\|
\end{aligned}$$

The result given in part (c) follows immediately from that of part (b) by noting that when $\tilde{A} = A + E$ is also nonsingular; then,

$$\frac{\|(A + E)^+ - A^{-1}\|}{\|A^{-1}\|} = \frac{\|(A + E)^{-1} - A^{-1}\|}{\|A^{-1}\|}.$$

Finally, part (d) gives the result stated in Theorem 2.5 of Stewart and Sun (1990). For a proof, please see page 118 of Stewart and Sun (1990).

Lemma B-2: Suppose that part (a) of Assumption 4-2 holds. Let $\underline{\lambda}_{1,1}, \underline{\lambda}_{1,2}, \dots, \underline{\lambda}_{1,\kappa}$ be those eigenvalues which are the diagonal elements of $\underline{\Delta}_1$ as defined in part (a) of Assumption 4-2

and let $\lambda_1 = \lambda_{\max}(\bar{\Sigma}_X)$. Further define

$$\begin{aligned} \underline{V}_{1 \times 1} &= 1 \text{ for } \kappa = 1 \\ \underline{V}_{2 \times 2} &= \begin{pmatrix} 1 & 1 \\ 1 & \underline{\Delta}_{1,2}/\lambda_1 \end{pmatrix} \text{ for } \kappa = 2, \text{ and} \\ \underline{V}_{\kappa \times \kappa} &= \begin{pmatrix} 1 & \underline{\Delta}_{1,1}/\lambda_1 & \underline{\Delta}_{1,1}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,1}^{\kappa-1}/\lambda_1^{\kappa-1} \\ 1 & \underline{\Delta}_{1,2}/\lambda_1 & \underline{\Delta}_{1,2}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^{\kappa-1}/\lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underline{\Delta}_{1,\kappa}/\lambda_1 & \underline{\Delta}_{1,\kappa}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,\kappa}^{\kappa-1}/\lambda_1^{\kappa-1} \end{pmatrix} \text{ for integer } \kappa \geq 3 \end{aligned}$$

Then, there exists a constant $\underline{C} > 0$ such that

$$\mu_{\min}(\underline{V}'\underline{V}) \geq \underline{C} > 0,$$

where $\mu_{\min}(\underline{V}'\underline{V})$ here denotes the smallest eigenvalue of the matrix $\underline{V}'\underline{V}$.

Proof of Lemma B-2: For $\kappa = 1$, the result is trivial since

$$\mu_{\min}(\underline{V}'\underline{V}) = \underline{V}^2 = 1 = \underline{C} > 0$$

To show the result for integer $\kappa \geq 2$, first rewrite

$$\begin{aligned} \underline{V}_{\kappa \times \kappa} &= \begin{pmatrix} 1 & \underline{\Delta}_{1,1}/\lambda_1 & \underline{\Delta}_{1,1}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,1}^{\kappa-1}/\lambda_1^{\kappa-1} \\ 1 & \underline{\Delta}_{1,2}/\lambda_1 & \underline{\Delta}_{1,2}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^{\kappa-1}/\lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underline{\Delta}_{1,\kappa}/\lambda_1 & \underline{\Delta}_{1,\kappa}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,\kappa}^{\kappa-1}/\lambda_1^{\kappa-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \underline{\Delta}_{1,1}/\lambda_1 & (\underline{\Delta}_{1,1}/\lambda_1)^2 & \cdots & (\underline{\Delta}_{1,1}/\lambda_1)^{\kappa-1} \\ 1 & \underline{\Delta}_{1,2}/\lambda_1 & (\underline{\Delta}_{1,2}/\lambda_1)^2 & \cdots & (\underline{\Delta}_{1,2}/\lambda_1)^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underline{\Delta}_{1,\kappa}/\lambda_1 & (\underline{\Delta}_{1,\kappa}/\lambda_1)^2 & \cdots & (\underline{\Delta}_{1,\kappa}/\lambda_1)^{\kappa-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{\kappa-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_\kappa & x_\kappa^2 & \cdots & x_\kappa^{\kappa-1} \end{pmatrix} \end{aligned}$$

where $x_g = \underline{\Delta}_{1,g}/\lambda_1$ for $g = 1, \dots, \kappa$. It is clear from the above representation that $\underline{V}$ is a Vandermonde matrix. Now, from the well-known formula for the determinant of the

Vandermonde matrix (see, for example, page 29 of Horn and Johnson, 1985), we have

$$\det(\underline{V}) = \prod_{\substack{g,h=1 \\ g>h}}^{\kappa} \left(\frac{\lambda_{1,g}}{\lambda_1} - \frac{\lambda_{1,h}}{\lambda_1} \right)$$

so that

$$\begin{aligned} \det(\underline{V}'\underline{V}) &= \det(\underline{V}') \det(\underline{V}) \\ &= [\det(\underline{V})]^2 \\ &= \prod_{\substack{g,h=1 \\ g>h}}^{\kappa} \left(\frac{\lambda_{1,g}}{\lambda_1} - \frac{\lambda_{1,h}}{\lambda_1} \right)^2 \\ &= \prod_{\substack{g,h=1 \\ g>h}}^{\kappa} \left(\frac{\lambda_{1,h} - \lambda_{1,g}}{\lambda_{1,h}} \right)^2 \left(\frac{\lambda_{1,h}}{\lambda_{1,1}} \right)^2 \left(\frac{\lambda_{1,1}}{\lambda_1} \right)^2 \end{aligned}$$

Now, by part (a) of Assumption 4-2, there exists some positive constant $\underline{c}$

$$\frac{\lambda_{1,h} - \lambda_{1,g}}{\lambda_{1,h}} \geq \underline{c} > 0 \text{ for all } N, T \text{ sufficiently large}$$

and also by part (a) of Assumption 4-2, $\lambda_{1,h} \asymp \lambda_{1,1}$ so that there exists a positive constant $c_{\#}$ such that

$$\frac{\lambda_{1,h}}{\lambda_{1,1}} \geq c_{\#} > 0 \text{ for all } N, T \text{ sufficiently large.}$$

In addition, part (a) of Lemma D-3 implies that there exists some constant $c^* > 1$ such that

$$\frac{\lambda_{1,1}}{\lambda_1} \geq \frac{1}{c^*} > 0 \text{ for all } N, T \text{ sufficiently large.}$$

It follows that

$$\begin{aligned}
\det(\underline{V}'\underline{V}) &= \prod_{\substack{g,h=1 \\ g>h}}^{\kappa} \left(\frac{\lambda_{1,g}}{\lambda_1} - \frac{\lambda_{1,h}}{\lambda_1} \right)^2 \\
&= \prod_{\substack{g,h=1 \\ g>h}}^{\kappa} \left(\frac{\lambda_{1,h} - \lambda_{1,g}}{\lambda_{1,h}} \right)^2 \left(\frac{\lambda_{1,h}}{\lambda_{1,1}} \right)^2 \left(\frac{\lambda_{1,1}}{\lambda_1} \right)^2 \\
&\geq \frac{\kappa(\kappa-1)}{2} \left(\frac{c c_{\#}}{c^*} \right)^2 \\
&= \underline{C}_1 > 0 \text{ for } \kappa \geq 2 \left(\text{where we have set } \underline{C}_1 = \frac{\kappa(\kappa-1)}{2} \left(\frac{c c_{\#}}{c^*} \right)^2 \right).
\end{aligned}$$

Next, let $\mu_g(\underline{V}'\underline{V})$ ($g = 1, \dots, \kappa$) be the eigenvalues of $\underline{V}'\underline{V}$ which can be ordered, without loss of generality, as follows

$$\mu_1(\underline{V}'\underline{V}) \geq \mu_2(\underline{V}'\underline{V}) \geq \dots \geq \mu_{\kappa}(\underline{V}'\underline{V}) = \mu_{\min}(\underline{V}'\underline{V})$$

By the well-known equality between the determinant of a matrix and the product of its eigenvalues, we have

$$\prod_{g=1}^{\kappa} \mu_g(\underline{V}'\underline{V}) = \det(\underline{V}'\underline{V}),$$

which, in turn, implies that

$$\mu_{\min}(\underline{V}'\underline{V}) = \mu_{\kappa}(\underline{V}'\underline{V}) = \frac{\det(\underline{V}'\underline{V})}{\prod_{g=1}^{\kappa-1} \mu_g(\underline{V}'\underline{V})}.$$

In addition, note that, for $g = 1, \dots, \kappa - 1$,

$$\begin{aligned}
& \mu_g(\underline{V}'\underline{V}) \\
& \leq \operatorname{tr} \{ \underline{V}'\underline{V} \} \\
& = \operatorname{tr} \left\{ \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \underline{\lambda}_{1,1}/\lambda_1 & \underline{\lambda}_{1,2}/\lambda_1 & \cdots & \underline{\lambda}_{1,\kappa}/\lambda_1 \\ (\underline{\lambda}_{1,1}/\lambda_1)^2 & (\underline{\lambda}_{1,2}/\lambda_1)^2 & \cdots & (\underline{\lambda}_{1,\kappa}/\lambda_1)^2 \\ \vdots & \vdots & \cdots & \vdots \\ (\underline{\lambda}_{1,1}/\lambda_1)^{\kappa-1} & (\underline{\lambda}_{1,2}/\lambda_1)^{\kappa-1} & \cdots & (\underline{\lambda}_{1,\kappa}/\lambda_1)^{\kappa-1} \end{pmatrix} \right. \\
& \quad \times \left. \begin{pmatrix} 1 & \underline{\lambda}_{1,1}/\lambda_1 & (\underline{\lambda}_{1,1}/\lambda_1)^2 & \cdots & (\underline{\lambda}_{1,1}/\lambda_1)^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2}/\lambda_1 & (\underline{\lambda}_{1,2}/\lambda_1)^2 & \cdots & (\underline{\lambda}_{1,2}/\lambda_1)^{\kappa-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \underline{\lambda}_{1,\kappa}/\lambda_1 & (\underline{\lambda}_{1,\kappa}/\lambda_1)^2 & \cdots & (\underline{\lambda}_{1,\kappa}/\lambda_1)^{\kappa-1} \end{pmatrix} \right\} \\
& = \kappa + \sum_{g=1}^{\kappa} \left(\frac{\underline{\lambda}_{1,g}}{\lambda_1} \right)^2 + \sum_{g=1}^{\kappa} \left(\frac{\underline{\lambda}_{1,g}}{\lambda_1} \right)^4 + \cdots + \sum_{g=1}^{\kappa} \left(\frac{\underline{\lambda}_{1,g}}{\lambda_1} \right)^{2(\kappa-1)} \\
& \leq \kappa + \sum_{g=1}^{\kappa} \left(\frac{\underline{\lambda}_{1,g}}{\underline{\lambda}_{1,1}} \right)^2 \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right)^2 + \sum_{g=1}^{\kappa} \left(\frac{\underline{\lambda}_{1,g}}{\underline{\lambda}_{1,1}} \right)^4 \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right)^4 + \cdots \\
& \quad \cdots + \sum_{g=1}^{\kappa} \left(\frac{\underline{\lambda}_{1,g}}{\underline{\lambda}_{1,1}} \right)^{2(\kappa-1)} \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right)^{2(\kappa-1)} \\
& \leq \kappa + \sum_{g=1}^{\kappa} (c^*)^2 + \sum_{g=1}^{\kappa} (c^*)^4 + \cdots + \sum_{g=1}^{\kappa} (c^*)^{2(\kappa-1)} \text{ for } N, T \text{ sufficiently large} \\
& \quad \text{(by part (a) of Lemma D-3 and by the fact that } \underline{\lambda}_{1,g} \leq \underline{\lambda}_{1,1} \text{ for } g = 1, \dots, \kappa) \\
& \leq \overline{C}_2 \text{ (for some constant } \overline{C}_2 < \infty \text{ which exists since } \kappa \text{ and } c^* \text{ are both finite).}
\end{aligned}$$

from which we further deduce that, there exists a positive constant $\underline{C} = \underline{C}_1/\overline{C}_2$ such that

$$\begin{aligned}
\mu_{\min}(\underline{V}'\underline{V}) &= \frac{\det(\underline{V}'\underline{V})}{\prod_{g=1}^{\kappa-1} \mu_g(\underline{V}'\underline{V})} \\
&\geq \frac{1}{\prod_{g=1}^{\kappa-1} \mu_g(\underline{V}'\underline{V})} \underline{C}_1 \\
&\quad \text{(since } \det(\underline{V}'\underline{V}) \geq \underline{C}_1 \text{ as shown earlier)} \\
&\geq \frac{\underline{C}_1}{\overline{C}_2} \\
&= \underline{C} > 0 \text{ (since } \underline{C}_1 > 0 \text{ and } 0 < \overline{C}_2 < \infty) \square
\end{aligned}$$

Lemma B-3: Under Assumptions 4-1, 4-2, 4-3, 4-4, 6-1, 6-2, and 6-3; the following statements are true.

(a) There exists a positive constant C such that

$$\left\| \frac{\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 = \frac{\bar{\sigma}'_{Xy} \bar{\sigma}_{Xy}}{\lambda_1} \leq C \text{ for all } N \text{ and } T \text{ sufficiently large.}$$

(b) For any integer j such that $2 \leq j \leq 2\kappa + 1$,

$$\left| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{\beta' \left(\bar{\Sigma}_X^j - [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^j \right) \beta}{\lambda_1^{j-1}} \right| = O \left(\frac{1}{\lambda_1} \right)$$

Moreover,

$$\left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right\|_F = O \left(\frac{1}{\lambda_1} \right),$$

where

$$\begin{aligned} G_W &= \begin{pmatrix} [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \beta & [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^2 \beta & \cdots & [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^\kappa \beta \end{pmatrix}, \\ D_{\lambda,\kappa} &= \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{\kappa - \frac{1}{2}} \end{pmatrix}, \quad G = \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{pmatrix}, \text{ and} \\ \bar{\Sigma}_X &= \bar{\Sigma}_W + \bar{\Sigma}_u = L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2 + \bar{\Sigma}_u \end{aligned}$$

(c) Let $D_{\lambda,\kappa}$ and G_W be as defined in part (b) above and let

$$\begin{aligned} D_{\phi,\lambda} &= \begin{pmatrix} \underline{\Delta}_{1,1}^3 \phi_1^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2}^3 \phi_2^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,\kappa}^3 \phi_\kappa^2 / \lambda_1^3 \end{pmatrix} \text{ and} \\ \underline{V} &= \begin{pmatrix} 1 & \underline{\Delta}_{1,1} / \lambda_1 & \underline{\Delta}_{1,1}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,1}^{\kappa-1} / \lambda_1^{\kappa-1} \\ 1 & \underline{\Delta}_{1,2} / \lambda_1 & \underline{\Delta}_{1,2}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^{\kappa-1} / \lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underline{\Delta}_{1,\kappa} / \lambda_1 & \underline{\Delta}_{1,\kappa}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,\kappa}^{\kappa-1} / \lambda_1^{\kappa-1} \end{pmatrix} \end{aligned}$$

Then,

$$\left\| \left(\frac{\Delta_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \underline{V}' D_{\phi,\lambda} \underline{V} \right\|_2 = O \left(\frac{1}{\lambda_1} \right) = o(1).$$

and there exists a positive constants $C_{\#}$ and C_* such that

$$\lambda_{\min} (\underline{V}' D_{\phi,\lambda} \underline{V}) \geq C_{\#} > 0 \text{ and } \lambda_{\min} \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \geq C_* > 0$$

for all T and N sufficiently large. Moreover,

$$\left\| \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \underline{V}' \underline{D}_{\phi,\lambda} \underline{V} \right\|_2 = O \left(\frac{1}{\lambda_1} \right) = o(1),$$

where

$$\underline{D}_{\phi,\lambda} = \begin{pmatrix} \Delta_{1,1}^3 \phi_1^2 / (\lambda_1^2 \Delta_{1,1}) & 0 & \cdots & 0 \\ 0 & \Delta_{1,2}^3 \phi_2^2 / (\lambda_1^2 \Delta_{1,1}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Delta_{1,\kappa}^3 \phi_{\kappa}^2 / (\lambda_1^2 \Delta_{1,1}) \end{pmatrix}$$

(d)

$$\left| \frac{\hat{\sigma}_{Xy}^{S'} \hat{\sigma}_{Xy}^S}{\lambda_1} - \frac{\bar{\sigma}_{Xy}' \bar{\sigma}_{Xy}}{\lambda_1} \right| = O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right)$$

(e) For any fixed integer $k \geq 1$,

$$\begin{aligned} & \left| \frac{\hat{\sigma}_{Xy}^{S'} \left[\hat{\Sigma}_X^S \right]^k \hat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} - \frac{\bar{\sigma}_{Xy}' \bar{\Sigma}_X^k \bar{\sigma}_{Xy}}{\lambda_1^{k+1}} \right| \\ &= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \end{aligned}$$

(f) For any fixed integer $k \geq 1$

$$\left\| \frac{\left[\hat{\Sigma}_X^S \right]^k \hat{\sigma}_{Xy}^S - \bar{\Sigma}_X^k \bar{\sigma}_{Xy}}{\lambda_1^{k+\frac{1}{2}}} \right\|_2 = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

Proof of Lemma B-3: To show part (a), note that, by part (a) of Assumption 4-4 and the result of Lemma D-3, there exists a positive constant C such that, for all N and T sufficiently large,

$$\begin{aligned}
& \left\| \frac{\bar{\sigma}_{Xy}}{\lambda_1^{\frac{1}{2}}} \right\|_2^2 \\
&= \frac{\bar{\sigma}'_{Xy} \bar{\sigma}_{Xy}}{\lambda_1} \\
&= \frac{\beta' \bar{\Sigma}_X^2 \beta}{\lambda_1} \quad (\text{since } \bar{\sigma}_{Xy} = \bar{\Sigma}_X \beta) \\
&\leq \frac{\lambda_1^2}{\lambda_1} (\beta' \beta) \quad (\text{since } \lambda_1 = \lambda_{\max}(\bar{\Sigma}_X)) \\
&= \lambda_1 \left(\frac{\phi' L_1' L_1 \phi}{\underline{\lambda}_{1,1}} \right) \quad \left(\text{since } \beta = L_1 \frac{\phi}{\sqrt{\underline{\lambda}_{1,1}}} \text{ by Assumption 4-4 part (a)} \right) \\
&= \frac{\lambda_1}{\underline{\lambda}_{1,1}} \|\phi\|_2^2 \quad (\text{since } L_1' L_1 = I_\kappa) \\
&\leq c^* \|\phi\|_2^2 \quad (\text{since, by part (a) of Lemma D-3, there exists a constant } c^* > 1 \\
&\quad \text{such that } 0 < \frac{1}{c^*} \leq \frac{\lambda_1}{\underline{\lambda}_{1,1}} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large}) \\
&\leq c^* \kappa (\bar{c})^2 \quad (\text{by Assumption 4-4 part (a)}) \\
&\leq C < \infty \quad (\text{for some positive constant } C \geq c^* \kappa (\bar{c})^2)
\end{aligned}$$

To show part (b), note first that, for any integer j such that $2 \leq j \leq 2\kappa + 1$, we have

$$\begin{aligned}
& \left| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{\beta' \left(\bar{\Sigma}_X^j - [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^j \right)}{\lambda_1^{j-1}} \beta \right| \\
&= \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left| \frac{\beta' \left([L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2 + \bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)]^j - [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^j \right)}{\lambda_1^{j-1}} \right| \\
&\leq \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{1}{\lambda_1^{j-1}} \sum_{\ell=1}^j \binom{j}{\ell} \left\| \bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2) \right\|_2^\ell \|L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2\|_2^{j-\ell} \|\beta\|_2^2 \\
&= \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \lambda_1 \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)}{\lambda_1} \right\|_2^\ell \frac{\|L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2\|_2^{j-\ell}}{\lambda_1^{j-\ell}} \left\| L_1 \frac{\phi}{\sqrt{\underline{\lambda}_{1,1}}} \right\|_2^2 \\
&= \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)}{\lambda_1} \right\|_2^\ell \frac{\|L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2\|_2^{j-\ell}}{\lambda_1^{j-\ell}} \|L_1 \phi\|_2^2 \\
&\leq [c^* \max \{1, C_2\}]^{2\kappa+1} \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \|L_1 \phi\|_2^2 \quad (\text{since } \bar{\Sigma}_u = \bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)) \\
&\quad (\text{by Assumption 4-2(b) and Lemma D-3(a), there exist constants } c^* > 1 \text{ and } C_2 > 0 \text{ such} \\
&\quad \text{that } \|L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2\|_2 = \max \{\underline{\lambda}_{1,1}, \underline{\lambda}_{2,1}\} \leq c^* \max \{1, C_2\} \lambda_1 \text{ for } N, T \text{ sufficiently large}) \\
&\leq C^{2\kappa+1} \kappa (\bar{c})^2 \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \quad (\text{by setting } C \geq c^* \max \{1, C_2\} \text{ and noting that by} \\
&\quad \text{Assumption 4-4 part (a), } \kappa (\bar{c})^2 \geq \kappa \left(\max_{1 \leq g \leq \kappa} |\phi_g| \right)^2 \geq \|\phi\|_2^2 = \phi' L'_1 L_1 \phi = \|L_1 \phi\|_2^2) \\
&= C^{2\kappa+1} \kappa (\bar{c})^2 \sum_{\ell=1}^j \binom{j}{\ell} \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right)^\ell \quad (\text{by symmetry and positive semidefiniteness of } \bar{\Sigma}_u, \\
&\quad \text{so that } \|\bar{\Sigma}_u\|_2 = \sqrt{\lambda_{\max}(\bar{\Sigma}_u' \bar{\Sigma}_u)} = \sqrt{[\lambda_{\max}(\bar{\Sigma}_u)]^2} = \lambda_{\max}(\bar{\Sigma}_u)) \\
&\leq C^{2\kappa+1} \kappa (\bar{c})^2 [2\kappa + 1] [(2\kappa + 1)!] \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right) \\
&\quad \left(\text{since } j \leq 2\kappa + 1 \text{ and since } \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right)^\ell \ll \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right) \text{ for } \ell > 1 \text{ given that } \lambda_1 \rightarrow \infty \right. \\
&\quad \left. \text{while } \lambda_{\max}(\bar{\Sigma}_u) \leq \bar{C} < \infty \text{ for some constant } \bar{C} > 0 \text{ by Assumption 4-1 and Lemma D-2} \right) \\
&= O\left(\frac{1}{\lambda_1}\right) \quad (\text{given that } \kappa \text{ is a finite positive integer}) \\
&= o(1),
\end{aligned}$$

which shows the first result. Now, to show the second result, note that, by straightforward calculation

$$\begin{aligned}
& \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
&= \left\{ \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_{Xy} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \end{pmatrix} \frac{\bar{\Sigma}_X}{\lambda_1} \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{pmatrix} \right. \\
&\quad \times \left. \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \right\} \\
&= \begin{pmatrix} \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} / \lambda_1^2 & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} / \lambda_1^3 & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^\kappa \bar{\sigma}_{Xy} / \lambda_1^{\kappa+1} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} / \lambda_1^3 & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^3 \bar{\sigma}_{Xy} / \lambda_1^4 & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa+1} \bar{\sigma}_{Xy} / \lambda_1^{\kappa+2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^\kappa \bar{\sigma}_{Xy} / \lambda_1^{\kappa+1} & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa+1} \bar{\sigma}_{Xy} / \lambda_1^{\kappa+2} & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{2\kappa-1} \bar{\sigma}_{Xy} / \lambda_1^{2\kappa} \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{D_{\lambda,\kappa}^{-1} G'_W [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
&= \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \\ \vdots \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^\kappa \end{pmatrix} \frac{[L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]}{\lambda_1} \\
&\quad \times \begin{pmatrix} [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \beta & \cdots & [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^\kappa \beta \end{pmatrix} \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \\
&= \begin{pmatrix} \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^3 \beta / \lambda_1^2 & \cdots & \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+2} \beta / \lambda_1^{\kappa+1} \\ \vdots & \ddots & \vdots \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+2} \beta / \lambda_1^{\kappa+1} & \cdots & \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{2\kappa+1} \beta / \lambda_1^{2\kappa} \end{pmatrix} \quad (59)
\end{aligned}$$

It follows from these calculations that

$$\begin{aligned}
& \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \left\{ \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\} \\
&= \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \begin{pmatrix} \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} / \lambda_1^2 & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} / \lambda_1^3 & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^\kappa \bar{\sigma}_{Xy} / \lambda_1^{\kappa+1} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} / \lambda_1^3 & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^3 \bar{\sigma}_{Xy} / \lambda_1^4 & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa+1} \bar{\sigma}_{Xy} / \lambda_1^{\kappa+2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^\kappa \bar{\sigma}_{Xy} / \lambda_1^{\kappa+1} & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa+1} \bar{\sigma}_{Xy} / \lambda_1^{\kappa+2} & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{2\kappa-1} \bar{\sigma}_{Xy} / \lambda_1^{2\kappa} \end{pmatrix} \\
&- \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \begin{pmatrix} \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^3 \beta / \lambda_1^2 & \cdots & \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+2} \beta / \lambda_1^{\kappa+1} \\ \vdots & \ddots & \vdots \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+2} \beta / \lambda_1^{\kappa+1} & \cdots & \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{2\kappa+1} \beta / \lambda_1^{2\kappa} \end{pmatrix} \\
&= \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \begin{pmatrix} \beta' \bar{\Sigma}_X^3 \beta / \lambda_1^2 & \beta' \bar{\Sigma}_X^4 \beta / \lambda_1^3 & \cdots & \beta' \bar{\Sigma}_X^{\kappa+2} \beta / \lambda_1^{\kappa+1} \\ \beta' \bar{\Sigma}_X^4 \beta / \lambda_1^3 & \beta' \bar{\Sigma}_X^5 \beta / \lambda_1^4 & \cdots & \beta' \bar{\Sigma}_X^{\kappa+3} \beta / \lambda_1^{\kappa+2} \\ \vdots & \vdots & \ddots & \vdots \\ \beta' \bar{\Sigma}_X^{\kappa+2} \beta / \lambda_1^{\kappa+1} & \beta' \bar{\Sigma}_X^{\kappa+3} \beta / \lambda_1^{\kappa+2} & \cdots & \beta' \bar{\Sigma}_X^{2\kappa+1} \beta / \lambda_1^{2\kappa} \end{pmatrix} \quad (\text{since } \bar{\sigma}_{Xy} = \bar{\Sigma}_X \beta) \\
&- \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \begin{pmatrix} \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^3 \beta / \lambda_1^2 & \cdots & \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+2} \beta / \lambda_1^{\kappa+1} \\ \vdots & \ddots & \vdots \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+2} \beta / \lambda_1^{\kappa+1} & \cdots & \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{2\kappa+1} \beta / \lambda_1^{2\kappa} \end{pmatrix} \\
&= \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \begin{pmatrix} A_{11} & \cdots & A_{1\kappa} \\ \vdots & \ddots & \vdots \\ A_{\kappa 1} & \cdots & A_{\kappa\kappa} \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
A_{11} &= \beta' \left(\bar{\Sigma}_X^3 - [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^3 \right) \beta / \lambda_1^2, \\
A_{1\kappa} &= \beta' \left(\bar{\Sigma}_X^{\kappa+2} - [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+2} \right) \beta / \lambda_1^{\kappa+1}, \\
A_{\kappa 1} &= \beta' \left(\bar{\Sigma}_X^{\kappa+2} - [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+2} \right) \beta / \lambda_1^{\kappa+1}, \\
A_{\kappa\kappa} &= \beta' \left(\bar{\Sigma}_X^{2\kappa+1} - [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{2\kappa+1} \right) \beta / \lambda_1^{2\kappa}.
\end{aligned}$$

from which we further deduce that

$$\begin{aligned}
& \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right\|_F^2 \\
&= \sum_{j=1}^{\kappa} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{\beta' \left(\bar{\Sigma}_X^{2+j} - [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{2+j} \right) \beta}{\lambda_1^{2+j-1}} \right)^2 \\
&\quad + \sum_{j=1}^{\kappa} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{\beta' \left(\bar{\Sigma}_X^{3+j} - [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{3+j} \right) \beta}{\lambda_1^{3+j-1}} \right)^2 \\
&\quad + \cdots + \sum_{j=1}^{\kappa} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{\beta' \left(\bar{\Sigma}_X^{\kappa+1+j} - [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+1+j} \right) \beta}{\lambda_1^{\kappa+j}} \right)^2 \\
&= \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{\beta' \left(\bar{\Sigma}_X^{i+j+1} - [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{i+j+1} \right) \beta}{\lambda_1^{i+j}} \right)^2 \\
&\leq \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \left[C^{2\kappa+1} \kappa (\bar{c})^2 \sum_{\ell=1}^{i+j+1} \binom{i+j+1}{\ell} \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right)^\ell \right]^2 \\
&\leq \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \left[C^{2\kappa+1} \kappa (\bar{c})^2 \sum_{\ell=1}^{i+j+1} \binom{i+j+1}{\ell} \left(\frac{\bar{C}}{\lambda_1} \right)^\ell \right]^2 \quad (\text{since, by Assumption 4-1 and} \\
&\quad \text{Lemma D-2, there exists a constant } \bar{C} > 0 \text{ such that } \lambda_{\max}(\bar{\Sigma}_u) \leq \bar{C} < \infty) \\
&\leq \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} C^{4\kappa+2} \kappa^2 (\bar{c})^4 (i+j+1) \sum_{\ell=1}^{i+j+1} \binom{i+j+1}{\ell}^2 \left(\frac{\bar{C}}{\lambda_1} \right)^{2\ell} \\
&\quad \left(\text{by the inequality } \left| \sum_{i=1}^m a_i \right|^2 \leq m \sum_{i=1}^m |a_i|^2 \right) \\
&\leq C^{4\kappa+2} \kappa^4 (\bar{c})^4 (2\kappa+1)^2 [(2\kappa+1)!]^2 \left(\frac{\bar{C}}{\lambda_1} \right)^2 \quad \text{for all } N, T \text{ sufficiently large} \\
&\quad \left(\text{given that } \left(\frac{\bar{C}}{\lambda_1} \right)^\ell \ll \left(\frac{\bar{C}}{\lambda_1} \right) \text{ for integer } \ell \geq 2 \text{ since, by Lemma D-3(a)} \right. \\
&\quad \left. \lambda_1 \asymp \underline{\lambda}_{1,1} \text{ so that } \lambda_1 \rightarrow \infty \text{ as } N, T \rightarrow \infty \right) \\
&\leq C^\bullet \left(\frac{1}{\lambda_1} \right)^2 \quad \left(\text{for some constant } C^\bullet \text{ such that } C^\bullet \geq C^{4\kappa+2} \kappa^2 (\bar{c})^4 (2\kappa+1)^2 [(2\kappa+1)!]^2 \bar{C}^2 \right) \\
&= O\left(\frac{1}{\lambda_1^2} \right),
\end{aligned}$$

so that

$$\left\| \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right\|_F = O \left(\frac{1}{\lambda_1} \right)$$

To show part (c), note first that, for integer $k \geq 3$,

$$\begin{aligned} \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{\beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^k \beta}{\lambda_1^{k-1}} &= \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{\phi' L'_1}{\sqrt{\lambda_{1,1}}} \frac{[L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^k}{\lambda_1^{k-1}} \frac{L_1 \phi}{\sqrt{\lambda_{1,1}}} \\ &\quad \left(\text{since } \beta = \frac{L_1 \phi}{\sqrt{\lambda_{1,1}}} \text{ by Assumption 4-4} \right) \\ &= \frac{\phi' L'_1 [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^k L_1 \phi}{\lambda_1^k} \\ &= \frac{\phi' L'_1 (L_1 \underline{\Delta}_1^k L'_1 + L_2 \underline{\Delta}_2^k L'_2) L_1 \phi}{\lambda_1^k} \\ &\quad \left(\text{since } [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^k = L_1 \underline{\Delta}_1^k L'_1 + L_2 \underline{\Delta}_2^k L'_2 \right) \\ &= \frac{\phi' \underline{\Delta}_1^k \phi}{\lambda_1^k} \\ &= \frac{1}{\lambda_1^k} \sum_{g=1}^{\kappa} \lambda_{1,g}^k \phi_g^2 \end{aligned}$$

Making use of the above identity and of expression (59) in the proof of part (b) above, we further obtain

$$\begin{aligned} &\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G'_W [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \\ &= \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \begin{pmatrix} \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^3 \beta / \lambda_1^2 & \cdots & \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+2} \beta / \lambda_1^{\kappa+1} \\ \vdots & \ddots & \vdots \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+2} \beta / \lambda_1^{\kappa+1} & \cdots & \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{2\kappa+1} \beta / \lambda_1^{2\kappa} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{g=1}^{\kappa} \lambda_{1,g}^3 \phi_g^2 / \lambda_1^3 & \sum_{g=1}^{\kappa} \lambda_{1,g}^4 \phi_g^2 / \lambda_1^4 & \cdots & \sum_{g=1}^{\kappa} \lambda_{1,g}^{\kappa+2} \phi_g^2 / \lambda_1^{\kappa+2} \\ \sum_{g=1}^{\kappa} \lambda_{1,g}^4 \phi_g^2 / \lambda_1^4 & \sum_{g=1}^{\kappa} \lambda_{1,g}^5 \phi_g^2 / \lambda_1^5 & \cdots & \sum_{g=1}^{\kappa} \lambda_{1,g}^{\kappa+3} \phi_g^2 / \lambda_1^{\kappa+3} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{g=1}^{\kappa} \lambda_{1,g}^{\kappa+2} \phi_g^2 / \lambda_1^{\kappa+2} & \sum_{g=1}^{\kappa} \lambda_{1,g}^{\kappa+3} \phi_g^2 / \lambda_1^{\kappa+3} & \cdots & \sum_{g=1}^{\kappa} \lambda_{1,g}^{2\kappa+1} \phi_g^2 / \lambda_1^{2\kappa+1} \end{pmatrix} \end{aligned}$$

Now,

$$\begin{aligned}
& \begin{pmatrix} \sum_{g=1}^{\kappa} \Delta_{1,g}^3 \phi_g^2 / \lambda_1^3 & \sum_{g=1}^{\kappa} \Delta_{1,g}^4 \phi_g^2 / \lambda_1^4 & \cdots & \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+2} \phi_g^2 / \lambda_1^{\kappa+2} \\ \sum_{g=1}^{\kappa} \Delta_{1,g}^4 \phi_g^2 / \lambda_1^4 & \sum_{g=1}^{\kappa} \Delta_{1,g}^5 \phi_g^2 / \lambda_1^5 & \cdots & \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+3} \phi_g^2 / \lambda_1^{\kappa+3} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+2} \phi_g^2 / \lambda_1^{\kappa+2} & \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+3} \phi_g^2 / \lambda_1^{\kappa+3} & \cdots & \sum_{g=1}^{\kappa} \Delta_{1,g}^{2\kappa+1} \phi_g^2 / \lambda_1^{2\kappa+1} \end{pmatrix} \\
&= \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \right. \\
&\quad \times \begin{pmatrix} \sum_{g=1}^{\kappa} \Delta_{1,g}^3 \phi_g^2 / \lambda_1^3 & \sum_{g=1}^{\kappa} \Delta_{1,g}^4 \phi_g^2 / \lambda_1^3 & \cdots & \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+2} \phi_g^2 / \lambda_1^3 \\ \sum_{g=1}^{\kappa} \Delta_{1,g}^4 \phi_g^2 / \lambda_1^3 & \sum_{g=1}^{\kappa} \Delta_{1,g}^5 \phi_g^2 / \lambda_1^3 & \cdots & \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+3} \phi_g^2 / \lambda_1^3 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+2} \phi_g^2 / \lambda_1^3 & \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+3} \phi_g^2 / \lambda_1^3 & \cdots & \sum_{g=1}^{\kappa} \Delta_{1,g}^{2\kappa+1} \phi_g^2 / \lambda_1^3 \end{pmatrix} \\
&\quad \times \left. \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \begin{pmatrix} \Delta_{1,1}^3 \phi_1^2 / \lambda_1^3 & \Delta_{1,2}^3 \phi_2^2 / \lambda_1^3 & \cdots & \Delta_{1,\kappa}^3 \phi_{\kappa}^2 / \lambda_1^3 \\ \Delta_{1,1}^4 \phi_1^2 / \lambda_1^3 & \Delta_{1,2}^4 \phi_2^2 / \lambda_1^3 & \cdots & \Delta_{1,\kappa}^4 \phi_{\kappa}^2 / \lambda_1^3 \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{1,1}^{\kappa+2} \phi_1^2 / \lambda_1^3 & \Delta_{1,2}^{\kappa+2} \phi_2^2 / \lambda_1^3 & \cdots & \Delta_{1,\kappa}^{\kappa+2} \phi_{\kappa}^2 / \lambda_1^3 \end{pmatrix} \right. \\
&\quad \times \left. \begin{pmatrix} 1 & \Delta_{1,1} & \Delta_{1,1}^2 & \cdots & \Delta_{1,1}^{\kappa-1} \\ 1 & \Delta_{1,2} & \Delta_{1,2}^2 & \cdots & \Delta_{1,2}^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \Delta_{1,\kappa} & \Delta_{1,\kappa}^2 & \cdots & \Delta_{1,\kappa}^{\kappa-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \underline{\lambda}_{1,1} & \underline{\lambda}_{1,2} & \cdots & \underline{\lambda}_{1,\kappa} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\lambda}_{1,1}^{\kappa-1} & \underline{\lambda}_{1,2}^{\kappa-1} & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} \end{pmatrix} \right. \\
&\quad \times \begin{pmatrix} \underline{\lambda}_{1,1}^3 \phi_1^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \phi_2^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa}^3 \phi_\kappa^2 / \lambda_1^3 \end{pmatrix} \\
&\quad \times \left. \begin{pmatrix} 1 & \underline{\lambda}_{1,1} & \underline{\lambda}_{1,1}^2 & \cdots & \underline{\lambda}_{1,1}^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2} & \underline{\lambda}_{1,2}^2 & \cdots & \underline{\lambda}_{1,2}^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\lambda}_{1,\kappa} & \underline{\lambda}_{1,\kappa}^2 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \right\} \\
&= \underline{V}' D_{\phi,\lambda} \underline{V}
\end{aligned}$$

where

$$\begin{aligned}
D_{\phi,\lambda} &= \begin{pmatrix} \underline{\lambda}_{1,1}^3 \phi_1^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \phi_2^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa}^3 \phi_\kappa^2 / \lambda_1^3 \end{pmatrix} \\
\text{and } \underline{V} &= \begin{pmatrix} 1 & \underline{\lambda}_{1,1} & \underline{\lambda}_{1,1}^2 & \cdots & \underline{\lambda}_{1,1}^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2} & \underline{\lambda}_{1,2}^2 & \cdots & \underline{\lambda}_{1,2}^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\lambda}_{1,\kappa} & \underline{\lambda}_{1,\kappa}^2 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \\
&= \begin{pmatrix} 1 & \underline{\lambda}_{1,1} / \lambda_1 & \underline{\lambda}_{1,1}^2 / \lambda_1^2 & \cdots & \underline{\lambda}_{1,1}^{\kappa-1} / \lambda_1^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2} / \lambda_1 & \underline{\lambda}_{1,2}^2 / \lambda_1^2 & \cdots & \underline{\lambda}_{1,2}^{\kappa-1} / \lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\lambda}_{1,\kappa} / \lambda_1 & \underline{\lambda}_{1,\kappa}^2 / \lambda_1^2 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} / \lambda_1^{\kappa-1} \end{pmatrix}.
\end{aligned}$$

It follows by the second result given in part (b) of this lemma that

$$\begin{aligned}
& \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \underline{V}' D_{\phi,\lambda} \underline{V} \right\|_2 \\
&= \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right\|_2 \\
&\leq \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right\|_F \\
&= O\left(\frac{1}{\lambda_1}\right). \tag{60}
\end{aligned}$$

Moreover, applying the result of Lemma B-2 above as well as Assumptions 4-2(a) and 4-4, we further obtain

$$\begin{aligned}
\lambda_{\min}(\underline{V}' D_{\phi,\lambda} \underline{V}) &\geq \min_{1 \leq g \leq \kappa} (\underline{\lambda}_{1,g}^3 \phi_g^2 / \lambda_1^3) \lambda_{\min}(\underline{V}' \underline{V}) \\
&\geq \left(\frac{\underline{\lambda}_{1,\kappa}}{\lambda_1} \right)^3 \left(\min_{1 \leq g \leq \kappa} |\phi_g| \right)^2 \lambda_{\min}(\underline{V}' \underline{V}) \\
&\geq \left(\frac{\underline{\lambda}_{1,\kappa}}{\lambda_1} \right)^3 \left(\frac{1}{\bar{c}} \right)^2 \underline{C} \text{ (by Lemma B-2 and Assumption 4-4 part (a))} \\
&\geq \left(\frac{1}{c^*} \right)^3 \left(\frac{1}{\bar{c}} \right)^2 \underline{C} \text{ for all } N, T \text{ sufficiently large} \\
&\quad \text{(since Lemma D-3 part (a) implies that there exists some constant } c^* > 1 \\
&\quad \text{such that } \frac{\underline{\lambda}_{1,\kappa}}{\lambda_1} \geq \frac{1}{c^*} > 0 \text{ for all } N, T \text{ sufficiently large)} \\
&\geq C_{\#} \left(\text{for some positive constant } C_{\#} > 0 \text{ such that } C_{\#} \leq \left(\frac{1}{c^*} \right)^3 \left(\frac{1}{\bar{c}} \right)^2 \underline{C} \right)
\end{aligned}$$

In addition, by Lemma D-4 and by expression (60) above, we see that

$$\begin{aligned}
\left| \lambda_{\min} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) - \lambda_{\min}(\underline{V}' D_{\phi,\lambda} \underline{V}) \right| &\leq \left\| \left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \underline{V}' D_{\phi,\lambda} \underline{V} \right\|_2 \\
&= O\left(\frac{1}{\lambda_1}\right) = o(1) \tag{61}
\end{aligned}$$

Hence, applying the result of Lemma B-2 and the result given in expression (61) above, as well as Assumptions 4-2(a) and 4-4, we also obtain

$$\begin{aligned}
& \lambda_{\min} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \\
&= \lambda_{\min} (\underline{V}' D_{\phi,\lambda} \underline{V}) + \lambda_{\min} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) - \lambda_{\min} (\underline{V}' D_{\phi,\lambda} \underline{V}) \\
&\geq \lambda_{\min} (\underline{V}' D_{\phi,\lambda} \underline{V}) - \left| \lambda_{\min} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) - \lambda_{\min} (\underline{V}' D_{\phi,\lambda} \underline{V}) \right| \\
&\geq \min_{1 \leq g \leq \kappa} (\underline{\lambda}_{1,g}^3 \phi_g^2 / \lambda_1^3) \lambda_{\min} (\underline{V}' \underline{V}) - \left| \lambda_{\min} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) - \lambda_{\min} (\underline{V}' D_{\phi,\lambda} \underline{V}) \right| \\
&\geq \left(\frac{\underline{\lambda}_{1,\kappa}}{\lambda_1} \right)^3 \left(\min_{1 \leq g \leq \kappa} |\phi_g| \right)^2 \lambda_{\min} (\underline{V}' \underline{V}) + o(1) \\
&\geq C_{\#} \quad (\text{for some constant } C_{\#} > 0) \\
&\quad \text{and for all } N, T \text{ sufficiently large given Assumptions 4-2(a) and 4-4} \tag{62}
\end{aligned}$$

Furthermore, note that, for all N, T sufficiently large,

$$\begin{aligned}
& \lambda_{\min} \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \\
&= \frac{\lambda_1}{\underline{\lambda}_{1,1}} \lambda_{\min} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \\
&\geq \frac{C_{\#}}{c^*} \quad (\text{since by Lemma D-3 part (a), there exists a constant } c^* > 1 \\
&\quad \text{such that } 0 < \frac{1}{c^*} \leq \frac{\lambda_1}{\underline{\lambda}_{1,1}} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large}) \\
&\geq C_* \text{ for some constant } C_* > 0 \text{ such that } C_* \leq \frac{C_{\#}}{c^*}.
\end{aligned}$$

Now, to show the last result of part (c), note first that

$$\begin{aligned}
\left(\frac{\lambda_1}{\underline{\lambda}_{1,1}}\right) D_{\phi,\lambda} &= \left(\frac{\lambda_1}{\underline{\lambda}_{1,1}}\right) \begin{pmatrix} \underline{\lambda}_{1,1}^3 \phi_1^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \phi_2^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa}^3 \phi_\kappa^2 / \lambda_1^3 \end{pmatrix} \\
&= \begin{pmatrix} (\lambda_1 / \underline{\lambda}_{1,1}) \underline{\lambda}_{1,1}^3 \phi_1^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & (\lambda_1 / \underline{\lambda}_{1,1}) \underline{\lambda}_{1,2}^3 \phi_2^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (\lambda_1 / \underline{\lambda}_{1,1}) \underline{\lambda}_{1,\kappa}^3 \phi_\kappa^2 / \lambda_1^3 \end{pmatrix} \\
&= \begin{pmatrix} \underline{\lambda}_{1,1}^3 \phi_1^2 / (\lambda_1^2 \underline{\lambda}_{1,1}) & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \phi_2^2 / (\lambda_1^2 \underline{\lambda}_{1,1}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa}^3 \phi_\kappa^2 / (\lambda_1^2 \underline{\lambda}_{1,1}) \end{pmatrix} \\
&= \underline{D}_{\phi,\lambda}
\end{aligned}$$

It follows, by applying part (a) of Lemma D-3 and the first result of part (c) here, we have

$$\begin{aligned}
&\left\| \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \underline{V}' \underline{D}_{\phi,\lambda} \underline{V} \right\|_2 \\
&= \left\| \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \left(\frac{\lambda_1}{\underline{\lambda}_{1,1}}\right) \underline{V}' \underline{D}_{\phi,\lambda} \underline{V} \right\|_2 \\
&= \left(\frac{\lambda_1}{\underline{\lambda}_{1,1}}\right) \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1}\right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \underline{V}' \underline{D}_{\phi,\lambda} \underline{V} \right\|_2 \\
&\leq c^* \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1}\right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \underline{V}' \underline{D}_{\phi,\lambda} \underline{V} \right\|_2 \quad \left(\text{since, by Lemma D-3 part (a), there exists} \right. \\
&\quad \left. \text{a constant } c^* > 1 \text{ such that } 0 < \frac{1}{c^*} \leq \frac{\lambda_1}{\underline{\lambda}_{1,1}} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large} \right) \\
&= O\left(\frac{1}{\lambda_1}\right) \quad (\text{by the first result of this part of Lemma B-3})
\end{aligned}$$

Now, to show part (d), note that, by simple calculation, we have

$$\begin{aligned}
& \left| \frac{\hat{\sigma}_{Xy}^{S'} \hat{\sigma}_{Xy}^S}{\lambda_1} - \frac{\bar{\sigma}_{Xy}' \bar{\sigma}_{Xy}}{\lambda_1} \right| \\
&= \left| \frac{[\bar{\sigma}_{Xy} + (\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy})]' [\bar{\sigma}_{Xy} + (\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy})]}{\lambda_1} - \frac{\bar{\sigma}_{Xy}' \bar{\sigma}_{Xy}}{\lambda_1} \right| \\
&= \left| \frac{\bar{\sigma}_{Xy}' \bar{\sigma}_{Xy}}{\lambda_1} + 2 \frac{(\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy})' \bar{\sigma}_{Xy}}{\lambda_1} + \frac{(\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy})' (\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy})}{\lambda_1} - \frac{\bar{\sigma}_{Xy}' \bar{\sigma}_{Xy}}{\lambda_1} \right| \\
&= \left| 2 \frac{(\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy})' \bar{\sigma}_{Xy}}{\lambda_1} + \frac{(\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy})' (\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy})}{\lambda_1} \right| \\
&\leq 2 \frac{|(\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy})' \bar{\sigma}_{Xy}|}{\lambda_1} + \frac{(\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy})' (\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy})}{\lambda_1} \quad (\text{by triangle inequality}) \\
&\leq 2 \sqrt{\frac{\|\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}\|_2^2}{\lambda_1}} \sqrt{\frac{\bar{\sigma}_{Xy}' \bar{\sigma}_{Xy}}{\lambda_1}} + \left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\lambda_1^{\frac{1}{2}}} \right\|_2^2 \quad (\text{by Cauchy Schwarz inequality}) \\
&\quad \text{and the fact that } \|\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}\|_2^2 = (\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy})' (\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}) \\
&= 2 \left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \left\| \frac{\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 + \left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2
\end{aligned}$$

Now, Lemma C-9 shows that

$$\left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 = O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right)$$

It follows by Lemma C-9 and by part (a) of this lemma that

$$\begin{aligned}
& \left| \frac{\widehat{\sigma}_{Xy}^{S'} \widehat{\sigma}_{Xy}^S}{\lambda_1} - \frac{\overline{\sigma}_{Xy}' \overline{\sigma}_{Xy}}{\lambda_1} \right| \\
& \leq 2 \left\| \frac{\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \left\| \frac{\overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 + \left\| \frac{\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 \\
& = O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right) O(1) + O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \right) \\
& = O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right)
\end{aligned}$$

for $0 \leq q < 1$.

Turning our attention to part (e), note first that, for any fixed integer $k \geq 1$,

$$\begin{aligned}
& \left| \frac{\widehat{\sigma}_{Xy}^{S'} \left[\widehat{\Sigma}_X^S \right]^k \widehat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} - \frac{\overline{\sigma}_{Xy}' \overline{\Sigma}_X^k \overline{\sigma}_{Xy}}{\lambda_1^{k+1}} \right| \\
& = \left| \frac{\widehat{\sigma}_{Xy}^{S'} \left(\overline{\Sigma}_X + \widehat{\Sigma}_X^S - \overline{\Sigma}_X \right)^k \widehat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} - \frac{\overline{\sigma}_{Xy}' \overline{\Sigma}_X^k \overline{\sigma}_{Xy}}{\lambda_1^{k+1}} \right| \\
& = \left| \frac{\widehat{\sigma}_{Xy}^{S'} \left(\overline{\Sigma}_X + \widehat{\Sigma}_X^S - \overline{\Sigma}_X \right)^k \widehat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} - \frac{\widehat{\sigma}_{Xy}^{S'} \overline{\Sigma}_X^k \widehat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} \right| \\
& \quad + \left| \frac{\widehat{\sigma}_{Xy}^{S'} \overline{\Sigma}_X^k \widehat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} - \frac{\overline{\sigma}_{Xy}' \overline{\Sigma}_X^k \overline{\sigma}_{Xy}}{\lambda_1^{k+1}} \right| \\
& \leq \left| \frac{\widehat{\sigma}_{Xy}^{S'} \left(\overline{\Sigma}_X + \widehat{\Sigma}_X^S - \overline{\Sigma}_X \right)^k \widehat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} - \frac{\widehat{\sigma}_{Xy}^{S'} \overline{\Sigma}_X^k \widehat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} \right| \\
& \quad + \left| \frac{\widehat{\sigma}_{Xy}^{S'} \overline{\Sigma}_X^k \widehat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} - \frac{\overline{\sigma}_{Xy}' \overline{\Sigma}_X^k \overline{\sigma}_{Xy}}{\lambda_1^{k+1}} \right|
\end{aligned} \tag{63}$$

To analyze the first term on the majorant side of the expression (63) given above, note that

$$\begin{aligned}
& \left| \frac{\hat{\sigma}_{Xy}^{S'} \left(\bar{\Sigma}_X + \hat{\Sigma}_X^S - \bar{\Sigma}_X \right)^k \hat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} - \frac{\hat{\sigma}_{Xy}^{S'} \bar{\Sigma}_X^k \hat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} \right| \\
&= \frac{1}{\lambda_1^{k+1}} \left| \hat{\sigma}_{Xy}^{S'} \left(\bar{\Sigma}_X + \hat{\Sigma}_X^S - \bar{\Sigma}_X \right)^k \hat{\sigma}_{Xy}^S - \hat{\sigma}_{Xy}^{S'} \bar{\Sigma}_X^k \hat{\sigma}_{Xy}^S \right| \\
&\leq \frac{1}{\lambda_1^{k+1}} \sum_{j=1}^k \binom{k}{j} \left\| \hat{\Sigma}_X^S - \bar{\Sigma}_X \right\|_2^j \left\| \bar{\Sigma}_X \right\|_2^{k-j} \left\| \hat{\sigma}_{Xy}^S \right\|_2^2 \\
&= \sum_{j=1}^k \binom{k}{j} \left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2^j \frac{\left\| \bar{\Sigma}_X \right\|_2^{k-j}}{\lambda_1^{k-j}} \left\| \frac{\hat{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2^2 \\
&= \sum_{j=1}^k \binom{k}{j} \left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2^j \left\| \frac{\hat{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2^2 \\
&\quad \left(\text{since } \left\| \bar{\Sigma}_X \right\|_2^{k-j} / \lambda_1^{k-j} = 1 \text{ given that } \left\| \bar{\Sigma}_X \right\|_2 = \sqrt{\lambda_{\max} \left(\bar{\Sigma}_X^2 \right)} = \sqrt{\lambda_1^2} = \lambda_1 \right) \\
&\leq \sum_{j=1}^k \binom{k}{j} \left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2^j \left(\left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 + \left\| \frac{\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \right)^2
\end{aligned}$$

so that

$$\begin{aligned}
& \left| \frac{\hat{\sigma}_{Xy}^{S'} \left(\bar{\Sigma}_X + \hat{\Sigma}_X^S - \bar{\Sigma}_X \right)^k \hat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} - \frac{\hat{\sigma}_{Xy}^{S'} \bar{\Sigma}_X^k \hat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} \right| \\
&\leq \sum_{j=1}^k \binom{k}{j} \left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2^j \left(\left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 + \left\| \frac{\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \right)^2 \\
&= O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) O_p(1) + \dots + O_p \left(\frac{c_0^k(N)}{\lambda_1^k} \left[\frac{\ln N}{T} \right]^{\frac{k}{2}(1-q)} \right) O_p(1) \quad (\text{given Lemma C-6,} \\
&\quad \text{Lemma C-9, and part (a) of this lemma and given that } k \text{ is a finite positive integer}) \\
&= O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \quad (\text{by Assumption 6-3}).
\end{aligned}$$

Now, for the second term in expression (63), note that

$$\begin{aligned}
& \left| \frac{\widehat{\sigma}_{Xy}^{S'} \overline{\Sigma}_X^k \widehat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} - \frac{\overline{\sigma}'_{Xy} \overline{\Sigma}_X^k \overline{\sigma}_{Xy}}{\lambda_1^{k+1}} \right| \\
&= \left| \frac{(\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy} + \overline{\sigma}_{Xy})' \overline{\Sigma}_X^k (\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy} + \overline{\sigma}_{Xy})}{\lambda_1^{k+1}} - \frac{\overline{\sigma}'_{Xy} \overline{\Sigma}_X^k \overline{\sigma}_{Xy}}{\lambda_1^{k+1}} \right| \\
&= \left| \frac{(\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})' \overline{\Sigma}_X^k (\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})}{\lambda_1^{k+1}} + \frac{2\overline{\sigma}'_{Xy} \overline{\Sigma}_X^k (\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})}{\lambda_1^{k+1}} \right| \\
&\leq \left| \frac{(\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})' \overline{\Sigma}_X^k (\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})}{\lambda_1^{k+1}} \right| + 2 \left| \frac{\overline{\sigma}'_{Xy} \overline{\Sigma}_X^k (\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})}{\lambda_1^{k+1}} \right| \\
&\leq \left| \frac{(\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})' \overline{\Sigma}_X^k (\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})}{\lambda_1^{k+1}} \right| + 2 \sqrt{\frac{\overline{\sigma}'_{Xy} \overline{\Sigma}_X^{2k} \overline{\sigma}_{Xy}}{\lambda_1^{2k+1}}} \sqrt{\frac{(\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})' (\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})}{\lambda_1}} \\
&\leq \frac{\lambda_1^k}{\lambda_1^k} \left| \frac{(\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})' (\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})}{\lambda_1} \right| + 2 \frac{\lambda_1^k}{\lambda_1^k} \sqrt{\frac{\overline{\sigma}'_{Xy} \overline{\sigma}_{Xy}}{\lambda_1}} \sqrt{\frac{(\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})' (\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy})}{\lambda_1}} \\
&= \left\| \frac{\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 + 2 \sqrt{\frac{\overline{\sigma}'_{Xy} \overline{\sigma}_{Xy}}{\lambda_1}} \left\| \frac{\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\
&= O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \right) + O(1) O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right) \\
&\quad (\text{by Lemma C-9 and part (a) of this lemma}) \\
&= O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right)
\end{aligned}$$

Putting things together, we see that

$$\begin{aligned}
& \left| \frac{\widehat{\sigma}_{Xy}^{S'} \left[\widehat{\Sigma}_X^S \right]^k \widehat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} - \frac{\overline{\sigma}_{Xy}' \overline{\Sigma}_X^k \overline{\sigma}_{Xy}}{\lambda_1^{k+1}} \right| \\
\leq & \left| \frac{\widehat{\sigma}_{Xy}^{S'} \left[\overline{\Sigma}_X + \widehat{\Sigma}_X^S - \overline{\Sigma}_X \right]^k \widehat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} - \frac{\widehat{\sigma}_{Xy}^{S'} \overline{\Sigma}_X^k \widehat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} \right| \\
& + \left| \frac{\widehat{\sigma}_{Xy}^{S'} \overline{\Sigma}_X^k \widehat{\sigma}_{Xy}^S}{\lambda_1^{k+1}} - \frac{\overline{\sigma}_{Xy}' \overline{\Sigma}_X^k \overline{\sigma}_{Xy}}{\lambda_1^{k+1}} \right| \\
= & O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) + O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right) \\
= & O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).
\end{aligned}$$

Finally, to show part (f), note that, for finite integer $k \geq 1$,

$$\begin{aligned}
& \left\| \frac{[\hat{\Sigma}_X^S]^k \hat{\sigma}_{Xy}^S - \bar{\Sigma}_X^k \bar{\sigma}_{Xy}}{\lambda_1^{k+\frac{1}{2}}} \right\|_2 \\
&= \left\| \frac{[\hat{\Sigma}_X^S - \bar{\Sigma}_X + \bar{\Sigma}_X]^k \hat{\sigma}_{Xy}^S - \bar{\Sigma}_X^k \hat{\sigma}_{Xy}^S + \bar{\Sigma}_X^k \hat{\sigma}_{Xy}^S - \bar{\Sigma}_X^k \bar{\sigma}_{Xy}}{\lambda_1^{k+\frac{1}{2}}} \right\|_2 \\
&= \left\| \frac{[\hat{\Sigma}_X^S - \bar{\Sigma}_X + \bar{\Sigma}_X]^k \hat{\sigma}_{Xy}^S - \bar{\Sigma}_X^k \hat{\sigma}_{Xy}^S}{\lambda_1^{k+\frac{1}{2}}} + \frac{\bar{\Sigma}_X^k (\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy})}{\lambda_1^{k+\frac{1}{2}}} \right\|_2 \\
&\leq \left\| \frac{[\hat{\Sigma}_X^S - \bar{\Sigma}_X + \bar{\Sigma}_X]^k \hat{\sigma}_{Xy}^S - \bar{\Sigma}_X^k \hat{\sigma}_{Xy}^S}{\lambda_1^{k+\frac{1}{2}}} \right\|_2 + \left\| \frac{\bar{\Sigma}_X^k [\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}]}{\lambda_1^{k+\frac{1}{2}}} \right\|_2 \\
&\leq \sum_{j=1}^k \binom{k}{j} \left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2^j \left\| \frac{\bar{\Sigma}_X}{\lambda_1} \right\|_2^{k-j} \left\| \frac{\hat{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2 + \left\| \frac{\bar{\Sigma}_X}{\lambda_1} \right\|_2^k \left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\
&= \sum_{j=1}^k \binom{k}{j} \left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2^j \left\| \frac{\hat{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2 + \left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\
&\quad \left(\text{since } \left\| \frac{\bar{\Sigma}_X}{\lambda_1} \right\|_2^{k-j} = \frac{\|\bar{\Sigma}_X\|_2^{k-j}}{\lambda_1^{k-j}} = \left(\frac{\lambda_1}{\lambda_1} \right)^{k-j} = 1 \text{ and, similarly, } \left\| \frac{\bar{\Sigma}_X}{\lambda_1} \right\|_2^k = 1 \right) \\
&\leq \sum_{j=1}^k \binom{k}{j} \left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2^j \left(\left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 + \left\| \frac{\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \right) + \left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\
&= O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) O_p(1) + \dots + O_p \left(\frac{c_0^k(N)}{\lambda_1^k} \left[\frac{\ln N}{T} \right]^{\frac{k}{2}(1-q)} \right) O_p(1) \\
&\quad + O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right) \quad (\text{by Lemmas C6, C9, and part(a) of this lemma}) \\
&= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \quad (\text{by Assumption 6-3}). \quad \square
\end{aligned}$$

Lemma B-4: Let

$$D_{\lambda,\kappa} = \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{\kappa-\frac{1}{2}} \end{pmatrix} \text{ and } \widehat{D}_{\lambda,\kappa} = \begin{pmatrix} \widehat{\lambda}_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \widehat{\lambda}_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \widehat{\lambda}_1^{\kappa-\frac{1}{2}} \end{pmatrix}$$

Then, under Assumptions 4-1, 4-2, 4-3, 4-4, 6-1, 6-2, and 6-3; the following statements are true.

(a) There exists some positive constant $\overline{C}$ such that

$$\lambda_{\max} \left(D_{\lambda,\kappa}^{-1} G' G D_{\lambda,\kappa}^{-1} \right) = \|G D_{\lambda,\kappa}^{-1}\|_2^2 \leq \|G D_{\lambda,\kappa}^{-1}\|_F^2 \leq \overline{C} < \infty$$

and

$$\left\| \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \leq \overline{C} < \infty$$

for all N and T sufficiently large.

(b) There exists some positive constant $\overline{C}$ such that

$$\left\| \frac{D_{\lambda,\kappa}^{-1} G' \overline{\sigma}_{Xy}}{\lambda_1^{\frac{1}{2}}} \right\|_2 \leq \overline{C} < \infty$$

for all N and T sufficiently large.

(c) For $0 \leq q < 1$,

$$\left\| \left(\widehat{G}_S - G \right) D_{\lambda,\kappa}^{-1} \right\|_2 = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)$$

and

$$\left\| \left(\widehat{G}_S - G \right) \widehat{D}_{\lambda,\kappa}^{-1} \right\|_2 = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

(d) For $0 \leq q < 1$,

$$\left\| \frac{D_{\lambda,\kappa}^{-1} \left(\widehat{G}_S' \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy} \right)}{\sqrt{\lambda_1}} \right\|_2 = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

and

$$\left\| \frac{\widehat{D}_{\lambda,\kappa}^{-1} \left(\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy} \right)}{\sqrt{\lambda_1}} \right\|_2 = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

(e)

$$\begin{aligned} & \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \\ &= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right). \end{aligned}$$

and

$$\begin{aligned} & \left\| \frac{\widehat{D}_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \\ &= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \end{aligned}$$

(f)

$$\begin{aligned} & \left\| \left(\frac{\widehat{D}_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \\ &= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right). \end{aligned}$$

(g) For any fixed real number $v > 0$,

$$\begin{aligned} & \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \widehat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \\ &= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{v+1}} \right\} \right). \end{aligned}$$

(h) For any fixed real number $v > 0$,

$$\begin{aligned}
& \left\| \left(\frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \widehat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} - \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \\
&= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{v+1}} \right\} \right).
\end{aligned}$$

Proof of Lemma B-4:

To show the first result of part (a), note that

$$\begin{aligned}
& \lambda_{\max} (D_{\lambda,\kappa}^{-1} G' G D_{\lambda,\kappa}^{-1}) \\
\leq & \|GD_{\lambda,\kappa}^{-1}\|_F^2 = \text{tr} \{D_{\lambda,\kappa}^{-1} G' G D_{\lambda,\kappa}^{-1}\} \\
= & \text{tr} \left\{ \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_{Xy} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \end{pmatrix} \right. \\
& \quad \times \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{pmatrix} \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \Bigg\} \\
= & \text{tr} \left\{ \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_{Xy} \bar{\sigma}_{Xy} & \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^\kappa \bar{\sigma}_{Xy} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^\kappa \bar{\sigma}_{Xy} & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{2(\kappa-1)} \bar{\sigma}_{Xy} \end{pmatrix} \right. \\
& \quad \times \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \Bigg\} \\
= & \text{tr} \left\{ \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \right. \\
& \quad \times \begin{pmatrix} \lambda_1^{-\frac{1}{2}} \bar{\sigma}'_{Xy} \bar{\sigma}_{Xy} & \lambda_1^{-\frac{3}{2}} \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \lambda_1^{-(\kappa-\frac{1}{2})} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \\ \lambda_1^{-\frac{1}{2}} \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} & \lambda_1^{-\frac{3}{2}} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} & \cdots & \lambda_1^{-(\kappa-\frac{1}{2})} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^\kappa \bar{\sigma}_{Xy} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{-\frac{1}{2}} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} & \lambda_1^{-\frac{3}{2}} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^\kappa \bar{\sigma}_{Xy} & \cdots & \lambda_1^{-(\kappa-\frac{1}{2})} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{2(\kappa-1)} \bar{\sigma}_{Xy} \end{pmatrix} \Bigg\}
\end{aligned}$$

$$\begin{aligned}
&= \text{tr} \left\{ \begin{pmatrix} \lambda_1^{-1} \bar{\sigma}'_{Xy} \bar{\sigma}_{Xy} & \lambda_1^{-2} \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \lambda_1^{-\kappa} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \\ \lambda_1^{-2} \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} & \lambda_1^{-3} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} & \cdots & \lambda_1^{-(\kappa+1)} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^\kappa \bar{\sigma}_{Xy} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{-\kappa} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} & \lambda_1^{-(\kappa+1)} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^\kappa \bar{\sigma}_{Xy} & \cdots & \lambda_1^{-(2\kappa-1)} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{2(\kappa-1)} \bar{\sigma}_{Xy} \end{pmatrix} \right\} \\
&= \sum_{j=1}^{\kappa} \frac{\bar{\sigma}'_{Xy} \bar{\Sigma}_X^{2(j-1)} \bar{\sigma}_{Xy}}{\lambda_1^{2j-1}} \\
&\leq \sum_{j=1}^{\kappa} \left(\frac{\lambda_1}{\lambda_1} \right)^{2(j-1)} \frac{\bar{\sigma}'_{Xy} \bar{\sigma}_{Xy}}{\lambda_1} \quad (\text{since } \lambda_1 = \lambda_{\max}(\bar{\Sigma}_X)) \\
&= \kappa \frac{\bar{\sigma}'_{Xy} \bar{\sigma}_{Xy}}{\lambda_1} \\
&\leq \kappa C \quad (\text{given that, by part (a) of Lemma B-3, there exists some constant } C > 0 \\
&\quad \text{such that } \frac{\bar{\sigma}'_{Xy} \bar{\sigma}_{Xy}}{\lambda_1} \leq C \text{ for all } N, T \text{ sufficiently large}) \\
&\leq \bar{C} < \infty \text{ for some constant } \bar{C} \geq \kappa C \text{ and for all } N \text{ and } T \text{ sufficiently large.}
\end{aligned}$$

Now, to show the second result of part (a), note that

$$\begin{aligned}
& \left\| \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \\
&= \sqrt{\lambda_{\max} \left[\left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^2 \right]} \quad \left(\text{given that } \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \text{ is symmetric} \right) \\
&= \sqrt{\left[\lambda_{\max} \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right]^2} \\
&= \lambda_{\max} \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \quad \left(\text{given that } \lambda_{\max} \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) > 0 \text{ since} \right. \\
&\quad \left. \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \text{ is positive definite as shown in the proof of part (f) below} \right) \\
&\leq \frac{\lambda_{\max}(\bar{\Sigma}_X)}{\lambda_1} \lambda_{\max}(D_{\lambda,\kappa}^{-1} G' G D_{\lambda,\kappa}^{-1}) \\
&\leq \lambda_{\max}(D_{\lambda,\kappa}^{-1} G' G D_{\lambda,\kappa}^{-1}) \quad \left(\frac{\lambda_{\max}(\bar{\Sigma}_X)}{\lambda_1} = 1 \text{ since } \lambda_1 = \lambda_{\max}(\bar{\Sigma}_X) \right) \\
&\leq \bar{C} < \infty \quad (\text{by the first result given above for part (a)})
\end{aligned}$$

To show part (b), write

$$\frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} = \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_{Xy} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \end{pmatrix} \frac{\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} = \begin{pmatrix} \bar{\sigma}'_{Xy} \bar{\sigma}_{Xy} / \lambda_1 \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} / \lambda_1^2 \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} / \lambda_1^\kappa \end{pmatrix} \quad (64)$$

so that

$$\begin{aligned}
& \left\| \frac{D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 \\
&= \sum_{g=1}^{\kappa} \left(\frac{\bar{\sigma}'_{Xy} \bar{\Sigma}_X^{g-1} \bar{\sigma}_{Xy}}{\lambda_1^g} \right)^2 \\
&= \sum_{g=1}^{\kappa} \left(\frac{\beta'_{Xy} \bar{\Sigma}_X^{g+1} \beta}{\lambda_1^g} \right)^2 \quad (\text{since } \bar{\sigma}_{Xy} = \bar{\Sigma}_X \beta) \\
&\leq \sum_{g=1}^{\kappa} \left(\frac{\lambda_1}{\lambda_1} \right)^{2(g+1)} (\beta' \beta)^2 \lambda_1^2 \quad (\text{since } \lambda_1 = \lambda_{\max}(\bar{\Sigma}_X)) \\
&= \sum_{g=1}^{\kappa} \left(\frac{\phi'}{\sqrt{\underline{\lambda}_{1,1}}} L'_1 L_1 \frac{\phi}{\sqrt{\underline{\lambda}_{1,1}}} \right)^2 \lambda_1^2 \\
&\quad \left(\text{since } \beta = L_1 \left(\frac{\phi}{\sqrt{\underline{\lambda}_{1,1}}} \right) \text{ by Assumption 4-4(a)} \right) \\
&= \left(\frac{\lambda_1}{\underline{\lambda}_{1,1}} \right)^2 \sum_{g=1}^{\kappa} \|\phi\|_2^4 \quad (\text{since } L'_1 L_1 = I_{\kappa}) \\
&\leq (c^*)^2 \sum_{g=1}^{\kappa} \|\phi\|_2^4 \quad (\text{since, by part (a) of Lemma D-3, there exists a constant} \\
&\quad c^* > 1 \text{ such that } 0 < \frac{1}{c^*} \leq \frac{\lambda_1}{\underline{\lambda}_{1,1}} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large}) \\
&\leq (c^*)^2 \kappa^3 \left(\max_{1 \leq g \leq \kappa} |\phi_g| \right)^4 \\
&\leq (c^*)^2 \kappa^3 \bar{c}^4 \quad (\text{given Assumption 4-4 (a)}) \\
&\leq \bar{C} < \infty \quad (\text{for some positive constant } \bar{C} \geq (c^*)^2 \kappa^3 \bar{c}^4)
\end{aligned}$$

To show the first result of part (c), first write

$$\begin{aligned}
& \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \\
&= \begin{pmatrix} \widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy} & \widehat{\Sigma}_X^S \widehat{\sigma}_{Xy}^S - \overline{\Sigma}_X \overline{\sigma}_{Xy} & \cdots & \left[\widehat{\Sigma}_X^S \right]^{\kappa-1} \widehat{\sigma}_{Xy}^S - \overline{\Sigma}_X^{\kappa-1} \overline{\sigma}_{Xy} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \\
&= \begin{pmatrix} [\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}] / \sqrt{\lambda_1} & \cdots & \left[\left[\widehat{\Sigma}_X^S \right]^{\kappa-1} \widehat{\sigma}_{Xy}^S - \overline{\Sigma}_X^{\kappa-1} \overline{\sigma}_{Xy} \right] / \lambda_1^{\kappa-\frac{1}{2}} \end{pmatrix}
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left\| \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \right\|_2^2 \\
&\leq \left\| \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \right\|_F^2 \\
&= tr \left\{ \begin{pmatrix} \lambda_1^{-\frac{1}{2}} [\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}]' \\ \vdots \\ \lambda_1^{-(\kappa-\frac{1}{2})} \left[\left[\widehat{\Sigma}_X^S \right]^{\kappa-1} \widehat{\sigma}_{Xy}^S - \overline{\Sigma}_X^{\kappa-1} \overline{\sigma}_{Xy} \right]' \end{pmatrix} \right. \\
&\quad \times \begin{pmatrix} [\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}] / \sqrt{\lambda_1} & \cdots & \left[\left[\widehat{\Sigma}_X^S \right]^{\kappa-1} \widehat{\sigma}_{Xy}^S - \overline{\Sigma}_X^{\kappa-1} \overline{\sigma}_{Xy} \right] / \lambda_1^{(\kappa-\frac{1}{2})} \end{pmatrix} \left. \right\} \\
&= \left\| \frac{\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 + \left\| \frac{\widehat{\Sigma}_X^S \widehat{\sigma}_{Xy}^S - \overline{\Sigma}_X \overline{\sigma}_{Xy}}{\lambda_1^{\frac{3}{2}}} \right\|_2^2 + \cdots + \left\| \frac{\left[\widehat{\Sigma}_X^S \right]^{\kappa-1} \widehat{\sigma}_{Xy}^S - \overline{\Sigma}_X^{\kappa-1} \overline{\sigma}_{Xy}}{\lambda_1^{\kappa-\frac{1}{2}}} \right\|_2^2 \\
&= O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \right) + O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{1-q}, \frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \right\} \right) \\
&\quad + \cdots + O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{1-q}, \frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \right\} \right) \\
&\quad \text{(given Lemma C-9 and part (f) of Lemma B-3)} \\
&= O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^2} \left(\frac{\ln N}{T} \right)^{1-q}, \frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \right\} \right)
\end{aligned}$$

so that

$$\begin{aligned} \left\| \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \right\|_2 &\leq \left\| \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \right\|_F \\ &= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right). \end{aligned}$$

Now, for the second result of part (c), write

$$\begin{aligned} \left(\widehat{G}_S - G \right) \widehat{D}_{\lambda, \kappa}^{-1} &= \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} D_{\lambda, \kappa} \widehat{D}_{\lambda, \kappa}^{-1} \\ &= \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} + \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \left(D_{\lambda, \kappa} \widehat{D}_{\lambda, \kappa}^{-1} - I_\kappa \right) \end{aligned}$$

It then follows from the first result of part (c) given above as well as the result given in part (d) of Lemma D-9 that

$$\begin{aligned} &\left\| \left(\widehat{G}_S - G \right) \widehat{D}_{\lambda, \kappa}^{-1} \right\|_2 \\ &\leq \left\| \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \right\|_2 + \left\| \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \left(D_{\lambda, \kappa} \widehat{D}_{\lambda, \kappa}^{-1} - I_\kappa \right) \right\|_2 \\ &\quad (\text{by the triangle inequality}) \\ &\leq \left\| \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \right\|_2 + \left\| \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \right\|_2 \left\| D_{\lambda, \kappa} \widehat{D}_{\lambda, \kappa}^{-1} - I_\kappa \right\|_2 \\ &\quad (\text{by the submultiplicative property of the spectral norm}) \\ &= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\ &\quad + O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right\} \right) \\ &= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right). \end{aligned}$$

To show the first result of part (d), note that, by making use of parts (d) and (e) of Lemma B-3, we have

$$\begin{aligned}
& \left\| \frac{D_{\lambda, \kappa}^{-1} \left(\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy} \right)}{\sqrt{\lambda_1}} \right\|_2^2 \\
&= \left\| \begin{pmatrix} \lambda_1^{-1} [\widehat{\sigma}_{Xy}^{S'} \widehat{\sigma}_{Xy}^S - \overline{\sigma}'_{Xy} \overline{\sigma}_{Xy}] \\ \lambda_1^{-2} [\widehat{\sigma}_{Xy}^{S'} \widehat{\Sigma}_X^S \widehat{\sigma}_{Xy}^S - \overline{\sigma}'_{Xy} \overline{\Sigma}_X \overline{\sigma}_{Xy}] \\ \vdots \\ \lambda_1^{-\kappa} [\widehat{\sigma}_{Xy}^{S'} [\widehat{\Sigma}_X^S]^{\kappa-1} \widehat{\sigma}_{Xy}^S - \overline{\sigma}'_{Xy} \overline{\Sigma}_X^{\kappa-1} \overline{\sigma}_{Xy}] \end{pmatrix} \right\|_2^2 \\
&= \left(\frac{\widehat{\sigma}_{Xy}^{S'} \widehat{\sigma}_{Xy}^S - \overline{\sigma}'_{Xy} \overline{\sigma}_{Xy}}{\lambda_1} \right)^2 + \left(\frac{\widehat{\sigma}_{Xy}^{S'} \widehat{\Sigma}_X^S \widehat{\sigma}_{Xy}^S - \overline{\sigma}'_{Xy} \overline{\Sigma}_X \overline{\sigma}_{Xy}}{\lambda_1^2} \right)^2 + \dots \\
&\quad \dots + \left(\frac{\widehat{\sigma}_{Xy}^{S'} [\widehat{\Sigma}_X^S]^{\kappa-1} \widehat{\sigma}_{Xy}^S - \overline{\sigma}'_{Xy} \overline{\Sigma}_X^{\kappa-1} \overline{\sigma}_{Xy}}{\lambda_1^\kappa} \right)^2 \\
&= O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \right) + O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^2} \left(\frac{\ln N}{T} \right)^{(1-q)}, \left(\frac{a(N)}{\lambda_1} \right) \left(\frac{\ln N}{T} \right)^{(1-\frac{q}{2})} \right\} \right) \\
&\quad + \dots + O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^2} \left(\frac{\ln N}{T} \right)^{(1-q)}, \left(\frac{a(N)}{\lambda_1} \right) \left(\frac{\ln N}{T} \right)^{(1-\frac{q}{2})} \right\} \right) \\
&= O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^2} \left(\frac{\ln N}{T} \right)^{(1-q)}, \left(\frac{a(N)}{\lambda_1} \right) \left(\frac{\ln N}{T} \right)^{(1-\frac{q}{2})} \right\} \right).
\end{aligned}$$

from which it follows that

$$\left\| \frac{D_{\lambda, \kappa}^{-1} \left(\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy} \right)}{\sqrt{\lambda_1}} \right\|_2 = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

Now, to show the second result of part (d), first write

$$\begin{aligned}
\frac{\widehat{D}_{\lambda, \kappa}^{-1} \left(\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy} \right)}{\sqrt{\lambda_1}} &= \frac{\widehat{D}_{\lambda, \kappa}^{-1} D_{\lambda, \kappa} D_{\lambda, \kappa}^{-1} \left(\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy} \right)}{\sqrt{\lambda_1}} \\
&= \frac{\left(\widehat{D}_{\lambda, \kappa}^{-1} D_{\lambda, \kappa} - I_\kappa \right) D_{\lambda, \kappa}^{-1} \left(\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy} \right)}{\sqrt{\lambda_1}} + \frac{D_{\lambda, \kappa}^{-1} \left(\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy} \right)}{\sqrt{\lambda_1}}
\end{aligned}$$

It then follows from the first result of part (d) given above as well as the result given in part (d) of Lemma D-9 that

$$\begin{aligned}
& \left\| \frac{\widehat{D}_{\lambda,\kappa}^{-1} \left(\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy} \right)}{\sqrt{\lambda_1}} \right\|_2 \\
& \leq \left\| \frac{\left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right) D_{\lambda,\kappa}^{-1} \left(\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy} \right)}{\sqrt{\lambda_1}} \right\|_2 + \left\| \frac{D_{\lambda,\kappa}^{-1} \left(\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy} \right)}{\sqrt{\lambda_1}} \right\|_2 \\
& \quad (\text{by triangle inequality}) \\
& \leq \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \left\| \frac{D_{\lambda,\kappa}^{-1} \left(\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy} \right)}{\sqrt{\lambda_1}} \right\|_2 + \left\| \frac{D_{\lambda,\kappa}^{-1} \left(\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy} \right)}{\sqrt{\lambda_1}} \right\|_2 \\
& \quad (\text{by the submultiplicative property of the spectral norm}) \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right\} \right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& \quad + O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)
\end{aligned}$$

Now, to show the first result of part (e), note first that

$$\begin{aligned}
& \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
= & \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1} - D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
= & \left\{ \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \widehat{\sigma}_{Xy}^{S'} \\ \widehat{\sigma}_{Xy}^{S'} \widehat{\Sigma}_X^S \\ \vdots \\ \widehat{\sigma}_{Xy}^{S'} [\widehat{\Sigma}_X^S]^{\kappa-1} \end{pmatrix} \frac{\widehat{\Sigma}_X^S}{\lambda_1} \right. \\
& \times \begin{pmatrix} \widehat{\sigma}_{Xy}^S & \widehat{\Sigma}_X^S \widehat{\sigma}_{Xy}^S & \cdots & [\widehat{\Sigma}_X^S]^{\kappa-1} \widehat{\sigma}_{Xy}^S \end{pmatrix} \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \Bigg\} \\
& - \left\{ \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \overline{\sigma}_{Xy}' \\ \overline{\sigma}_{Xy}' \overline{\Sigma}_X \\ \vdots \\ \overline{\sigma}_{Xy}' \overline{\Sigma}_X^{\kappa-1} \end{pmatrix} \frac{\overline{\Sigma}_X}{\lambda_1} \right. \\
& \times \begin{pmatrix} \overline{\sigma}_{Xy} & \overline{\Sigma}_X \overline{\sigma}_{Xy} & \cdots & \overline{\Sigma}_X^{\kappa-1} \overline{\sigma}_{Xy} \end{pmatrix} \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \Bigg\} \\
= & \begin{pmatrix} \lambda_1^{-2} \left\{ \widehat{\sigma}_{Xy}^{S'} \widehat{\Sigma}_X^S \widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}' \overline{\Sigma}_X \overline{\sigma}_{Xy} \right\} & \cdots & \lambda_1^{-(\kappa+1)} \left\{ \widehat{\sigma}_{Xy}^{S'} [\widehat{\Sigma}_X^S]^\kappa \widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}' \overline{\Sigma}_X^\kappa \overline{\sigma}_{Xy} \right\} \\ \vdots & \ddots & \vdots \\ \lambda_1^{-(\kappa+1)} \left\{ \widehat{\sigma}_{Xy}^{S'} [\widehat{\Sigma}_X^S]^\kappa \widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}' \overline{\Sigma}_X^\kappa \overline{\sigma}_{Xy} \right\} & \cdots & \lambda_1^{-2\kappa} \left\{ \widehat{\sigma}_{Xy}^{S'} [\widehat{\Sigma}_X^S]^{2\kappa-1} \widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}' \overline{\Sigma}_X^{2\kappa-1} \overline{\sigma}_{Xy} \right\} \end{pmatrix},
\end{aligned}$$

so that

$$\begin{aligned}
& \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1} - D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2^2 \\
& \leq \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\underline{\lambda},\kappa}^{-1} - D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\underline{\lambda},\kappa}^{-1}}{\lambda_1} \right\|_F^2 \\
& = \sum_{j=0}^{\kappa-1} \sum_{k=1}^{\kappa} \left(\frac{\widehat{\sigma}_{Xy}^{S'} [\widehat{\Sigma}_X^S]^{j+k} \widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}^{S'} \overline{\Sigma}_X^{j+k} \overline{\sigma}_{Xy}}{\lambda_1^{j+k+1}} \right)^2 \\
& = O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^2} \left(\frac{\ln N}{T} \right)^{(1-q)}, \left(\frac{a(N)}{\lambda_1} \right) \left(\frac{\ln N}{T} \right)^{(1-\frac{q}{2})} \right\} \right) \\
& \quad \text{(applying part (e) of Lemma B-3 and noting that } \kappa \text{ is a fixed positive integer)}
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1} - D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \\
& \leq \left\| \frac{D_{\underline{\lambda},\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1} - D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_F \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right). \tag{65}
\end{aligned}$$

To show the second result of part (e), note that

$$\begin{aligned}
& \frac{\widehat{D}_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
= & \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} \left[\frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right] D_{\lambda,\kappa} \widehat{D}_{\lambda,\kappa}^{-1} \\
& + \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} D_{\lambda,\kappa} \widehat{D}_{\lambda,\kappa}^{-1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
= & \left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa + I_\kappa \right) \left[\frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right] \left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa + I_\kappa \right) \\
& + \left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa + I_\kappa \right) \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa + I_\kappa \right) - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
= & \left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right) \left[\frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right] \left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right) \\
& + \left[\frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right] \left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right) \\
& + \left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right) \left[\frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right] \\
& + \left[\frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right] \\
& + \left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right) \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right) + \left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right) \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
& + \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right)
\end{aligned}$$

It follows by applying the submultiplicative property of spectral norm and the triangle inequality as well as part (a) of this lemma, the first result given in part (e) of this lemma,

and the result given in part (d) of Lemma D-9 that

$$\begin{aligned}
& \left\| \frac{\widehat{D}_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S \widehat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \\
\leq & \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \\
& + \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \\
& + \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \\
& + \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 + \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \left\| \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \\
& + \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \left\| \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \\
& + \left\| \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 \\
= & O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right\} \right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& \times O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right\} \right) \\
& + O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right\} \right) \\
& + O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right\} \right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& + O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& + O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right\} \right) O(1) \\
& + O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right\} \right) O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right\} \right)
\end{aligned}$$

$$\begin{aligned}
& +O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right\} \right) \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).
\end{aligned}$$

Next, to show part (f), note first that $D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1} / \lambda_1$ is clearly a symmetric matrix. Moreover, the result of part (c) of Lemma B-3 further shows that there exists a positive constant C_* such that

$$\lambda_{\min} \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \geq C_* > 0$$

for all N and T sufficiently large, so that $D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1} / \lambda_1$ is also positive definite and, thus, non-singular for all N and T sufficiently large. Making use of these properties of the

matrix $D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_XGD_{\lambda,\kappa}^{-1}/\lambda_1$, we see that for N and T sufficiently large

$$\begin{aligned}
& \left\| \left(\frac{D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_XGD_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \\
&= \sqrt{\lambda_{\max} \left\{ \left[\left(\frac{D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_XGD_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right]' \left(\frac{D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_XGD_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\}} \\
&= \sqrt{\lambda_{\max} \left\{ \left[\left(\frac{D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_XGD_{\lambda,\kappa}^{-1}}{\lambda_1} \right)' \right]^{-1} \left(\frac{D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_XGD_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\}} \\
&\quad (\text{since we can interchange the transposition and the inverse operations}) \\
&= \sqrt{\lambda_{\max} \left\{ \left(\frac{D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_XGD_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \left(\frac{D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_XGD_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\}} \\
&\quad \left(\text{since } \frac{D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_XGD_{\lambda,\kappa}^{-1}}{\lambda_1} \text{ is a symmetric matrix} \right) \\
&= \sqrt{\left[\lambda_{\max} \left\{ \left(\frac{D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_XGD_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \right]^2} \\
&= \sqrt{\left(\frac{1}{\lambda_{\min} (D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_XGD_{\lambda,\kappa}^{-1}/\lambda_1)} \right)^2} \\
&= \frac{1}{\lambda_{\min} (D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_XGD_{\lambda,\kappa}^{-1}/\lambda_1)}. \\
&\leq \frac{1}{C_*} \quad (\text{given that } \lambda_{\min} (D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_XGD_{\lambda,\kappa}^{-1}/\lambda_1) \geq C_*) \\
&< \infty \quad (\text{since } C_* > 0)
\end{aligned} \tag{66}$$

Next, define

$$E^\# = \frac{\hat{D}_{\lambda,\kappa}^{-1}\hat{G}'_S\hat{\Sigma}_X^S\hat{G}_S\hat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\underline{\lambda},\kappa}^{-1}G'\bar{\Sigma}_XGD_{\underline{\lambda},\kappa}^{-1}}{\lambda_1}$$

and note that by applying the result given in expression (66) above as well as the second result given in part (e) of this lemma, we see that

$$\begin{aligned}
& \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \|E^\#\|_2 \\
& \leq \frac{1}{C_*} \|E^\#\|_2 \\
& = O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& = o_p(1) \text{ (by Assumption 6-3)}. \tag{67}
\end{aligned}$$

It follows from the inequality given in part (b) of Lemma B-1 and the results given in expressions (66) and (67) above that

$$\begin{aligned}
& \left\| \left(\frac{\hat{D}_{\lambda,\kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S \hat{D}_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^+ - \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \\
& \leq \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \\
& \quad \times \left(1 - \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \|E^\#\|_2 \right)^{-1} \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \|E^\#\|_2 \\
& = O(1) \left[1 - O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \right]^{-1} \\
& \quad \times O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).
\end{aligned}$$

Turning our attention now to part (g), note first that

$$\begin{aligned}
& \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \widehat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
&= \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1} - D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \widehat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
&= \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1} - D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{\widehat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-2}}{\lambda_1} \\
&\quad \left(\text{since } \widehat{D}_{\lambda,\kappa} \text{ and } D_{\lambda,\kappa} \text{ are diagonal matrices} \right).
\end{aligned}$$

Now, by the result given above in expression (65) in the proof of part (e) of this lemma, we have

$$\begin{aligned}
& \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1} - D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \\
&\leq \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1} - D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_F \\
&= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).
\end{aligned}$$

Moreover, by applying the result given in part (c) of Lemma D-9, we have, for any fixed real number $v > 0$

$$\begin{aligned}
\frac{1}{\lambda_1} \left\| \frac{1}{\widehat{\lambda}_1^v} \widehat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-2} \right\|_2 &= O \left(\frac{1}{\lambda_1} \right) O_p \left(\frac{1}{\lambda_1^v} \right) \\
&= O_p \left(\frac{1}{\lambda_1^{v+1}} \right)
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \widehat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \\
& \leq \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 + \left\| \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \widehat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \\
& \leq \left\| \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1} - D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_F + \frac{1}{\lambda_1} \left\| \frac{1}{\widehat{\lambda}_1^v} \widehat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-2} \right\|_2 \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}} \right\} \right) + O_p \left(\frac{1}{\lambda_1^{v+1}} \right) \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}}, \frac{1}{\lambda_1^{v+1}} \right\} \right).
\end{aligned}$$

Finally, to show part (h), first define

$$E^* = \frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \widehat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1}$$

Next, recall that from expression (66) given in the proof of part (f) of this lemma, we have that there exists a constant $C_* > 0$ such that

$$\left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \leq \frac{1}{C_*} < \infty \text{ for all } N \text{ and } T \text{ sufficiently large.}$$

Making use of this result and following arguments similar to that given in the proof of part (g) above, we see that

$$\begin{aligned}
& \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} E^* \right\|_2 \\
& \leq \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \|E^*\|_2 \\
& = \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \left\| \frac{D_{\lambda,\kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \hat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \\
& \leq \frac{1}{C_*} \left[\left\| \frac{D_{\lambda,\kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 + \left\| \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \hat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 \right] \\
& = \frac{1}{C_*} \left[\left\| \frac{D_{\lambda,\kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right\|_2 + \frac{1}{\lambda_1} \left\| \frac{1}{\widehat{\lambda}_1^v} \hat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-2} \right\|_2 \right] \\
& = O(1) \left[O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) + O \left(\frac{1}{\lambda_1} \right) O_p \left(\frac{1}{\lambda_1^v} \right) \right] \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{v+1}} \right\} \right) \\
& = o_p(1) \text{ for } q \in [0, 1) \text{ and for any fixed real number } v > 0. \tag{68}
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} E^* \right\|_2 \\
& = \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \left(\frac{D_{\lambda,\kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \hat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right\|_2 \\
& < 1 \text{ w.p.a.1 as } N, T \rightarrow \infty.
\end{aligned}$$

so that part (d) of Lemma B-1 implies that

$$\frac{D_{\lambda,\kappa}^{-1} \hat{G}'_S \hat{\Sigma}_X^S \hat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \hat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1}$$

is nonsingular w.p.a.1 as $N, T \rightarrow \infty$. Now, applying the last inequality given in part (d) of Lemma B-1 as well as the results given in expressions (66) and (68) above, we further obtain

$$\begin{aligned}
& \left\| \left(\frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \widehat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} - \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \\
& \leq \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \frac{\left\| (D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1} / \lambda_1)^{-1} E^* \right\|_2}{1 - \left\| (D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1} / \lambda_1)^{-1} E^* \right\|_2} \\
& \leq O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{v+1}} \right\} \right) \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{v+1}} \right\} \right) \cdot \square
\end{aligned}$$

Lemma B-5: Under Assumptions 4-1, 4-2, and 4-4; the following statements hold.

(a)

$$\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G'_W [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \beta}{\sqrt{\lambda_1}} = \underline{V}' \underline{w}_\phi,$$

where

$$\begin{aligned}
D_{\lambda,\kappa} &= \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{\kappa-\frac{1}{2}} \end{pmatrix}, \underline{w}_\phi = \begin{pmatrix} \underline{\Delta}_{1,1}^2 \phi_1^2 / \lambda_1^2 \\ \underline{\Delta}_{1,2}^2 \phi_2^2 / \lambda_1^2 \\ \vdots \\ \underline{\Delta}_{1,\kappa}^2 \phi_\kappa^2 / \lambda_1^2 \end{pmatrix}, \\
\underline{V} &= \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \underline{\Delta}_{1,2} / \lambda_1 & \underline{\Delta}_{1,2}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^{\kappa-1} / \lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\Delta}_{1,\kappa} / \lambda_1 & \underline{\Delta}_{1,\kappa}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,\kappa}^{\kappa-1} / \lambda_1^{\kappa-1} \end{pmatrix} \\
G_W &= \left([L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \beta \quad [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^2 \beta \quad \cdots \quad [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^\kappa \beta \right)
\end{aligned}$$

(b)

$$\left\| \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\phi \right\|_2 = O \left(\frac{1}{\lambda_1} \right).$$

where

$$G = \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{pmatrix}$$

(c)

$$\left\| \sqrt{\frac{\underline{\Delta}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - L_1 D_{\phi,\lambda}^* \underline{V} \right\|_2 = O\left(\frac{1}{\lambda_1}\right).$$

where

$$D_{\phi,\lambda}^* = \begin{pmatrix} \underline{\Delta}_{1,1}\phi_1/\lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2}\phi_2/\lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,\kappa}\phi_\kappa/\lambda_1 \end{pmatrix}.$$

(d)

$$\left\| \left(\left[\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} - (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\|_2 = O\left(\frac{1}{\lambda_1}\right),$$

where $\underline{V}$ is the Vandermonde matrix defined in part (a) above and where

$$D_{\phi,\lambda} = \begin{pmatrix} \underline{\Delta}_{1,1}^3 \phi_1^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2}^3 \phi_2^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,\kappa}^3 \phi_\kappa^2 / \lambda_1^3 \end{pmatrix}.$$

Proof of Lemma B-5:

To show part (a), note first that

$$\begin{aligned} & \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G'_W [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \beta}{\lambda_1^{\frac{1}{2}}} \\ &= \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^2 \\ \vdots \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^\kappa \end{pmatrix} \frac{[L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \beta}{\lambda_1^{\frac{1}{2}}} \\ &= \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \begin{pmatrix} \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^2 \beta / \lambda_1 \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^3 \beta / \lambda_1^2 \\ \vdots \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+1} \beta / \lambda_1^\kappa \end{pmatrix} \end{aligned} \tag{69}$$

Next, note that, for finite positive integer k , we have, as shown in the proof of part (c) of Lemma B-3,

$$\left(\frac{\lambda_{1,1}}{\lambda_1}\right) \frac{\beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^k \beta}{\lambda_1^{k-1}} = \frac{1}{\lambda_1^k} \sum_{g=1}^{\kappa} \lambda_{1,g}^k \phi_g^2$$

from which we further obtain

$$\left(\frac{\lambda_{1,1}}{\lambda_1}\right) \begin{pmatrix} \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^2 \beta / \lambda_1 \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^3 \beta / \lambda_1^2 \\ \vdots \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+1} \beta / \lambda_1^{\kappa} \end{pmatrix} = \begin{pmatrix} \sum_{g=1}^{\kappa} \lambda_{1,g}^2 \phi_g^2 / \lambda_1^2 \\ \sum_{g=1}^{\kappa} \lambda_{1,g}^3 \phi_g^2 / \lambda_1^3 \\ \vdots \\ \sum_{g=1}^{\kappa} \lambda_{1,g}^{\kappa+1} \phi_g^2 / \lambda_1^{\kappa+1} \end{pmatrix} \quad (70)$$

Making use of expression (70), we see that, by straightforward calculation,

$$\begin{aligned} & \left(\frac{\lambda_{1,1}}{\lambda_1}\right) \frac{D_{\lambda,\kappa}^{-1} G'_W [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \beta}{\lambda_1^{\frac{1}{2}}} \\ &= \begin{pmatrix} \sum_{g=1}^{\kappa} \lambda_{1,g}^2 \phi_g^2 / \lambda_1^2 \\ \sum_{g=1}^{\kappa} \lambda_{1,g}^3 \phi_g^2 / \lambda_1^3 \\ \vdots \\ \sum_{g=1}^{\kappa} \lambda_{1,g}^{\kappa+1} \phi_g^2 / \lambda_1^{\kappa+1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \begin{pmatrix} \sum_{g=1}^{\kappa} \lambda_{1,g}^2 \phi_g^2 / \lambda_1^2 \\ \sum_{g=1}^{\kappa} \lambda_{1,g}^3 \phi_g^2 / \lambda_1^3 \\ \vdots \\ \sum_{g=1}^{\kappa} \lambda_{1,g}^{\kappa+1} \phi_g^2 / \lambda_1^{\kappa+1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,\kappa} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1,1}^{\kappa-1} & \lambda_{1,2}^{\kappa-1} & \cdots & \lambda_{1,\kappa}^{\kappa-1} \end{pmatrix} \begin{pmatrix} \lambda_{1,1}^2 \phi_1^2 / \lambda_1^2 \\ \lambda_{1,2}^2 \phi_2^2 / \lambda_1^2 \\ \vdots \\ \lambda_{1,\kappa}^2 \phi_{\kappa}^2 / \lambda_1^2 \end{pmatrix} \\ &= \underline{V}' \underline{w}_{\phi}, \end{aligned}$$

where $\underline{w}_{\phi} = (\lambda_{1,1}^2 \phi_1^2 / \lambda_1^2, \lambda_{1,2}^2 \phi_2^2 / \lambda_1^2, \dots, \lambda_{1,\kappa}^2 \phi_{\kappa}^2 / \lambda_1^2)'$.

To show part (b), note first that, by making use of expression (64) given in the proof of Lemma B-4 part (b) and expression (69) in part (a) above, we have

$$\begin{aligned}
& \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\phi \\
= & \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\lambda_1^{\frac{1}{2}}} - \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G'_W [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \beta}{\lambda_1^{\frac{1}{2}}} \\
& \text{(since } (\lambda_{1,1}/\lambda_1) D_{\lambda,\kappa}^{-1} G'_W [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \beta = \underline{V}' \underline{w}_\phi \text{ by the result given in part (a))} \\
= & \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \begin{pmatrix} \bar{\sigma}'_{Xy} \bar{\sigma}_{Xy} / \lambda_1 \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} / \lambda_1^2 \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} / \lambda_1^\kappa \end{pmatrix} - \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \begin{pmatrix} \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^2 \beta / \lambda_1 \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^3 \beta / \lambda_1^2 \\ \vdots \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+1} \beta / \lambda_1^\kappa \end{pmatrix} \\
= & \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \begin{pmatrix} \beta' \bar{\Sigma}_X^2 \beta / \lambda_1 \\ \beta' \bar{\Sigma}_X^3 \beta / \lambda_1^2 \\ \vdots \\ \beta' \bar{\Sigma}_X^{\kappa+1} \beta / \lambda_1^\kappa \end{pmatrix} - \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \begin{pmatrix} \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^2 \beta / \lambda_1 \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^3 \beta / \lambda_1^2 \\ \vdots \\ \beta' [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^{\kappa+1} \beta / \lambda_1^\kappa \end{pmatrix} \quad (\text{since } \bar{\sigma}_{Xy} = \bar{\Sigma}_X \beta) \\
= & \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \begin{pmatrix} \beta' \left[\bar{\Sigma}_X^2 - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)^2 \right] \beta / \lambda_1 \\ \beta' \left[\bar{\Sigma}_X^3 - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)^3 \right] \beta / \lambda_1^2 \\ \vdots \\ \beta' \left[\bar{\Sigma}_X^{\kappa+1} - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)^{\kappa+1} \right] \beta / \lambda_1^\kappa \end{pmatrix}
\end{aligned}$$

Moreover, let $\bar{C}_2 \geq \max\{1, C_2\}$ where C_2 is the positive constant given in part (b) of Assumption 4-2. Then, by a similar proof as that given for part (b) of Lemma B-3, we can

show that, for any fixed integer $j \geq 2$

$$\begin{aligned}
& \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left| \frac{\beta' \left(\bar{\Sigma}_X^j - [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^j \right) \beta}{\lambda_1^{j-1}} \right| \\
&= \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left| \frac{\beta' \left([L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2 + \bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)]^j - [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^j \right) \beta}{\lambda_1^{j-1}} \right| \\
&\leq \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{1}{\lambda_1^{j-1}} \sum_{\ell=1}^j \binom{j}{\ell} \left\| \bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2) \right\|_2^\ell \|L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2\|_2^{j-\ell} \|\beta\|_2^2 \\
&= \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \lambda_1 \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)}{\lambda_1} \right\|_2^\ell \frac{\|L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2\|_2^{j-\ell}}{\lambda_1^{j-\ell}} \left\| L_1 \frac{\phi}{\sqrt{\lambda_{1,1}}} \right\|_2^2 \\
&\leq \frac{\underline{\lambda}_{1,1}}{\underline{\lambda}_{1,1}} \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \frac{\|L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2\|_2^{j-\ell}}{\lambda_1^{j-\ell}} \|L_1 \phi\|_2^2 \quad (\text{since } \bar{\Sigma}_u = \bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)) \\
&= \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \frac{\|L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2\|_2^{j-\ell}}{\lambda_1^{j-\ell}} \|L_1 \phi\|_2^2 \\
&\leq [c^* \max \{1, C_2\}]^j \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \|L_1 \phi\|_2^2
\end{aligned}$$

(since Assumption 4-2(b) and Lemma D-3(a) imply that there exist constants $c^* > 1$ and $C_2 > 0$ su

$\|L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2\|_2 = \lambda_{\max} \{\bar{\Sigma}_W\} = \max \{\underline{\lambda}_{1,1}, \underline{\lambda}_{2,1}\} \leq c^* \max \{1, C_2\} \lambda_1$ for all N, T sufficiently

$$\begin{aligned}
&\leq \bar{C}_2^j \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \|L_1 \phi\|_2^2 \quad (\text{for some constant } \bar{C}_2 \geq c^* \max\{1, C_2\}) \\
&\leq \bar{C}_2^j \kappa \bar{c}^2 \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \quad (\text{since, by Assumption 4-4 (a), there exists a constant } \bar{c} \\
&\quad \text{such that } \bar{c} \geq \max_{1 \leq g \leq \kappa} |\phi_g| \text{ so that } \kappa \bar{c}^2 \geq \kappa \left(\max_{1 \leq g \leq \kappa} |\phi_g| \right)^2 \geq \|\phi\|_2^2 = \phi' L_1' L_1 \phi = \|L_1 \phi\|_2^2) \\
&= \bar{C}_2^j \kappa \bar{c}^2 \sum_{\ell=1}^j \binom{j}{\ell} \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right)^\ell \quad (\text{by symmetry and positive semidefiniteness of } \bar{\Sigma}_u, \\
&\quad \text{we have that } \|\bar{\Sigma}_u\|_2 = \sqrt{\lambda_{\max}(\bar{\Sigma}_u' \bar{\Sigma}_u)} = \sqrt{[\lambda_{\max}(\bar{\Sigma}_u)]^2} = \lambda_{\max}(\bar{\Sigma}_u)) \\
&\leq \bar{C}_2^j \kappa \bar{c}^2 \sum_{\ell=1}^j \binom{j}{\ell} \left(\frac{\bar{C}}{\lambda_1} \right)^\ell \quad (\text{since Assumption 4-1 and Lemma D-2 together imply that} \\
&\quad \text{there exists a constant } \bar{C} > 0 \text{ such that } \lambda_{\max}(\bar{\Sigma}_u) \leq \bar{C} < \infty) \\
&= O\left(\frac{1}{\lambda_1}\right) \left(\text{given that } \left(\frac{\bar{C}}{\lambda_1} \right)^\ell \ll \left(\frac{\bar{C}}{\lambda_1} \right) \text{ for all } \ell > 1 \text{ since } \frac{\bar{C}}{\lambda_1} = o(1) \right. \\
&\quad \left. \text{and given that } j \text{ is a finite positive integer so that } \sum_{\ell=1}^j \binom{j}{\ell} \text{ is finite} \right),
\end{aligned}$$

from which we see that

$$\begin{aligned}
&\left\| \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{Xy}}{\lambda_1^{\frac{1}{2}}} - \underline{V}' \underline{w}_\phi \right\|_2^2 \\
&= \sum_{j=2}^{\kappa+1} \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \left(\frac{\beta' [\bar{\Sigma}_X^j - (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2')^j] \beta}{\lambda_1^{j-1}} \right)^2 \\
&\leq \sum_{j=2}^{\kappa+1} \left[\bar{C}_2^j \kappa \bar{c}^2 \sum_{\ell=1}^j \binom{j}{\ell} \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right)^\ell \right]^2 \\
&\leq \bar{C}_2^{2\kappa+2} \kappa^3 \bar{c}^4 [\kappa+1]^2 [(\kappa+1)!]^2 \left(\frac{\bar{C}}{\lambda_1} \right)^2 \quad \text{for all } N \text{ and } T \text{ sufficiently large} \\
&= O\left(\frac{1}{\lambda_1^2}\right).
\end{aligned}$$

It follows that

$$\left\| \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\lambda_1^{\frac{1}{2}}} - \underline{V}' \underline{w}_\phi \right\|_2 = O \left(\frac{1}{\lambda_1} \right).$$

Turning our attention to part (c), note first that, by straightforward calculation

$$\begin{aligned} & \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} \\ &= \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{pmatrix} \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \\ &= \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} \begin{pmatrix} \bar{\sigma}_{Xy}/\lambda_1^{\frac{1}{2}} & \bar{\Sigma}_X \bar{\sigma}_{Xy}/\lambda_1^{\frac{3}{2}} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy}/\lambda_1^{\kappa-\frac{1}{2}} \end{pmatrix} \\ &= \sqrt{\lambda_{1,1}} \begin{pmatrix} \bar{\Sigma}_X \beta/\lambda_1 & \bar{\Sigma}_X^2 \beta/\lambda_1^2 & \cdots & \bar{\Sigma}_X^\kappa \beta/\lambda_1^\kappa \end{pmatrix} \quad (\text{since } \bar{\sigma}_{Xy} = \bar{\Sigma}_X \beta) \end{aligned} \quad (71)$$

and

$$\begin{aligned} & \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G_W D_{\lambda,\kappa}^{-1} \\ &= \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} \begin{pmatrix} [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \beta & [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^2 \beta & \cdots & [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^\kappa \beta \end{pmatrix} \\ & \quad \times \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \\ &= \sqrt{\lambda_{1,1}} \begin{pmatrix} [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \beta/\lambda_1 & \cdots & [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^\kappa \beta/\lambda_1^\kappa \end{pmatrix} \end{aligned} \quad (72)$$

It follows immediately from expressions (71) and (72) that

$$\begin{aligned} & \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G_W D_{\lambda,\kappa}^{-1} \\ &= \sqrt{\lambda_{1,1}} \begin{pmatrix} [\bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)] \beta/\lambda_1 & \cdots & [\bar{\Sigma}_X^\kappa - (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)^\kappa] \beta/\lambda_1^\kappa \end{pmatrix} \end{aligned}$$

Now, again let $\bar{C}_2 \geq \max\{1, C_2\}$ where C_2 is the positive constant given in part (b) of Assumption 4-2; and, for any fixed integer $1 \leq j \leq \kappa$, we have

$$\begin{aligned}
& \left\| \sqrt{\lambda_{1,1}} \frac{(\bar{\Sigma}_X^j - [L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2']^j) \beta}{\lambda_1^j} \right\|_2 \\
&= \sqrt{\lambda_{1,1}} \left\| \frac{([L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2' + \bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2')]^j - [L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2']^j) \beta}{\lambda_1^j} \right\|_2 \\
&\leq \frac{\sqrt{\lambda_{1,1}}}{\lambda_1^j} \sum_{\ell=1}^j \binom{j}{\ell} \|\bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2')\|_2^\ell \|L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2'\|_2^{j-\ell} \|\beta\|_2 \\
&= \sqrt{\lambda_{1,1}} \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2')}{\lambda_1} \right\|_2^\ell \frac{\|L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2'\|_2^{j-\ell}}{\lambda_1^{j-\ell}} \left\| L_1 \frac{\phi}{\lambda_{1,1}^{\frac{1}{2}}} \right\|_2 \\
&= \frac{\lambda_{1,1}^{\frac{1}{2}}}{\lambda_{1,1}^{\frac{1}{2}}} \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \frac{\|L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2'\|_2^{j-\ell}}{\lambda_1^{j-\ell}} \|L_1 \phi\|_2 \quad (\text{since } \bar{\Sigma}_u = \bar{\Sigma}_X - (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2')) \\
&\leq [c^* \max\{1, C_2\}]^j \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \|L_1 \phi\|_2 \\
&\quad (\text{since Assumption 4-2(b) and Lemma D-3(a) imply that there exist constants } c^* > 1 \text{ and } C_2 > 0 \text{ such that } \\
&\quad \|L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2'\|_2 = \lambda_{\max}\{\bar{\Sigma}_W\} = \max\{\lambda_{1,1}, \lambda_{2,1}\} \leq c^* \max\{1, C_2\} \lambda_1 \text{ for all } N, T \text{ sufficiently large}) \\
&\leq \bar{C}_2^j \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \|L_1 \phi\|_2 \quad (\text{for some constant } \bar{C}_2 \geq c^* \max\{1, C_2\}) \\
&\leq \bar{C}_2^\kappa \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \|\phi\|_2 \quad \left(\text{since } \|L_1 \phi\|_2 = \sqrt{\phi' L_1' L_1 \phi} = \sqrt{\phi' \phi} = \|\phi\|_2 \text{ and } \right. \\
&\quad \left. \text{since } \bar{C}_2^\kappa > \bar{C}_2^j \text{ for } 1 \leq j \leq \kappa \text{ given that } \bar{C}_2 > 1 \right) \\
&\leq \bar{C}_2^\kappa \sqrt{\kappa \bar{C}} \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \quad (\text{since, by Assumption 4-4 (a), there exists a constant } \bar{c} \\
&\quad \text{such that } \bar{c} \geq \max_{1 \leq g \leq \kappa} |\phi_g| \text{ so that } \sqrt{\kappa \bar{C}} \geq \sqrt{\kappa \left(\max_{1 \leq g \leq \kappa} |\phi_g| \right)^2} \geq \sqrt{\|\phi\|_2^2} = \|\phi\|_2) \\
&= \bar{C}_2^\kappa \sqrt{\kappa \bar{C}} \sum_{\ell=1}^j \binom{j}{\ell} \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right)^\ell \quad (\text{by symmetry and positive semidefiniteness of } \bar{\Sigma}_u, \\
&\quad \text{so that } \|\bar{\Sigma}_u\|_2 = \sqrt{\lambda_{\max}(\bar{\Sigma}_u' \bar{\Sigma}_u)} = \sqrt{[\lambda_{\max}(\bar{\Sigma}_u)]^2} = \lambda_{\max}(\bar{\Sigma}_u)) \\
&\leq \bar{C}_2^\kappa \sqrt{\kappa \bar{C}} \sum_{\ell=1}^j \binom{j}{\ell} \left(\frac{\bar{C}}{\lambda_1} \right)^\ell \quad (\text{since } \lambda_{\max}(\bar{\Sigma}_u) \leq \bar{C} < \infty \text{ for some constant } \bar{C} > 0 \\
&\quad \text{by Assumption 4-1 and Lemma D-2})
\end{aligned}$$

$$\begin{aligned}
&\leq \bar{C}_2^\kappa \sqrt{\kappa \bar{C}} \left(\frac{\bar{C}}{\lambda_1} \right) \sum_{\ell=1}^j \binom{j}{\ell} \text{ for all } N \text{ and } T \text{ sufficiently large} \\
&\quad \left(\text{since } \lambda_1 \rightarrow \infty \text{ so } \left(\frac{\bar{C}}{\lambda_1} \right)^\ell \ll \left(\frac{\bar{C}}{\lambda_1} \right) \text{ for integer } \ell \text{ such that } 1 < \ell \leq j \leq \kappa \right) \\
&\leq \bar{C}_2^\kappa \sqrt{\kappa \bar{C}} (\kappa!) \left(\frac{\bar{C}}{\lambda_1} \right) \text{ (given that } j \leq \kappa) \\
&= O\left(\frac{1}{\lambda_1}\right)
\end{aligned}$$

Making use of this result, we see that

$$\begin{aligned}
\left\| \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G_W D_{\lambda,\kappa}^{-1} \right\|_2^2 &\leq \left\| \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} [G D_{\lambda,\kappa}^{-1} - G_W D_{\lambda,\kappa}^{-1}] \right\|_F^2 \\
&= \frac{\lambda_{1,1}}{\lambda_1} \text{tr} \left\{ (G D_{\lambda,\kappa}^{-1} - G_W D_{\lambda,\kappa}^{-1})' (G D_{\lambda,\kappa}^{-1} - G_W D_{\lambda,\kappa}^{-1}) \right\} \\
&= \frac{\lambda_{1,1}}{\lambda_1} \sum_{j=1}^{\kappa} \left\| \frac{(\bar{\Sigma}_X^j - [L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2']^j) \beta}{\lambda_1^j} \right\|_2^2 \\
&= O(1) O\left(\frac{1}{\lambda_1^2}\right) \text{ (given that } \kappa \text{ is a fixed finite integer and} \\
&\quad \text{given that } \frac{\lambda_{1,1}}{\lambda_1} = O(1) \text{ by the result of Lemma D-3(a))} \\
&= O\left(\frac{1}{\lambda_1^2}\right),
\end{aligned}$$

from which it follows that

$$\left\| \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G_W D_{\lambda,\kappa}^{-1} \right\|_2 = O\left(\frac{1}{\lambda_1}\right). \quad (73)$$

Moreover, note that, from expression (72), we see that

$$\begin{aligned}
& \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G_W D_{\lambda,\kappa}^{-1} \\
&= \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} \left([L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \beta / \lambda_1^{\frac{1}{2}} \quad \cdots \quad [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^\kappa \beta / \lambda_1^{\kappa - \frac{1}{2}} \right) \\
&= \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} \left([L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] \left(L_1 / \lambda_1^{\frac{1}{2}} \right) \left(\phi / \lambda_{1,1}^{\frac{1}{2}} \right) \quad \cdots \quad [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^\kappa \left(L_1 / \lambda_1^{\kappa - \frac{1}{2}} \right) \left(\phi / \lambda_{1,1}^{\frac{1}{2}} \right) \right) \\
&\quad \left(\text{given } \beta = \frac{L_1 \phi}{\sqrt{\lambda_{1,1}}} \right) \\
&= \left([L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2] L_1 \phi / \lambda_1 \quad \cdots \quad [L_1 \underline{\Delta}_1^\kappa L'_1 + L_2 \underline{\Delta}_2^\kappa L'_2] L_1 \phi / \lambda_1^\kappa \right) \\
&\quad \left(\text{since } [L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2]^j = L_1 \underline{\Delta}_1^j L'_1 + L_2 \underline{\Delta}_2^j L'_2 \text{ for positive integer } j \right) \\
&= \left(L_1 \underline{\Delta}_1 \phi / \lambda_1 \quad L_1 \underline{\Delta}_1^2 \phi / \lambda_1^2 \quad \cdots \quad L_1 \underline{\Delta}_1^\kappa \phi / \lambda_1^\kappa \right) \\
&= L_1 \begin{pmatrix} \underline{\Delta}_{1,1} \phi_1 / \lambda_1 & \underline{\Delta}_{1,1}^2 \phi_1 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,1}^\kappa \phi_1 / \lambda_1^\kappa \\ \underline{\Delta}_{1,2} \phi_2 / \lambda_1 & \underline{\Delta}_{1,2}^2 \phi_2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^\kappa \phi_2 / \lambda_1^\kappa \\ \vdots & \vdots & & \vdots \\ \underline{\Delta}_{1,\kappa} \phi_\kappa / \lambda_1 & \underline{\Delta}_{1,\kappa}^2 \phi_\kappa / \lambda_1^2 & \cdots & \underline{\Delta}_{1,\kappa}^\kappa \phi_\kappa / \lambda_1^\kappa \end{pmatrix} \\
&= L_1 \begin{pmatrix} \underline{\Delta}_{1,1} \phi_1 / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2} \phi_2 / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,\kappa} \phi_\kappa / \lambda_1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \underline{\Delta}_{1,2} / \lambda_1 & \underline{\Delta}_{1,2}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^{\kappa-1} / \lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\Delta}_{1,\kappa} / \lambda_1 & \underline{\Delta}_{1,\kappa}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,\kappa}^{\kappa-1} / \lambda_1^{\kappa-1} \end{pmatrix} \\
&= L_1 D_{\phi,\lambda}^* \underline{V}
\end{aligned}$$

where

$$\begin{aligned}
D_{\phi,\lambda}^* &= \begin{pmatrix} \underline{\Delta}_{1,1} \phi_1 / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2} \phi_2 / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,\kappa} \phi_\kappa / \lambda_1 \end{pmatrix}, \\
\underline{V} &= \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \underline{\Delta}_{1,2} / \lambda_1 & \underline{\Delta}_{1,2}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^{\kappa-1} / \lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\Delta}_{1,\kappa} / \lambda_1 & \underline{\Delta}_{1,\kappa}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,\kappa}^{\kappa-1} / \lambda_1^{\kappa-1} \end{pmatrix}.
\end{aligned}$$

Hence, the result given in expression (73) can be restated as

$$\left\| \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - L_1 D_{\phi,\lambda}^* \underline{V} \right\|_2 = O\left(\frac{1}{\lambda_1}\right).$$

Finally, to show part (d), note first that $\underline{V}' D_{\phi,\lambda} \underline{V}$ is clearly a symmetric matrix. Moreover, the result of part (c) of Lemma B-3 further shows that there exists a positive constant $C_\#$ such that

$$\lambda_{\min}(\underline{V}' D_{\phi,\lambda} \underline{V}) \geq C_\# > 0$$

for all N and T sufficiently large, so that $\underline{V}' D_{\phi,\lambda} \underline{V}$ is also positive definite and, thus, non-singular for all N and T sufficiently large. Making use of these properties of the matrix $\underline{V}' D_{\phi,\lambda} \underline{V}$, we see that for N and T sufficiently large

$$\begin{aligned} \left\| (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\|_2 &= \sqrt{\lambda_{\max} \left\{ \left[(\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right]' (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\}} \\ &= \sqrt{\lambda_{\max} \left\{ \left[(\underline{V}' D_{\phi,\lambda} \underline{V})' \right]^{-1} (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\}} \\ &\quad \text{(since we can interchange the transposition and the inverse operations)} \\ &= \sqrt{\lambda_{\max} \left\{ (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\}} \\ &\quad \text{(since } \underline{V}' D_{\phi,\lambda} \underline{V} \text{ is a symmetric matrix)} \\ &= \sqrt{\left[\lambda_{\max} \left\{ (\underline{V}' D_{\phi,\lambda} \underline{V})^{-1} \right\} \right]^2} \\ &= \sqrt{\left(\frac{1}{\lambda_{\min}(\underline{V}' D_{\phi,\lambda} \underline{V})} \right)^2} \\ &= \frac{1}{\lambda_{\min}(\underline{V}' D_{\phi,\lambda} \underline{V})} \\ &\leq \frac{1}{C_*} \quad \text{(given that } \lambda_{\min}(\underline{V}' D_{\phi,\lambda} \underline{V}) \geq C_*) \\ &< \infty \quad \text{(since } C_* > 0) \end{aligned}$$

Next, define

$$E^\dagger = \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \underline{V}' D_{\phi,\lambda} \underline{V}$$

and applying the result of part (c) of Lemma B-3, we have

$$\begin{aligned}
\left\| (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} E^\dagger \right\|_2 &\leq \left\| (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} \right\|_2 \|E^\dagger\|_2 \\
&\leq \frac{1}{C_\#} \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} - \underline{V}' D_{\phi, \lambda} \underline{V} \right\|_2 \\
&= O(1) O\left(\frac{1}{\lambda_1}\right) \\
&= O\left(\frac{1}{\lambda_1}\right) = o(1)
\end{aligned}$$

Also, by expression (62) given in the proof of part (c) of Lemma B-3, we have that $(\underline{\lambda}_{1,1}/\lambda_1) D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}$ is positive definite for all N and T sufficiently large. Finally, applying the last inequality given in part (d) of Lemma B-1, we further deduce that

$$\begin{aligned}
&\left\| \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} - (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} \right\|_2 \\
&\leq \left\| (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} \right\|_2 \frac{\left\| (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} E^\dagger \right\|_2}{1 - \left\| (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} E^\dagger \right\|_2} \\
&\leq \frac{1}{C_\#} \frac{\left\| (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} E^\dagger \right\|_2}{1 - \left\| (\underline{V}' D_{\phi, \lambda} \underline{V})^{-1} E^\dagger \right\|_2} \\
&= O(1) O\left(\frac{1}{\lambda_1}\right) \\
&= O\left(\frac{1}{\lambda_1}\right). \square
\end{aligned}$$

Appendix C: Lemmas Used for Proving the Consistency of the Generalized Thresholding Estimators for $\bar{\Sigma}_X$ and $\bar{\sigma}_{Xy}$

Lemma C-1 Let $(Z_t)_{t \in \mathbb{Z}}$ be a sequence of centered real valued random variables and let $\mathbb{V}$ be defined by

$$\mathbb{V} = \sup_{M > 0} \sup_{s > 0} \left\{ \text{Var}(\psi_M(Z_s)) + 2 \sum_{t > s} |\text{Cov}(\psi_M(Z_s), \psi_M(Z_t))| \right\}$$

where

$$\psi_M(x) = (x \wedge M) \vee (-M).$$

Assume the following three conditions

1. For some constants $\gamma_1 \in (0, +\infty)$ and $b_1 \in (0, +\infty)$, the following tail condition is satisfied: for any positive s ,

$$\sup_{t \in \mathbb{Z}} \mathbb{P}(|Z_s| > s) \leq \exp \left\{ 1 - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\} := H(s).$$

2. Define $\mathcal{F}_{-\infty}^t = \sigma(\dots, X_{t-2}, X_{t-1}, X_t)$ and $\mathcal{F}_{t+m}^\infty = \sigma(X_{t+m}, X_{t+m+1}, X_{t+m+2}, \dots)$. Suppose that there exist positive constants γ_2 , $a \geq 1$, and c such that

$$\alpha(m) = \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty) \leq a \exp(-cm^{\gamma_2}) \text{ for any } m \geq 1$$

where

$$\alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty) = \sup_{G \in \mathcal{F}_{-\infty}^t, H \in \mathcal{F}_{t+m}^\infty} |P(G \cap H) - P(G)P(H)|.$$

3. Let

$$\gamma < 1 \text{ where } \gamma \text{ is defined by } \frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.$$

Under these conditions, $\mathbb{V}$ is a finite positive constant (which we can denote by rewriting $\mathbb{V} = C_3$), and, for any $T \geq 4$, there exist (in addition to C_3) positive constants C_1, C_2, C_4 , and

C_5 depending on a, b, c, γ and γ_2 such that, for any positive z ,

$$\begin{aligned} \mathbb{P} \left(\sup_{t \leq T} |S_t| \geq z \right) &= \mathbb{P} \left(\sup_{t \leq T} \left| \sum_{s=1}^t Z_s \right| \geq z \right) \\ &\leq T \exp \left\{ -\frac{z^\gamma}{C_1} \right\} + \exp \left\{ -\frac{z^2}{C_2(1+TC_3)} \right\} \\ &\quad + \exp \left\{ -\frac{z^2}{C_4 T} \exp \left(\frac{z^{\gamma(1-\gamma)}}{C_5 (\ln z)^\gamma} \right) \right\}. \end{aligned}$$

Proof of Lemma C-1: This is essentially Theorem 1 of Merlevède, Peligrad, and Rio (2011). The original Bernstein-type inequality given in Theorem 1 of Merlevède, Peligrad, and Rio (2011) applies when the underlying process is either τ -mixing or α -mixing. We have stated the result above only for the case of α -mixing because α -mixing seems to be more commonly used. For a proof of this result, see the proof of Theorem 1 given in Merlevède, Peligrad, and Rio (2011).

Lemma C-2: Suppose that Assumptions 4-3 and 4-4 hold, and let γ_1 and b_1 be the positive constants defined in part (a) of Assumption 4-3. Then, for some constant $b_2 \geq 2(1 + \ln 2)^{\frac{1}{\gamma_1}} b_1 > 0$ such that for all $s > 0$, for any integer i such that $1 \leq i \leq N$, and for all $t \in \mathbb{Z}$; the following statement is true:

$$P(|X_{it} - \mu_{X,i}| > s) \leq \exp \left\{ 1 - \left(\frac{s}{b_2} \right)^{\gamma_1} \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{b_2} \right)^{\gamma_1} \right\}$$

for some positive constant $\bar{C} \geq \exp\{1\}$.

Proof of Lemma C-2: To proceed, first let $e_{i,N}$ denote an $N \times 1$ elementary vector with its i^{th} component equal to 1 and all other components equal to 0, and note that $\|e_{i,N}\|_2 = 1$. Hence, in this case, we can represent $u_{it} = u'_t e_{i,N} = u'_t \delta$ by setting $\delta = e_{i,N}$. It follows then, by part (a) of Assumption 4-3, there exist constants $\gamma_1 > 0$ and $b_1 > 0$, such that $\forall s > 0$, for all integer i such that $1 \leq i \leq N$, and for all $t \in \mathbb{Z}$

$$\begin{aligned} P(|X_{it} - \mu_{X,i}| > s) &= P(|W_{it} + u_{it}| > s) \\ &\leq P(|W_{it}| + |u_{it}| > s) \\ &\leq P \left(\left\{ |W_{it}| > \frac{s}{2} \right\} \cup \left\{ |u_{it}| > \frac{s}{2} \right\} \right) \\ &\leq P \left(|W_{it}| > \frac{s}{2} \right) + P \left(|u_{it}| > \frac{s}{2} \right) \\ &\leq \exp \left\{ 1 - \left(\frac{s}{2b_1} \right)^{\gamma_1} \right\} + \exp \left\{ 1 - \left(\frac{s}{2b_1} \right)^{\gamma_1} \right\} \\ &= 2 \exp \left\{ 1 - \left(\frac{s}{2b_1} \right)^{\gamma_1} \right\}. \end{aligned}$$

Moreover, since, by definition, $b_2 \geq 2(1 + \ln 2)^{\frac{1}{\gamma_1}} b_1$, it follows that

$$\frac{b_2}{2b_1} \geq (1 + \ln 2)^{\frac{1}{\gamma_1}} \geq 1$$

This, in turn, implies that when $s > b_2$

$$\begin{aligned} P(|X_{it} - \mu_{X,i}| > s) &= P(|W_{it} + u_{it}| > s) \\ &\leq 2 \exp \left\{ 1 - \left(\frac{s}{2b_1} \right)^{\gamma_1} \right\} \\ &= 2 \exp \left\{ 1 - \left(\frac{s}{b_2} \right)^{\gamma_1} \left(\frac{b_2}{2b_1} \right)^{\gamma_1} \right\} \\ &\leq 2 \exp \left\{ 1 - \left(\frac{s}{b_2} (1 + \ln 2)^{\frac{1}{\gamma_1}} \right)^{\gamma_1} \right\} \\ &\quad \left(\text{given that } \frac{b_2}{2b_1} \geq (1 + \ln 2)^{\frac{1}{\gamma_1}} \right) \\ &= 2 \exp \left\{ 1 - \left(\frac{s}{b_2} \right)^{\gamma_1} (1 + \ln 2) \right\} \\ &= 2 \exp \left\{ 1 - \left(\frac{s}{b_2} \right)^{\gamma_1} - \left(\frac{s}{b_2} \right)^{\gamma_1} \ln 2 \right\} \\ &\leq 2 \exp \left\{ 1 - \left(\frac{s}{b_2} \right)^{\gamma_1} - \ln 2 \right\} \quad \left(\text{since } \frac{s}{b_2} > 1 \text{ and } \gamma_1 > 0 \right) \\ &= \frac{2}{2} \exp \left\{ 1 - \left(\frac{s}{b_2} \right)^{\gamma_1} \right\} \\ &= \exp \left\{ 1 - \left(\frac{s}{b_2} \right)^{\gamma_1} \right\} \\ &\leq \overline{C} \exp \left\{ - \left(\frac{s}{b_2} \right)^{\gamma_1} \right\} \quad (\text{since } \overline{C} \geq \exp \{1\}). \end{aligned}$$

On the other hand, when $0 < s \leq b_2$

$$\begin{aligned} P(|X_{it} - \mu_{X,i}| > s) &\leq 1 \leq \exp \left\{ 1 - \left(\frac{s}{b_2} \right)^{\gamma_1} \right\} \\ &= \exp \{1\} \exp \left\{ - \left(\frac{s}{b_2} \right)^{\gamma_1} \right\} \\ &\leq \overline{C} \exp \left\{ - \left(\frac{s}{b_2} \right)^{\gamma_1} \right\} \quad (\text{since } \overline{C} \geq \exp \{1\}). \end{aligned}$$

Combining the two cases, we obtain the desired result. $\square$

Lemma C-3: Suppose that Assumptions 4-3 and 4-4 hold, and let γ_1 and b_2 be the positive constants defined, respectively, in part (a) of Assumption 4-3 and in Lemma C-2. Then, for $r = \gamma_1/2 > 0$, for some constant $b \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} b_2^2 > 0$, and for all $s > 0$; the following statements are true.

(a) For integers i and j such that $1 \leq i, j \leq N$ and for all $t \in \mathbb{Z}$

$$P(|(X_{it} - \mu_{X,i})(X_{jt} - \mu_{X,j})| > s) \leq \exp\left\{1 - \left(\frac{s}{b}\right)^r\right\} \leq \bar{C} \exp\left\{-\left(\frac{s}{b}\right)^r\right\}.$$

where $\bar{C}$ is a positive constant such that $\bar{C} \geq \exp\{1\}$.

(b) Let $h \geq 1$ be a fixed finite integer. For any integer i such that $1 \leq i \leq N$ and for all $t \in \mathbb{Z}$

$$P(|(X_{it} - \mu_{X,i}) \varepsilon_{t+h}| > s) \leq \exp\left\{1 - \left(\frac{s}{b}\right)^r\right\}.$$

(c) There exists a positive constant $\bar{C} \geq \exp\{1\}$ such that, for any integer i with $1 \leq i \leq N$ and for all $t \in \mathbb{Z}$,

$$\begin{aligned} P(|(X_{it} - \mu_{X,i}) W'_t \beta| > s) &= P(|(W_{it} + u_{it}) W'_t \beta| > s) \\ &\leq \exp\left\{1 - \left(\frac{s}{b}\right)^r\right\} \\ &\leq \bar{C} \exp\left\{-\left(\frac{s}{b}\right)^r\right\}. \end{aligned}$$

(d) There exists a positive constant $\bar{C} \geq \exp\{1\}$ such that, for any integer i with $1 \leq i \leq N$ and for all $t \in \mathbb{Z}$,

$$\begin{aligned} P(|(X_{it} - \mu_{X,i}) u'_t \underline{\beta}| > s) &= P(|(W_{it} + u_{it}) u'_t \underline{\beta}| > s) \\ &\leq \exp\left\{1 - \left(\frac{s}{b}\right)^r\right\} \\ &\leq \bar{C} \exp\left\{-\left(\frac{s}{b}\right)^r\right\}, \text{ where } \underline{\beta} = \beta / \|\beta\|_2. \end{aligned}$$

Proof of Lemma C-3: To show part (a), note that, by the result given in Lemma C-2, there exist constants $\gamma_1 > 0$ and $b_2 > 0$, such that $\forall s > 0$, for any integers i and j with $1 \leq i, j \leq N$, and for all $t \in \mathbb{Z}$,

$$P(|X_{it} - \mu_{X,i}| > s) \leq \exp\left\{1 - \left(\frac{s}{b_2}\right)^{\gamma_1}\right\} \leq \bar{C} \exp\left\{-\left(\frac{s}{b_2}\right)^{\gamma_1}\right\},$$

and

$$P(|X_{jt} - \mu_{X,j}| > s) \leq \exp\left\{1 - \left(\frac{s}{b_2}\right)^{\gamma_1}\right\} \leq \bar{C} \exp\left\{-\left(\frac{s}{b_2}\right)^{\gamma_1}\right\}$$

Similar to the proof of Lemma A.2 of Fan, Liao, and Mincheva (2011), it follows that, for any $s > 0$, for all $t \in \mathbb{Z}$, and for any integers i and j such that $1 \leq i, j \leq N$,

$$\begin{aligned}
& P(|(X_{it} - \mu_{X,i})(X_{jt} - \mu_{X,j})| > s) \\
= & P(\{|X_{it} - \mu_{X,i}| |X_{jt} - \mu_{X,j}| > s\} \cap \{|X_{jt} - \mu_{X,j}| \leq \sqrt{s}\}) \\
& + P(\{|X_{it} - \mu_{X,i}| |X_{jt} - \mu_{X,j}| > s\} \cap \{|X_{jt} - \mu_{X,j}| > \sqrt{s}\}) \\
\leq & P(\sqrt{s} |X_{it} - \mu_{X,i}| > s) + P(\{|X_{it} - \mu_{X,i}| |X_{jt} - \mu_{X,j}| > s\} \cap \{|X_{jt} - \mu_{X,j}| > \sqrt{s}\}) \\
\leq & P(|X_{it} - \mu_{X,i}| > \sqrt{s}) + P(|X_{jt} - \mu_{X,j}| > \sqrt{s}) \\
\leq & \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_2} \right)^{\gamma_1} \right\} + \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_2} \right)^{\gamma_1} \right\} \\
= & 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_2} \right)^{\gamma_1} \right\}
\end{aligned}$$

Moreover, since, by definition, $b \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} b_2^2$, it follows that

$$\frac{b}{b_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \geq 1$$

This, in turn, implies that when $s > b$

$$\begin{aligned}
P(|(X_{it} - \mu_{X,i})(X_{jt} - \mu_{X,j})| > s) &\leq 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_2} \right)^{\gamma_1} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} \left(\frac{b}{b_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{b} (1 + \ln 2)^{\frac{2}{\gamma_1}} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\quad \left(\text{given that } \frac{b}{b_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \right) \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} (1 + \ln 2) \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} \ln 2 \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} - \ln 2 \right\} \quad \left(\text{since } \frac{s}{b} > 1 \text{ and } \frac{\gamma_1}{2} > 0 \right) \\
&= \frac{2}{2} \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= \exp \left\{ 1 - \left(\frac{s}{b} \right)^r \right\} \quad \left(\text{by setting } r = \frac{\gamma_1}{2} \right) \\
&\leq \overline{C} \exp \left\{ - \left(\frac{s}{b} \right)^r \right\} \quad \left(\text{since } \overline{C} \geq \exp \{1\} \right)
\end{aligned}$$

On the other hand, when $0 < s \leq b$

$$\begin{aligned}
P(|(X_{it} - \mu_{X,i})(X_{jt} - \mu_{X,j})| > s) &\leq 1 \leq \exp \left\{ 1 - \left(\frac{s}{b} \right)^r \right\} \\
&= \exp \{1\} \exp \left\{ - \left(\frac{s}{b} \right)^r \right\} \\
&\leq \overline{C} \exp \left\{ - \left(\frac{s}{b} \right)^r \right\} \quad \left(\text{since } \overline{C} \geq \exp \{1\} \right),
\end{aligned}$$

which shows part (a).

Next, to show part (b), note that, for any $s > 0$, for any integer i such that $1 \leq i \leq N$, and for all $t \in \mathbb{Z}$

$$\begin{aligned}
& P(|(X_{it} - \mu_{X,i}) \varepsilon_{t+h}| > s) \\
= & P(\{|X_{it} - \mu_{X,i}| |\varepsilon_{t+h}| > s\} \cap \{|\varepsilon_{t+h}| \leq \sqrt{s}\}) \\
& + P(\{|X_{it} - \mu_{X,i}| |\varepsilon_{t+h}| > s\} \cap \{|\varepsilon_{t+h}| > \sqrt{s}\}) \\
& \text{(given that } \{|\varepsilon_{t+h}| \leq \sqrt{s}\} \cap \{|\varepsilon_{t+h}| > \sqrt{s}\} = \emptyset \text{ in this case)} \\
\leq & P(\sqrt{s} |X_{it} - \mu_{X,i}| > s) + P(\{|X_{it} - \mu_{X,i}| |\varepsilon_{t+h}| > s\} \cap \{|\varepsilon_{t+h}| > \sqrt{s}\}) \\
\leq & P(|X_{it} - \mu_{X,i}| > \sqrt{s}) + P(|\varepsilon_{t+h}| > \sqrt{s}) \\
\leq & \exp\left\{1 - \left(\frac{\sqrt{s}}{b_2}\right)^{\gamma_1}\right\} + \exp\left\{1 - \left(\frac{\sqrt{s}}{b_1}\right)^{\gamma_1}\right\} \quad (\text{by Lemma C-2 and part (a) of Assumption 4-3}) \\
\leq & \exp\left\{1 - \left(\frac{\sqrt{s}}{b_2}\right)^{\gamma_1}\right\} + \exp\left\{1 - \left(\frac{\sqrt{s}}{b_2}\right)^{\gamma_1}\right\} \quad (\text{since } b_2 > b_1) \\
= & 2 \exp\left\{1 - \left(\frac{\sqrt{s}}{b_2}\right)^{\gamma_1}\right\}
\end{aligned}$$

When $s > b$, the inequality

$$\frac{b}{b_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \geq 1$$

further implies that

$$\begin{aligned}
P(|(X_{it} - \mu_{X,i}) \varepsilon_{t+h}| > s) &\leq 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_2} \right)^{\gamma_1} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} \left(\frac{b}{b_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{b} (1 + \ln 2)^{\frac{2}{\gamma_1}} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\quad \left(\text{given that } \frac{b}{b_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \right) \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} (1 + \ln 2) \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} \ln 2 \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} - \ln 2 \right\} \quad \left(\text{since } \frac{s}{b} > 1 \text{ and } \frac{\gamma_1}{2} > 0 \right) \\
&= \frac{2}{2} \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= \exp \left\{ 1 - \left(\frac{s}{b} \right)^r \right\} \quad \left(\text{by setting } r = \frac{\gamma_1}{2} \right) \\
&\leq \overline{C} \exp \left\{ - \left(\frac{s}{b} \right)^r \right\} \quad \left(\text{since } \overline{C} \geq \exp \{1\} \right)
\end{aligned}$$

On the other hand, when $0 < s \leq b$, we have

$$\begin{aligned}
P(|(X_{it} - \mu_{X,i}) \varepsilon_{t+h}| > s) &\leq 1 \leq \exp \left\{ 1 - \left(\frac{s}{b} \right)^r \right\} \\
&= \exp \{1\} \exp \left\{ - \left(\frac{s}{b} \right)^r \right\} \\
&= \overline{C} \exp \left\{ - \left(\frac{s}{b} \right)^r \right\} \quad \left(\text{since } \overline{C} \geq \exp \{1\} \right).
\end{aligned}$$

Next, to show part (c), note that. by part (b) of Assumption 4-4 and by the result given in Lemma C-2, there exist constants $\gamma_1 > 0$, $b_1 > 0$, and $b_2 > 0$ such that for any $s > 0$, for all $t \in \mathbb{Z}$, and for any integer i with $1 \leq i \leq N$,

$$P(|X_{it} - \mu_{X,i}| > s) = P(|W_{it} + u_{it}| > s) \leq \exp \left\{ 1 - \left(\frac{s}{b_2} \right)^{\gamma_1} \right\}$$

and

$$P(|W'_t\beta| > s) \leq \exp \left\{ 1 - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\}$$

Hence, for any $s > 0$, for all $t \in \mathbb{Z}$, and for any integer i such that $1 \leq i \leq N$,

$$\begin{aligned} & P(|(X_{it} - \mu_{X,i}) W'_t\beta| > s) \\ = & P(\{|X_{it} - \mu_{X,i}| |W'_t\beta| > s\} \cap \{|W'_t\beta| \leq \sqrt{s}\}) \\ & + P(\{|X_{it} - \mu_{X,i}| |W'_t\beta| > s\} \cap \{|W'_t\beta| > \sqrt{s}\}) \\ \leq & P(\sqrt{s} |X_{it} - \mu_{X,i}| > s) + P(\{|X_{it} - \mu_{X,i}| |W'_t\beta| > s\} \cap \{|W'_t\beta| > \sqrt{s}\}) \\ \leq & P(|X_{it} - \mu_{X,i}| > \sqrt{s}) + P(|W'_t\beta| > \sqrt{s}) \\ \leq & \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_2} \right)^{\gamma_1} \right\} + \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_1} \right)^{\gamma_1} \right\} \\ \leq & 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_2} \right)^{\gamma_1} \right\} \end{aligned}$$

given that

$$\frac{b_2}{2b_1} \geq (1 + \ln 2)^{\frac{1}{\gamma_1}} \geq 1 \text{ implying that } b_2 \geq b_1.$$

Moreover, by definition, the positive constant b satisfies the inequality

$$\frac{b}{b_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \geq 1.$$

This, in turn, implies that when $s > b$

$$\begin{aligned}
P(|(X_{it} - \mu_{X,i}) W'_t \beta| > s) &\leq 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_2} \right)^{\gamma_1} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} \left(\frac{b}{b_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{b} (1 + \ln 2)^{\frac{2}{\gamma_1}} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\quad \left(\text{given that } \frac{b}{b_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \right) \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} (1 + \ln 2) \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} \ln 2 \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} - \ln 2 \right\} \quad \left(\text{since } \frac{s}{b} > 1 \text{ and } \frac{\gamma_1}{2} > 0 \right) \\
&= \frac{2}{2} \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= \exp \left\{ 1 - \left(\frac{s}{b} \right)^r \right\} \quad \left(\text{by setting } r = \frac{\gamma_1}{2} \right) \\
&\leq \overline{C} \exp \left\{ - \left(\frac{s}{b} \right)^r \right\} \quad \left(\text{since } \overline{C} \geq \exp \{1\} \right)
\end{aligned}$$

On the other hand, when $0 < s \leq b$

$$\begin{aligned}
P(|(X_{it} - \mu_{X,i}) W'_t \beta| > s) &\leq 1 \leq \exp \left\{ 1 - \left(\frac{s}{b} \right)^r \right\} \\
&= \exp \{1\} \exp \left\{ - \left(\frac{s}{b} \right)^r \right\} \\
&= \overline{C} \exp \left\{ - \left(\frac{s}{b} \right)^r \right\} \quad \left(\text{since } \overline{C} \geq \exp \{1\} \right)
\end{aligned}$$

which shows part (c).

Finally, to show part (d), again note that, by the result of Lemma C-2, there exist constants $\gamma_1 > 0$ and $b_2 > 0$, such that for any $s > 0$, for all $t \in \mathbb{Z}$, and for any integer i with $1 \leq i \leq N$,

$$P(|X_{it} - \mu_{X,i}| > s) \leq \exp \left\{ 1 - \left(\frac{s}{b_2} \right)^{\gamma_1} \right\}.$$

Moreover, by definition, $\underline{\beta} = \beta / \|\beta\|_2$ so that $\|\underline{\beta}\|_2 = \|\beta\|_2 / \|\beta\|_2 = 1$. It, thus, follows by part (a) of Assumption 4-3 that there exist constants $\gamma_1 > 0$ and $b_1 > 0$, such that for any $s > 0$ and for all $t \in \mathbb{Z}$,

$$P(|u'_t \underline{\beta}| > s) \leq \exp \left\{ 1 - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\}.$$

Hence, for any $s > 0$, for all $t \in \mathbb{Z}$, and for any integer i such that $1 \leq i \leq N$; we have

$$\begin{aligned} & P(|(X_{it} - \mu_{X,i}) u'_t \underline{\beta}| > s) \\ = & P(\{|X_{it} - \mu_{X,i}| |u'_t \underline{\beta}| > s\} \cap \{|u'_t \underline{\beta}| \leq \sqrt{s}\}) \\ & + P(\{|X_{it} - \mu_{X,i}| |u'_t \underline{\beta}| > s\} \cap \{|u'_t \underline{\beta}| > \sqrt{s}\}) \\ \leq & P(\sqrt{s} |X_{it} - \mu_{X,i}| > s) + P(\{|X_{it} - \mu_{X,i}| |u'_t \underline{\beta}| > s\} \cap \{|u'_t \underline{\beta}| > \sqrt{s}\}) \\ \leq & P(|X_{it} - \mu_{X,i}| > \sqrt{s}) + P(|u'_t \underline{\beta}| > \sqrt{s}) \\ \leq & \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_2} \right)^{\gamma_1} \right\} + \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_1} \right)^{\gamma_1} \right\} \\ \leq & 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_2} \right)^{\gamma_1} \right\} \end{aligned}$$

given that

$$\frac{b_2}{2b_1} \geq (1 + \ln 2)^{\frac{1}{\gamma_1}} \geq 1 \text{ implying that } b_2 \geq b_1.$$

Moreover, as mentioned previously, by definition, the positive constant b satisfies the inequality

$$\frac{b}{b_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \geq 1.$$

This, in turn, implies that when $s > b$

$$\begin{aligned}
P(|(X_{it} - \mu_{X,i}) u'_t \underline{\beta}| > s) &\leq 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_2} \right)^{\gamma_1} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} \left(\frac{b}{b_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{b} (1 + \ln 2)^{\frac{2}{\gamma_1}} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\quad \left(\text{given that } \frac{b}{b_2^2} > (1 + \ln 2)^{\frac{2}{\gamma_1}} \right) \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} (1 + \ln 2) \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} \ln 2 \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} - \ln 2 \right\} \quad \left(\text{since } \frac{s}{b} > 1 \text{ and } \frac{\gamma_1}{2} > 0 \right) \\
&= \frac{2}{2} \exp \left\{ 1 - \left(\frac{s}{b} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= \exp \left\{ 1 - \left(\frac{s}{b} \right)^r \right\} \quad \left(\text{by setting } r = \frac{\gamma_1}{2} \right) \\
&\leq \overline{C} \exp \left\{ - \left(\frac{s}{b} \right)^r \right\} \quad \left(\text{since } \overline{C} \geq \exp \{1\} \right)
\end{aligned}$$

On the other hand, when $0 < s \leq b$

$$\begin{aligned}
P(|(X_{it} - \mu_{X,i}) u'_t \underline{\beta}| > s) &\leq 1 \leq \exp \left\{ 1 - \left(\frac{s}{b} \right)^r \right\} \\
&= \exp \{1\} \exp \left\{ - \left(\frac{s}{b} \right)^r \right\} \\
&= \overline{C} \exp \left\{ - \left(\frac{s}{b} \right)^r \right\} \quad \left(\text{since } \overline{C} \geq \exp \{1\} \right)
\end{aligned}$$

which shows part (d). $\square$

Lemma C-4 extends Lemma A.3 part (i) of Fan, Liao, and Mincheva (2011) to the case where the underlying process may not be strictly stationary, in particular, to the case where the covariance matrix of X_t may be time-varying.

Lemma C-4. Suppose that Assumption 4-3 holds. Let

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2}$$

where γ_1 and γ_2 are the positive constants defined in Assumption 4-3 and suppose that

$$\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \text{ as } N, T \rightarrow \infty.$$

Under these conditions, there exists a constant $B_{11} > 0$ that does not depend on N and T such that

$$P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \bar{\Sigma}_{X,(i,j)} \right| \geq B_{11} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

where $T_h = T - h$ and where

$$\bar{\Sigma}_{X,(i,j)} = \frac{1}{T_h} \sum_{t=1}^{T_h} E [(X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j})] \text{ is the } (i, j)^{th} \text{ element of } \bar{\Sigma}_X.$$

Proof of Lemma C-4:

To proceed, first write

$$\begin{aligned} & \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \bar{\Sigma}_{X,(i,j)} \\ &= \frac{1}{T_h} \sum_{t=1}^{T_h} \{ (X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - E [(X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j})] \} \end{aligned}$$

from which it follows trivially that the elements of the sum given above all have expectation equal to zero, i.e.,

$$\begin{aligned} & E \{ (X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - E [(X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j})] \} \\ &= E [(X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j})] - E [(X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j})] \\ &= 0 \end{aligned}$$

Next, note that the result given in part (a) of Lemma C-3 shows that, for any $s > 0$, for all integers i and j such that $1 \leq i, j \leq N$ and for all $t \in \mathbb{Z}$, there exists a positive constant

$\overline{C} \geq \exp\{1\}$ such that

$$\begin{aligned} P(|(X_{it} - \mu_{X,i})(X_{jt} - \mu_{X,j})| > s) &\leq \exp\left\{1 - \left(\frac{s}{b}\right)^r\right\} \\ &\leq \overline{C} \exp\left\{-\left(\frac{s}{b}\right)^r\right\}, \end{aligned}$$

where $r = \gamma_1/2 > 0$ and $b \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} b_2^2 > 0$ and where γ_1 and b_2 are positive constants as defined, respectively, in Assumption 4-3 and in Lemma C-2. Note also that

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} \iff \frac{1}{\gamma_*} = \frac{\gamma_1 + 2\gamma_2}{\gamma_1 \gamma_2} = \frac{2}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r} + \frac{1}{\gamma_2},$$

where γ_2 is a positive constant defined in part (b) of Assumption 4-3. In addition, note that under Assumption 4-3

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} = \gamma < 1.$$

Moreover, let $Z_{i,j,t} = (X_{it} - \mu_{X,i})(X_{jt} - \mu_{X,j}) = (W_{it} + u_{it})(W_{jt} + u_{jt})$, and let $V_t = (\varepsilon_t, W'_t, u'_t)'$ be as defined in part (b) of Assumption 4-3. Note that $\mathcal{F}_{Z,i,j,-\infty}^t = \sigma(\dots, Z_{i,j,t-2}, Z_{i,j,t-1}, Z_{i,j,t}) \subseteq \mathcal{F}_{-\infty}^t = \sigma(\dots, V_{t-2}, V_{t-1}, V_t)$ and $\mathcal{F}_{Z,i,j,t+m}^\infty = \sigma(Z_{i,j,t+m}, Z_{i,j,t+m+1}, Z_{i,j,t+m+2}, \dots) \subseteq \mathcal{F}_{t+m}^\infty = \sigma(V_{t+m}, V_{t+m+1}, V_{t+m+2}, \dots)$. Hence, it follows by part (b) of Assumption 4-3 and Theorem 14.1 of Davidson (1994) that $\{Z_{i,j,t}\}_{t \in \mathbb{Z}}$ is an α -mixing process with α coefficient $\alpha_{Z,i,j}(m) = \sup_t \alpha(\mathcal{F}_{Z,i,j,-\infty}^t, \mathcal{F}_{Z,i,j,t+m}^\infty)$ satisfying the rate of decay

$$\begin{aligned} \alpha_{Z,i,j}(m) &= \sup_t \alpha(\mathcal{F}_{Z,i,j,-\infty}^t, \mathcal{F}_{Z,i,j,t+m}^\infty) \\ &\leq \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1; \text{ for all } i, j; \end{aligned}$$

and for some positive constants γ_2 , a and c .

Hence, all the assumptions needed to apply the Bernstein inequality for weakly dependent processes, originally obtained by Merlevède, Peligrad, and Rio (2011) and restated here as Lemma C-1, are fulfilled⁶. It then follows from applying the Bernstein inequality, that there exist constants C_1, C_2, C_3, C_4 and C_5 depending on a, b, c, γ_* and γ_2 (where a and c are positive constants defined in part (b) of Assumption 4-3) such that, for all integers i and j

⁶A proof of this inequality is given in Merlevède, Peligrad, and Rio (2011). See the proof of Theorem 1 in that paper.

with $1 \leq i, j \leq N$,

$$\begin{aligned}
& P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [(X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \bar{\Sigma}_{X,(i,j)}] \right| \geq s \right) \\
&= P \left(\left| \sum_{t=1}^{T_h} [(X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \bar{\Sigma}_{X,(i,j)}] \right| \geq T_h s \right) \\
&\leq T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} + \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} + \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_* (1-\gamma_*)}}{C_5 (\ln T_h s)^{\gamma_*}} \right\} \right\}
\end{aligned}$$

Applying Bonferroni's inequality, we further obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [(X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \bar{\Sigma}_{X,(i,j)}] \right| \geq s \right) \\
&\leq N^2 \max_{1 \leq i, j \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [(X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \bar{\Sigma}_{X,(i,j)}] \right| \geq s \right)
\end{aligned}$$

For some $B_{11} > 0$, let

$$s = B_{11} \sqrt{\frac{\ln N}{T_h}}.$$

and, note that, by assumption, $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$. By choosing B_{11} to be large enough, we can obtain the following three results, which we will show in turn:

(i)

$$N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} = N^2 T_h \exp \left\{ -\frac{B_{11}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(ii)

$$N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_* (1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(iii)

$$N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = N^2 \exp \left\{ -\frac{B_{11}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = o \left(\frac{1}{N^2} \right).$$

To show result (i), first write

$$\begin{aligned} N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} &= N^4 T_h \exp \left\{ -\frac{B_{11}^{\gamma_*}}{C_1} T_h^{\gamma_*} \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma_*}{2}} \right\} \\ &= N^4 T_h \exp \left\{ -\frac{B_{11}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \end{aligned}$$

Taking the log transformation, we have

$$\ln \left[N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} \right] = 4 \ln N + \ln T_h - \frac{B_{11}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}$$

Since

$$\frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0 \quad (\text{given that } \gamma_* > 0)$$

and since

$$\begin{aligned} \frac{4 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} &= 4 \left(\frac{T}{T_h} \right)^{\frac{\gamma_*}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \right)^{\frac{\gamma_*}{2}} \rightarrow 0 \\ &\left(\text{given the assumption } \frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \text{ and given that} \right. \\ &\quad \left. \left| \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty, \text{ since } h \text{ is a fixed finite integer} \right) \right) \end{aligned}$$

we see that

$$\frac{4 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} = \frac{4 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned} N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} &= \exp \left\{ 4 \ln N + \ln T_h - \frac{B_{11}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \\ &= \exp \left\{ -\frac{B_{11}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \left[1 - \frac{C_1}{B_{11}^{\gamma_*}} \frac{4 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \right] \right\} \\ &= o(1) \end{aligned}$$

It follows that

$$N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} = \frac{1}{N^2} N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (ii), note that

$$\begin{aligned}
& N^4 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} \\
= & N^4 \exp \left\{ -\frac{T_h^2}{C_4 T_h} \left(B_{11} \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma_*(1-\gamma_*)}}{C_5 \left(\ln \left[T_h B_{11} \sqrt{\frac{\ln N}{T_h}} \right] \right)^{\gamma_*}} \left(B_{11} \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma_*(1-\gamma_*)} \right) \right\} \\
= & N^4 \exp \left\{ -\frac{B_{11}^2}{C_4} \ln N \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} B_{11}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\ln T_h + \ln B_{11} + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h \right)^{\gamma_*}} \right) \right\} \\
= & N^4 \exp \left\{ -\frac{B_{11}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} B_{11}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln B_{11} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) \right\} \\
= & \exp \left\{ -\ln N \left[\frac{B_{11}^2}{C_4} \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} B_{11}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln B_{11} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) - 4 \right] \right\} \\
= & o(1).
\end{aligned}$$

This, in turn, implies that

$$N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

Turning our attention now to showing result (iii), we first write

$$\begin{aligned}
N^4 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} &= N^4 \exp \left\{ -\frac{T_h^2}{C_2 (1 + T_h C_3)} \left(B_{11} \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\
&= N^4 \exp \left\{ -\frac{B_{11}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\}
\end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned}
\ln \left[N^4 \exp \left\{ -\frac{B_{11}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \right] &= 4 \ln N - \frac{B_{11}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \\
&= 4 \ln N - \frac{B_{11}^2}{C_2 C_3} \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1}
\end{aligned}$$

Now, let B_{11} be a positive constant such that $B_{11}^2 > 4C_2(1 + C_3)$, and set

$$C^* = \frac{B_{11}^2}{C_2 C_3} - 4.$$

Note that, by this choice of B_{11} and C^* , we have

$$C^* > \frac{4C_2(1 + C_3)}{C_2 C_3} - 4 = \frac{4}{C_3} > 0$$

It, thus, follows that

$$\begin{aligned} & N^4 \exp \left\{ -\frac{T_h^2 s^2}{C_2(1 + T_h C_3)} \right\} \\ = & N^4 \exp \left\{ -\frac{B_{11}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \\ = & \exp \left\{ \left(4 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - \frac{B_{11}^2}{C_2 C_3} \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ = & \exp \left\{ \left(4 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - [C^* + 4] \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ = & \exp \left\{ \left(\ln N \left[4C_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ = & \exp \left\{ -C^* \ln N \left[1 - \frac{4}{C^* C_3} \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ < & \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{4}{C^* C_3} < \frac{C_3 4}{4C_3} = 1 \right) \\ = & o(1) \end{aligned}$$

This, in turn, implies that

$$N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2(1 + T_h C_3)} \right\} = N^2 \exp \left\{ -\frac{B_{11}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (iii)

Combining the results given in (i)-(iii) above, we obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [(X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \bar{\Sigma}_{X,(i,j)}] \right| \geq B_{11} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq N^2 \max_{1 \leq i, j \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [(X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \bar{\Sigma}_{X,(i,j)}] \right| \geq B_{11} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} + N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} \\
& \quad + N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_* (1 - \gamma_*)}}{C_5 (\ln T_h s)^{\gamma_*}} \right\} \right\} \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right). \quad \square
\end{aligned}$$

Lemma C-5: Suppose that Assumption 4-3 holds. Let

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2}$$

where γ_1 and γ_2 are the positive constants defined in Assumption 4-3 and suppose that

$$\frac{(\ln N)^{\frac{2}{\gamma_*} - 1}}{T} \rightarrow 0 \text{ as } N, T \rightarrow \infty.$$

Under these conditions, the following statements are true.

(a) There exists a constant $B_{12} > 0$ not depending on (N, T) such that

$$\begin{aligned}
& P \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq \sqrt{B_{12}} \sqrt{\frac{\ln N}{T_h}} \right\} = o \left(\frac{1}{N^2} \right) \\
& P \left\{ \max_{1 \leq i \leq N} \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right)^2 \geq B_{12} \frac{\ln N}{T_h} \right\} = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

(b) There exists a constant $B_{12} > 0$ not depending on (N, T) such that

$$P \left\{ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) \frac{1}{T_h} \sum_{r=1}^{T_h} (X_{jr} - \mu_{X,j}) \right| \geq B_{12} \sqrt{\frac{\ln N}{T_h}} \right\} = o \left(\frac{1}{N^2} \right).$$

(c) There exists a constant $B_{13} \geq 2 \max \{B_{11}, B_{12}\} > 0$ not depending on (N, T) such that

$$P \left(\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X, (i, j)} - \overline{\Sigma}_{X, (i, j)} \right| \geq B_{13} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right),$$

where B_{11} is the positive constant defined in Lemma C-4 above.

Proof of Lemma C-5:

To show part (a), we first show that there exists a positive constant C_* such that

$$P \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X, i}] \right| \geq C_* \sqrt{\frac{\ln N}{T_h}} \right\} = O \left(\frac{1}{N^2} \right)$$

To proceed, note first that, by Assumption 4-1, $E(X_{it} - \mu_{X, i}) = E[W_{it}] + E[u_{it}] = 0$. Moreover, we can apply Lemma C-2 to obtain that, for any $s > 0$, for all integer i such that $1 \leq i \leq N$, and for all $t \in \mathbb{Z}$,

$$\begin{aligned} P(|X_{it} - \mu_{X, i}| > s) &\leq \exp \left\{ 1 - \left(\frac{s}{b_2} \right)^{\gamma_1} \right\} \\ &\leq \overline{C} \exp \left\{ - \left(\frac{s}{b_2} \right)^{\gamma_1} \right\} \end{aligned}$$

Next, let $\underline{Z}_{i, t} = X_{it} - \mu_{X, i} = W_{it} + u_{it}$, and let $V_t = (\varepsilon_t, W'_t, u'_t)'$ be as defined in part (b) of Assumption 4-3. Note that $\mathcal{F}_{\underline{Z}, i, -\infty}^t = \sigma(\dots, \underline{Z}_{i, t-2}, \underline{Z}_{i, t-1}, \underline{Z}_{i, t}) \subseteq \mathcal{F}_{-\infty}^t = \sigma(\dots, V_{t-2}, V_{t-1}, V_t)$ and $\mathcal{F}_{\underline{Z}, i, t+m}^\infty = \sigma(\underline{Z}_{i, t+m}, \underline{Z}_{i, t+m+1}, \underline{Z}_{i, t+m+2}, \dots) \subseteq \mathcal{F}_{t+m}^\infty = \sigma(V_{t+m}, V_{t+m+1}, V_{t+m+2}, \dots)$. Hence, it follows by part (b) of Assumption 4-3 and Theorem 14.1 of Davidson (1994) that $\{\underline{Z}_{i, t}\}_{t \in \mathbb{Z}}$ is an α -mixing process with α coefficient $\alpha_{\underline{Z}, i}(m) = \sup_t \alpha(\mathcal{F}_{\underline{Z}, i, -\infty}^t, \mathcal{F}_{\underline{Z}, i, t+m}^\infty)$ satisfying the rate of decay

$$\begin{aligned} \alpha_{\underline{Z}, i}(m) &= \sup_t \alpha(\mathcal{F}_{\underline{Z}, i, -\infty}^t, \mathcal{F}_{\underline{Z}, i, t+m}^\infty) \\ &\leq \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1, \text{ for all } i, \end{aligned}$$

and for some positive constants γ_2 , a and c . In addition, by Assumption 4-3, there exists a constant $\gamma \in (0, 1)$ such that

$$\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.$$

Hence, all the assumptions needed to obtain the Bernstein inequality given in Merlevède, Peligrad, and Rio (2011) (and restated here as Lemma C-1) are fulfilled. It follows from

applying this inequality that there exist constants C_1, C_2, C_3, C_4 and C_5 which depend on a, b_2, c, γ , and γ_2 such that, for any $s > 0$ and for all integer i with $1 \leq i \leq N$,

$$\begin{aligned}
& P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq s \right) \\
&= P \left(\left| \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq T_h s \right) \\
&\leq T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} + \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} + \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln T_h s)^\gamma} \right\} \right\}
\end{aligned}$$

Making use of Bonferroni's inequality, we further obtain

$$\begin{aligned}
P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq s \right) &\leq N \max_{1 \leq i \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq s \right) \\
&\leq N T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} + N \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} \\
&\quad + N \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln T_h s)^\gamma} \right\} \right\}
\end{aligned}$$

For some $B_{12} > 0$, define $C_* = \sqrt{B_{12}}$ and let

$$s = \sqrt{B_{12}} \sqrt{\frac{\ln N}{T_h}}.$$

By choosing C_* to be large enough, we can obtain the following three results, which we will show in turn:

(i)

$$N T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} = N T_h \exp \left\{ -\frac{B_{12}^{\frac{\gamma}{2}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}}{C_1} \right\} = o \left(\frac{1}{N^2} \right).$$

(ii)

$$N \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(iii)

$$N \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = N \exp \left\{ -\frac{B_{12} \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = O \left(\frac{1}{N^2} \right).$$

To show result (i), first write

$$N^3 T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} = N^3 T_h \exp \left\{ -\frac{B_{12}^{\frac{\gamma}{2}} T_h^\gamma \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma}{2}}}{C_1} \right\} = N^3 T_h \exp \left\{ -\frac{B_{12}^{\frac{\gamma}{2}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}}{C_1} \right\}$$

Taking the log transformation, we have

$$\ln \left[N^3 T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} \right] = 3 \ln N + \ln T_h - \frac{B_{12}^{\frac{\gamma}{2}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}}{C_1}$$

Now, by assumption, $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$, where

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2}.$$

Since

$$\gamma = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} > \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} = \gamma_*$$

it follows that $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$ implies that $(\ln N)^{\frac{2}{\gamma}-1} = o(T)$. This, in turn, implies that

$$\begin{aligned} \frac{3 (\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} &= 3 \left(\frac{T}{T_h} \right)^{\frac{\gamma}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \right)^{\frac{\gamma}{2}} \rightarrow 0 \\ &\left(\text{given that } \frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \rightarrow 0 \text{ and given that} \right. \\ &\left. \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ since } h \text{ is a fixed finite integer} \right), \end{aligned} \quad (74)$$

Making use of expression (74) and the fact that

$$\frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0 \quad (\text{given that } \gamma > 0),$$

we further deduce that

$$\frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} = \frac{3 (\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned}
N^3 T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} &= \exp \left\{ 3 \ln N + \ln T_h - \frac{B_{12}^{\frac{\gamma}{2}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}}{C_1} \right\} \\
&= \exp \left\{ -\frac{B_{12}^{\frac{\gamma}{2}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}}{C_1} \left[1 - \frac{C_1}{B_{12}^{\frac{\gamma}{2}}} \frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \right] \right\} \\
&= o(1)
\end{aligned}$$

It follows that

$$N T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} = \frac{1}{N^2} N^3 T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (ii), note that

$$\begin{aligned}
&N^3 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} \\
&= N^3 \exp \left\{ -\frac{T_h^2}{C_4 T_h} \left(\sqrt{B_{12}} \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma(1-\gamma)} \left[\sqrt{B_{12}} \sqrt{\frac{\ln N}{T_h}} \right]^{\gamma(1-\gamma)}}{C_5 \left(\ln \left[T_h \sqrt{B_{12}} \sqrt{\frac{\ln N}{T_h}} \right] \right)^\gamma} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{B_{12}}{C_4} \ln N \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} B_{12}^{\frac{\gamma(1-\gamma)}{2}} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 \left(\ln T_h + \frac{1}{2} \ln B_{12} + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h \right)^\gamma} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{B_{12}}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} B_{12}^{\frac{\gamma(1-\gamma)}{2}} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \frac{1}{2} \ln B_{12} + \frac{1}{2} \ln \ln N \right)^\gamma} \right) \right\} \\
&= \exp \left\{ -\ln N \left[\frac{B_{12}}{C_4} \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} B_{12}^{\frac{\gamma(1-\gamma)}{2}} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \frac{1}{2} \ln B_{12} + \frac{1}{2} \ln \ln N \right)^\gamma} \right) - 3 \right] \right\} \\
&= o(1),
\end{aligned}$$

which, in turn, implies that

$$\begin{aligned}
N \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} &= \frac{1}{N^2} N^3 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} \\
&= o \left(\frac{1}{N^2} \right)
\end{aligned}$$

Turning our attention now to showing result (iii), we first write

$$\begin{aligned} N^3 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} &= N^3 \exp \left\{ -\frac{T_h^2}{C_2 (1 + T_h C_3)} \left(\sqrt{B_{12}} \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\ &= N^3 \exp \left\{ -\frac{B_{12} \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned} \ln \left[N^3 \exp \left\{ -\frac{B_{12} \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \right] &= 3 \ln N - \frac{B_{12} \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \\ &= 3 \ln N - \frac{B_{12}}{C_2 C_3} \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \end{aligned}$$

Now, let B_{12} be a positive constant such that $B_{12} > 3C_2 (1 + C_3)$, and set

$$C^* = \frac{B_{12}}{C_2 C_3} - 3.$$

Note that, by choosing B_{12} and C^* in this way, we have

$$C^* > \frac{3C_2 (1 + C_3)}{C_2 C_3} - 3 = \frac{3}{C_3} > 0$$

It, thus, follows that

$$\begin{aligned}
& N^3 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} \\
= & N^3 \exp \left\{ -\frac{B_{12} \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \\
= & \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - \frac{B_{12}}{C_2 C_3} \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - [C^* + 3] \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(\ln N \left[3 C_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ -C^* \ln N \left[1 - \frac{3}{C^* C_3} \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
< & \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{3}{C^* C_3} < \frac{C_3 3}{3 C_3} = 1 \right) \\
= & o(1)
\end{aligned}$$

This, in turn, implies that

$$N \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = \left(\frac{1}{N^2} \right) N^3 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (iii).

Combining the results given in (i)-(iii) above, we obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq \sqrt{B_{12}} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq N \max_{1 \leq i \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq \sqrt{B_{12}} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq NT_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} + N \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln T_h s)^\gamma} \right\} \right\} \\
& \quad + N \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

Finally, note that

$$\begin{aligned}
& P \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right|^2 \geq B_{12} \frac{\ln N}{T_h} \right\} \\
& = P \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq \sqrt{B_{12}} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

To show part (b), note that

$$\begin{aligned}
& P \left\{ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) \frac{1}{T_h} \sum_{r=1}^{T_h} (X_{jr} - \mu_{X,j}) \right| \geq B_{12} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \leq P \left\{ \sqrt{\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) \right|^2} \sqrt{\max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{r=1}^{T_h} (X_{jr} - \mu_{X,j}) \right|^2} \geq B_{12} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& = P \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) \right|^2 \geq B_{12}^2 \frac{\ln N}{T_h} \right\} \\
& \leq P \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) \right|^2 \geq B_{12}^2 \frac{\ln N}{T_h} \right\} \text{ for all } N \text{ and } T \text{ sufficiently large} \\
& \quad \left(\text{since } \frac{\ln N}{T_h} < \frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T_h} \rightarrow 0 \text{ so } \frac{\ln N}{T_h} \leq \sqrt{\frac{\ln N}{T_h}} \text{ for all } N, T \text{ sufficiently large} \right) \\
& = o \left(\frac{1}{N^2} \right) \quad (\text{given the result obtained in part (a)}).
\end{aligned}$$

To show part (c), note first that

$$\begin{aligned}
\hat{\Sigma}_X - \bar{\Sigma}_X &= \frac{1}{T_h} \sum_{t=1}^{T_h} (X_t - \bar{X}_{T_h}) (X_t - \bar{X}_{T_h})' - \bar{\Sigma}_X \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} (X_t - \mu_X + \mu_X - \bar{X}_{T_h}) (X_t - \mu_X + \mu_X - \bar{X}_{T_h})' - \bar{\Sigma}_X \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} (X_t - \mu_X) (X_t - \mu_X)' - \bar{\Sigma}_X - \frac{1}{T_h} \sum_{t=1}^{T_h} (X_t - \mu_X) (\bar{X}_{T_h} - \mu_X)' \\
&\quad - \frac{1}{T_h} \sum_{t=1}^{T_h} (\bar{X}_{T_h} - \mu_X) (X_t - \mu_X)' + \frac{1}{T_h} \sum_{t=1}^{T_h} (\bar{X}_{T_h} - \mu_X) (\bar{X}_{T_h} - \mu_X)' \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} (X_t - \mu_X) (X_t - \mu_X)' - \bar{\Sigma}_X - \frac{1}{T_h} \sum_{t=1}^{T_h} (X_t - \mu_X) \frac{1}{T_h} \sum_{s=1}^{T_h} (X_s - \mu_X)'
\end{aligned}$$

so that

$$\begin{aligned}
& \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} (X_t - \mu_X) (X_t - \mu_X)' e_{j,N} - e'_{i,N} \overline{\Sigma}_X e_{j,N} \\
&\quad - \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} (X_t - \mu_X) \frac{1}{T_h} \sum_{s=1}^{T_h} (X_s - \mu_X)' e_{j,N} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \overline{\Sigma}_{X,(i,j)} - \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) \frac{1}{T_h} \sum_{s=1}^{T_h} (X_{js} - \mu_{X,j})
\end{aligned}$$

Hence, applying the triangle inequality and taking the maximum over all $i, j \in \{1, \dots, N\}$; we obtain

$$\begin{aligned}
\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| &\leq \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \overline{\Sigma}_{X,(i,j)} \right| \\
&\quad + \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) \frac{1}{T_h} \sum_{s=1}^{T_h} (X_{js} - \mu_{X,j}) \right|
\end{aligned}$$

from which it follows that

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \geq B_{13} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \overline{\Sigma}_{X,(i,j)} \right| \right. \\
& \quad \left. + \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) \frac{1}{T_h} \sum_{s=1}^{T_h} (X_{js} - \mu_{X,j}) \right| \geq B_{13} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq P \left(\left\{ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \overline{\Sigma}_{X,(i,j)} \right| \geq \frac{B_{13}}{2} \sqrt{\frac{\ln N}{T_h}} \right\} \right. \\
& \quad \left. \cup \left\{ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) \frac{1}{T_h} \sum_{s=1}^{T_h} (X_{js} - \mu_{X,j}) \right| \geq \frac{B_{13}}{2} \sqrt{\frac{\ln N}{T_h}} \right\} \right) \\
& \leq P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \overline{\Sigma}_{X,(i,j)} \right| \geq \frac{B_{13}}{2} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad + P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) \frac{1}{T_h} \sum_{r=1}^{T_h} (X_{jr} - \mu_{X,j}) \right| \geq \frac{B_{13}}{2} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) (X_{jt} - \mu_{X,j}) - \overline{\Sigma}_{X,(i,j)} \right| \geq B_{11} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad + P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (X_{it} - \mu_{X,i}) \frac{1}{T_h} \sum_{r=1}^{T_h} (X_{jr} - \mu_{X,j}) \right| \geq B_{12} \sqrt{\frac{\ln N}{T_h}} \right) \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

where the next-to-last equality comes from applying Lemma C-4 and part (b) of this lemma.

□

Lemma C-6: Let

$$\eta_1 = B_1 \sqrt{\frac{\ln N}{T_h}}$$

for some sufficiently large constant $B_1 > 0^7$. Suppose that Assumptions 4-2, 4-3, 6-1, and 6-2 hold, and suppose that as $N, T \rightarrow \infty$

$$\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} = o(1).$$

Under these conditions, we have

$$\left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2 = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) = o_p(1)$$

for $0 \leq q < 1$. Here, $\hat{\Sigma}_X^S$ denotes the generalized thresholding estimator of $\bar{\Sigma}_X$, as defined previously in Section 6.

Proof of Lemma C-6:

To proceed, we start with the decomposition

$$\begin{aligned} \left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2 &= \left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X^S + \bar{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2 \\ &\leq \left\| \frac{\bar{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2 + \left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X^S}{\lambda_1} \right\|_2. \end{aligned} \quad (75)$$

where $\bar{\Sigma}_X^S$ is an $N \times N$ matrix whose $(i, j)^{th}$ element is given by $\bar{\Sigma}_{X,(i,j)}^S = s_{\eta_1}(\bar{\Sigma}_{X,(i,j)})$. We shall consider separately the case where $0 < q < 1$ (Case 1) and the case where $q = 0$ (Case 2).

Case 1 $0 < q < 1$:

For this case, consider initially the first term on the right-hand side of expression (75) above, that is, the term

$$\left\| \frac{\bar{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2.$$

⁷See the proof of Lemma C-6 for a more detailed specification of the condition that the positive constant B_1 must satisfy.

Making use of conditions (ii) and (iii) of Assumption 6-1, we have

$$\begin{aligned}
& \left\| \frac{\bar{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2 \\
& \leq \sqrt{\left\| \frac{\bar{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_1 \left\| \frac{\bar{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_\infty} \\
& = \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \frac{s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) - \bar{\Sigma}_{X,(i,j)}}{\lambda_1} \right| \left(\text{given that } \frac{\bar{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \text{ is symmetric} \right) \\
& = \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \left(\sum_{j=1}^N |s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} + s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\} \right. \\
& \quad \left. - \bar{\Sigma}_{X,(i,j)} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} - \bar{\Sigma}_{X,(i,j)} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\} \right|) \\
& = \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \left(\sum_{j=1}^N |s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} - \bar{\Sigma}_{X,(i,j)} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} \right. \\
& \quad \left. - \bar{\Sigma}_{X,(i,j)} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\} \right|) \\
& \quad (\text{since } s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) = 0 \text{ for } |\bar{\Sigma}_{X,(i,j)}| \leq \eta_1 \text{ by condition (ii) of Assumption 6-1}) \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \left(\sum_{j=1}^N |s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) - \bar{\Sigma}_{X,(i,j)}| \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} + \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}| \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\} \right) \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \left(\sum_{j=1}^N \eta_1 \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} + \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}| \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\} \right) \\
& \quad (\text{since } |s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) - \bar{\Sigma}_{X,(i,j)}| \leq \eta_1 \text{ by condition (iii) of Assumption 6-1}) \\
& = \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \left(\sum_{j=1}^N \eta_1^q \eta_1^{1-q} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} + \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}|^q |\bar{\Sigma}_{X,(i,j)}|^{1-q} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\} \right) \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \left(\sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}|^q \eta_1^{1-q} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} + \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}|^q \eta_1^{1-q} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\} \right) \\
& = \frac{\eta_1^{1-q}}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}|^q
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left\| \frac{\bar{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2 \\
& \leq \frac{\eta_1^{1-q}}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}|^q \quad (\text{for } 0 < q < 1) \\
& \leq \frac{\eta_1^{1-q} c_0(N)}{\lambda_1} \quad (\text{since } \bar{\Sigma}_X \in \mathcal{U}_\Sigma(q, c_0(N), \mathbb{M}) \text{ by part (a) of Assumption 6-2}) \\
& = O\left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right) \quad \left(\text{since } \eta_1^{1-q} = (\bar{C})^{1-q} \left(\frac{\ln N}{T_h}\right)^{\frac{1-q}{2}} \asymp \left[\frac{\ln N}{T}\right]^{\frac{1-q}{2}}\right). \quad (76)
\end{aligned}$$

Next, consider the second term on the right-hand side of expression (75) above, i.e., the term

$$\left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X^S}{\lambda_1} \right\|_2.$$

Here, note that, by conditions (i) and (ii) of Assumption 6-1, we have

$$\begin{aligned}
& \left\| \frac{\widehat{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \right\|_2 \\
& \leq \sqrt{\left\| \frac{\widehat{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \right\|_1 \left\| \frac{\widehat{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \right\|_\infty} \\
& = \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \frac{s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)})}{\lambda_1} \right| \left(\text{given that } \frac{\widehat{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \text{ is also symmetric} \right) \\
& = \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \quad (\text{since, by} \\
& \quad \text{Assumption 6-1(ii), } s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) = 0 \text{ if } \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \text{ and } s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) = 0 \text{ if } \left| \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1) \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \quad (\text{since, by} \\
& \quad \text{Assumption 6-1 part (i), } \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) \right| \leq \left| \widehat{\Sigma}_{X,(i,j)} \right| \text{ and } \left| s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \leq \left| \overline{\Sigma}_{X,(i,j)} \right|)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \hat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \bar{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left[\left(\left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| + \left| s_{\eta_1} \left(\hat{\Sigma}_{X,(i,j)} \right) - \hat{\Sigma}_{X,(i,j)} \right| + \left| s_{\eta_1} \left(\bar{\Sigma}_{X,(i,j)} \right) - \bar{\Sigma}_{X,(i,j)} \right| \right) \right. \\
&\quad \quad \left. \times \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \right] \\
&\quad \text{(by the triangle inequality)} \\
&= (1) + (2) + (3) \tag{77}
\end{aligned}$$

To bound term (1), write

$$\begin{aligned}
(1) &= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \hat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} + \bar{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \bar{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \bar{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \tag{78}
\end{aligned}$$

To bound the second term in the expression above, note that

$$\begin{aligned}
& \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}| \mathbb{I} \{ |\bar{\Sigma}_{X,(i,j)}| \leq \eta_1 \} \\
&= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}|^{1-q} |\bar{\Sigma}_{X,(i,j)}|^q \mathbb{I} \{ |\bar{\Sigma}_{X,(i,j)}| \leq \eta_1 \} \\
&< \frac{\eta_1^{1-q}}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}|^q \mathbb{I} \{ |\bar{\Sigma}_{X,(i,j)}| \leq \eta_1 \} \\
&\leq \frac{\eta_1^{1-q}}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}|^q \\
&\leq \frac{\eta_1^{1-q} c_0(N)}{\lambda_1} \quad (\text{since, by Assumption 6-2 part (a), } \bar{\Sigma}_X \in \mathcal{U}_\Sigma(q, c_0(N), \mathbb{M})).
\end{aligned}$$

Next, take $\vartheta \in (0, 1)$, and we can bound the first term in expression (78) as follows

$$\begin{aligned}
& \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \vartheta \eta_1 < \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \frac{\left| \overline{\Sigma}_{X,(i,j)} \right|^q}{\left| \overline{\Sigma}_{X,(i,j)} \right|^q} \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \vartheta \eta_1 < \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1 (\vartheta \eta_1)^q} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \overline{\Sigma}_{X,(i,j)} \right|^q \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
&\quad + \frac{c_0(N)}{\lambda_1 (\vartheta \eta_1)^q} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \quad (\text{since by part (a) of Assumption 6-2} \\
&\quad \overline{\Sigma}_X \in \mathcal{U}_\Sigma(q, c_0(N), \mathbb{M}) = \left\{ \Sigma_{N \times N} : \max_{1 \leq i \leq N} \Sigma_{(i,i)} \leq \mathbb{M}, \max_{1 \leq i \leq N} \sum_{j=1}^N |\Sigma_{(i,j)}|^q \leq c_0(N) \right\})
\end{aligned}$$

Moreover, observe that

$$\begin{aligned}
& \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| + \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
& = \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 - \left| \overline{\Sigma}_{X,(i,j)} \right|, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 - \vartheta \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > (1 - \vartheta) \eta_1 \right\}.
\end{aligned}$$

Now, if we choose

$$\eta_1 = B_1 \sqrt{\frac{\ln N}{T_h}}$$

where $B_1 = B_{13}/(1 - \vartheta)$ and where B_{13} is the positive constant defined in part (c) of Lemma C-5; then, by Lemma C-5 part (c), we have

$$\begin{aligned}
& P \left\{ \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > (1 - \vartheta) \eta_1 \right\} > 0 \right\} \\
& = P \left[\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > (1 - \vartheta) \eta_1 \right] \\
& = P \left(\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \geq B_{13} \sqrt{\frac{\ln N}{T_h}} \right) \\
& = o \left(\frac{1}{N^2} \right),
\end{aligned}$$

uniformly on $\mathcal{U}_\Sigma(q, c_0(N), \mathbb{M})$. Making use of the fact that

$$P \left\{ \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > (1 - \vartheta) \eta_1 \right\} > 0 \right\} = o \left(\frac{1}{N^2} \right) = o(1) \text{ as } N, T \rightarrow \infty$$

we further obtain

$$\begin{aligned}
& \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i,j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
& \quad + \frac{c_0(N)}{\lambda_1 (\vartheta \eta_1)^q} \max_{1 \leq i,j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i,j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \Sigma_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > (1 - \vartheta) \eta_1 \right\} \\
& \quad + \frac{c_0(N)}{\lambda_1 (\vartheta \eta_1)^q} \max_{1 \leq i,j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \\
& = \frac{1}{\lambda_1} \max_{1 \leq i,j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > (1 - \vartheta) \eta_1 \right\} \\
& \quad + \frac{c_0(N)}{\lambda_1 \vartheta^q} \left(\frac{1}{\overline{C}} \right)^q \left(\frac{T_h}{\ln N} \right)^{\frac{q}{2}} \max_{1 \leq i,j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \\
& = o_p \left(\frac{1}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) + O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \\
& \quad \left(\text{given that } \max_{1 \leq i,j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \Sigma_{X,(i,j)} \right| = O_p \left(\sqrt{\frac{\ln N}{T}} \right) \right) \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right). \tag{79}
\end{aligned}$$

Combining the result given in expression (79) with the fact that

$$\begin{aligned}
\frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} & \leq \frac{\eta_1^{1-q} c_0(N)}{\lambda_1} \\
& = (\overline{C})^{1-q} \left(\frac{\ln N}{T_h} \right)^{\frac{1-q}{2}} \frac{c_0(N)}{\lambda_1} \\
& = O \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right)
\end{aligned}$$

we have

$$\begin{aligned}
(1) &= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \hat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \bar{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \bar{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&= O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) + O \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \\
&= O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right). \tag{80}
\end{aligned}$$

Next, for term (2), we have

$$\begin{aligned}
(2) &= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \bar{\Sigma}_{X,(i,j)} - \hat{\Sigma}_{X,(i,j)} + \hat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left[\left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| + \left| \hat{\Sigma}_{X,(i,j)} \right| \right] \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \hat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\quad + \frac{\eta_1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \left[\max_{1 \leq i, j \leq N} \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| + \eta_1 \right] \max_{1 \leq i \leq N} \sum_{j=1}^N \frac{|\bar{\Sigma}_{X,(i,j)}|^q}{|\bar{\Sigma}_{X,(i,j)}|^q} \mathbb{I} \left\{ |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \left[\max_{1 \leq i, j \leq N} \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| + \eta_1 \right] \frac{1}{\eta_1^q} \max_{1 \leq i \leq N} \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}|^q \\
&\leq \frac{c_0(N)}{\lambda_1 \eta_1^q} \max_{1 \leq i, j \leq N} \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| + \frac{\eta_1^{1-q}}{\lambda_1} c_0(N) \\
&\quad \left(\text{since } \bar{\Sigma}_X \in \mathcal{U}_\Sigma(q, c_0(N), \mathbb{M}) = \left\{ \Sigma_{N \times N} : \max_{1 \leq i \leq N} \Sigma_{(i,i)} \leq \mathbb{M}, \max_{1 \leq i \leq N} \sum_{j=1}^N |\Sigma_{(i,j)}|^q \leq c_0(N) \right\} \right) \\
&= O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) + O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \left(\text{since } \eta_1^q \asymp \left[\frac{\ln N}{T} \right]^{\frac{q}{2}} \right) \\
&\quad \text{and } \max_{1 \leq i, j \leq N} \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| = O_p \left(\sqrt{\frac{\ln N}{T}} \right) \text{ by Lemma C-5 part (c)} \\
&= O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right)
\end{aligned} \tag{81}$$

We now turn our attention to term (3) in expression (77), which we can rewrite as follow

$$\begin{aligned}
& (3) \\
&= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left[\left(\left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| + \left| s_{\eta_1} \left(\widehat{\Sigma}_{X,(i,j)} \right) - \widehat{\Sigma}_{X,(i,j)} \right| + \left| s_{\eta_1} \left(\overline{\Sigma}_{X,(i,j)} \right) - \overline{\Sigma}_{X,(i,j)} \right| \right) \right. \\
&\quad \left. \times \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \right] \\
&= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1} \left(\widehat{\Sigma}_{X,(i,j)} \right) - \widehat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1} \left(\overline{\Sigma}_{X,(i,j)} \right) - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \tag{82}
\end{aligned}$$

Focusing initially on the first-term on the right-hand side of expression (82) above, note that by applying part (c) of Lemma C-5, i.e.,

$$\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| = O_p \left(\sqrt{\frac{\ln N}{T}} \right),$$

we obtain

$$\begin{aligned}
& \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \frac{\left| \overline{\Sigma}_{X,(i,j)} \right|^q}{\left| \overline{\Sigma}_{X,(i,j)} \right|^q} \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \frac{\left| \overline{\Sigma}_{X,(i,j)} \right|^q}{\eta_1^q} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \frac{c_0(N)}{\eta_1^q} \\
& \quad \left(\text{since } \overline{\Sigma}_X \in \mathcal{U}_\Sigma(q, c_0(N), \mathbb{M}) = \left\{ \Sigma_{N \times N} : \max_{1 \leq i \leq N} \Sigma_{(i,i)} \leq \mathbb{M}, \max_{1 \leq i \leq N} \sum_{j=1}^N |\Sigma_{(i,j)}|^q \leq c_0(N) \right\} \right) \\
& = O_p \left(\frac{c_0(N)}{\lambda_1 \eta_1^q} \sqrt{\frac{\ln N}{T}} \right) \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \left(\text{since } \eta_1^q = \overline{C}^q \left(\frac{\ln N}{T_h} \right)^{\frac{q}{2}} \asymp \left[\frac{\ln N}{T} \right]^{\frac{q}{2}} \right). \tag{83}
\end{aligned}$$

Now, for the second term on the right-hand side of expression (82), note that, by applying condition (iii) of Assumption 6-1, we have

$$\begin{aligned}
& \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1} \left(\widehat{\Sigma}_{X,(i,j)} \right) - \widehat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \eta_1 \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad \left(\text{since, by condition (iii) of Assumption 6-1, } \left| s_{\eta_1} \left(\widehat{\Sigma}_{X,(i,j)} \right) - \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right) \\
& = \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \eta_1^q \eta_1^{1-q} \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \leq \frac{\eta_1^{1-q}}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \overline{\Sigma}_{X,(i,j)} \right|^q \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \leq \frac{\eta_1^{1-q}}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \overline{\Sigma}_{X,(i,j)} \right|^q \\
& \leq \frac{\eta_1^{1-q} c_0(N)}{\lambda_1} \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \left(\text{since } \eta_1^{1-q} = \overline{C}^{1-q} \left(\frac{\ln N}{T_h} \right)^{\frac{1-q}{2}} \asymp \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \quad (84)
\end{aligned}$$

Finally, for the third term on the right-hand side of expression (82), note that, similar to the argument given for the second term of expression (82),

$$\begin{aligned}
& \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N |s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) - \bar{\Sigma}_{X,(i,j)}| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \eta_1 \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad (\text{since, by condition (iii) of Assumption 6-1, } |s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) - \bar{\Sigma}_{X,(i,j)}| \leq \eta_1) \\
& = \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \eta_1^q \eta_1^{1-q} \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \leq \frac{\eta_1^{1-q}}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \bar{\Sigma}_{X,(i,j)} \right|^q \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \leq \frac{\eta_1^{1-q}}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \bar{\Sigma}_{X,(i,j)} \right|^q \\
& \leq \frac{\eta_1^{1-q} c_0(N)}{\lambda_1} \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \quad \left(\text{since } \eta_1^{1-q} = \bar{C}^{1-q} \left(\frac{\ln N}{T_h} \right)^{\frac{1-q}{2}} \asymp \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \quad (85)
\end{aligned}$$

Combining the results given in expressions (83)-(85), we obtain

$$\begin{aligned}
& (3) \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\hat{\Sigma}_{X,(i,j)}) - \hat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) - \bar{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) + O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) + O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \quad (86)
\end{aligned}$$

It further follows from expressions (77), (80), (81), and (86)

$$\begin{aligned}
& \left\| \frac{\widehat{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \right\|_2 \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left[\left(\left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| + \left| s_{\eta_1} \left(\widehat{\Sigma}_{X,(i,j)} \right) - \widehat{\Sigma}_{X,(i,j)} \right| + \left| s_{\eta_1} \left(\overline{\Sigma}_{X,(i,j)} \right) - \overline{\Sigma}_{X,(i,j)} \right| \right) \\
& \quad \quad \times \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \right] \quad (\text{by the triangle inequality}) \\
& = (1) + (2) + (3) \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) + O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) + O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right). \tag{87}
\end{aligned}$$

Finally, applying the results given in expressions (76) and (87) to expression (75), we obtain the desired result

$$\begin{aligned}
\left\| \frac{\widehat{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \right\|_2 & \leq \left\| \frac{\overline{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \right\|_2 + \left\| \frac{\widehat{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \right\|_2 \\
& = O \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) + O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right).
\end{aligned}$$

Case 2: $q = 0$:

For this case, consider again the term

$$\left\| \frac{\overline{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \right\|_2$$

on the right-hand side of expression (75). Under conditions (ii) and (iii) of Assumption 6-1, we have

$$\begin{aligned}
& \left\| \frac{\bar{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2 \\
& \leq \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \frac{s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) - \bar{\Sigma}_{X,(i,j)}}{\lambda_1} \right| \left(\text{given that } \frac{\bar{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \text{ is symmetric} \right) \\
& = \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N (|s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} + s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\} \\
& \quad - \bar{\Sigma}_{X,(i,j)} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} - \bar{\Sigma}_{X,(i,j)} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\}|) \\
& = \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \left(\sum_{j=1}^N |s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} - \bar{\Sigma}_{X,(i,j)} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} \right. \\
& \quad \left. - \bar{\Sigma}_{X,(i,j)} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\} \right) \\
& \quad (\text{since } s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) = 0 \text{ for } |\bar{\Sigma}_{X,(i,j)}| \leq \eta_1 \text{ by condition (ii) of Assumption 6-1}) \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \left(\sum_{j=1}^N |s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) - \bar{\Sigma}_{X,(i,j)}| \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} + \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}| \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\} \right) \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \left(\sum_{j=1}^N \eta_1 \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} + \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}| \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\} \right) \\
& \quad (\text{since } |s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) - \bar{\Sigma}_{X,(i,j)}| \leq \eta_1 \text{ by condition (iii) of Assumption 6-1}) \\
& = \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \left(\sum_{j=1}^N \eta_1 \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \neq 0\} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} \right. \\
& \quad \left. + \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}| \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \neq 0\} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\} \right) \\
& \leq \frac{\eta_1}{\lambda_1} \max_{1 \leq i \leq N} \left(\sum_{j=1}^N \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \neq 0\} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| > \eta_1\} + \sum_{j=1}^N \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \neq 0\} \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \leq \eta_1\} \right) \\
& = \frac{2\eta_1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I}\{|\bar{\Sigma}_{X,(i,j)}| \neq 0\}
\end{aligned}$$

It follows that, for $q = 0$,

$$\begin{aligned}
\left\| \frac{\bar{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2 &\leq \frac{2\eta_1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I}\{|\Sigma_{X,(i,j)}| \neq 0\} \\
&\leq \frac{2\eta_1 c_0(N)}{\lambda_1} \quad (\text{given part (a) of Assumption 6-2 for the case where } q = 0) \\
&= O\left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}}\right) \quad \left(\text{since } \eta_1 = \overline{C} \left(\frac{\ln N}{T_h}\right)^{\frac{1}{2}} \asymp \sqrt{\frac{\ln N}{T}}\right). \tag{88}
\end{aligned}$$

Next, consider, for the $q = 0$ case, the second term on the right-hand side of expression (75) above, i.e., the term

$$\left\| \frac{\widehat{\Sigma}_X^S - \bar{\Sigma}_X^S}{\lambda_1} \right\|_2.$$

By conditions (i) and (ii) of Assumption 6-1, we have

$$\begin{aligned}
& \left\| \frac{\widehat{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \right\|_2 \\
& \leq \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \frac{s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)})}{\lambda_1} \right| \left(\text{given that } \frac{\widehat{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \text{ is symmetric} \right) \\
& = \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \quad (\text{by condition (ii)}) \\
& \quad \text{of Assumption 6-1, } s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) = 0 \text{ if } \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \text{ and } s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) = 0 \text{ if } \left| \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) - s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad \left(\text{since, by condition (i) of Assumption 6-1, } \left| s_{\eta_1}(\widehat{\Sigma}_{X,(i,j)}) \right| \leq \left| \widehat{\Sigma}_{X,(i,j)} \right| \right. \\
& \quad \left. \text{and } \left| s_{\eta_1}(\overline{\Sigma}_{X,(i,j)}) \right| \leq \left| \overline{\Sigma}_{X,(i,j)} \right| \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left[\left(\left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| + \left| s_{\eta_1} \left(\widehat{\Sigma}_{X,(i,j)} \right) - \widehat{\Sigma}_{X,(i,j)} \right| + \left| s_{\eta_1} \left(\overline{\Sigma}_{X,(i,j)} \right) - \overline{\Sigma}_{X,(i,j)} \right| \right) \right. \\
&\quad \quad \left. \times \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \right] \quad (\text{by the triangle inequality}) \\
&= [1] + [2] + [3] \tag{89}
\end{aligned}$$

To bound term [1], write

$$\begin{aligned}
[1] &= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} + \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \tag{90}
\end{aligned}$$

To bound the second term in the expression above, note that

$$\begin{aligned}
&\frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \neq 0 \right\} \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&< \frac{\eta_1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \neq 0 \right\} \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&\leq \frac{\eta_1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \neq 0 \right\} \\
&\leq \frac{\eta_1 c_0(N)}{\lambda_1} \quad (\text{since } \overline{\Sigma}_X \in \mathcal{U}_\Sigma(0, c_0(N), \mathbb{M}) \text{ by Assumption 6-2 part (a)}) .
\end{aligned}$$

Next, take $\vartheta \in (0, 1)$, and we can bound the first term in expression (90) as follows

$$\begin{aligned}
& \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \vartheta \eta_1 < \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \neq 0 \right\} \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \vartheta \eta_1 < \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \neq 0 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
&\quad + \frac{c_0(N)}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \quad (\text{since } \overline{\Sigma}_X \in \mathcal{U}_\Sigma(0, c_0(N), \mathbb{M})) \\
&= \left\{ \sum_{N \times N} : \max_{1 \leq i \leq N} \Sigma_{(i,i)} \leq \mathbb{M}, \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \Sigma_{(i,j)} \right| \neq 0 \right\} \leq c_0(N) \right\}
\end{aligned} \tag{91}$$

Moreover, observe that

$$\begin{aligned}
& \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| + \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
& = \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 - \left| \overline{\Sigma}_{X,(i,j)} \right|, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 - \vartheta \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > (1 - \vartheta) \eta_1 \right\},
\end{aligned}$$

Again, if we choose

$$\eta_1 = B_1 \sqrt{\frac{\ln N}{T_h}}$$

where $B_1 = B_{13}/(1 - \vartheta)$ and where B_{13} is the positive constant defined in part (c) of Lemma C-5; then, by Lemma C-5 part (c), we have

$$\begin{aligned}
& P \left\{ \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > (1 - \vartheta) \eta_1 \right\} > 0 \right\} \\
& = P \left[\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > (1 - \vartheta) \eta_1 \right] \\
& = P \left(\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \geq B_{13} \sqrt{\frac{\ln N}{T_h}} \right) \\
& = o \left(\frac{1}{N^2} \right),
\end{aligned}$$

uniformly on

$$\mathcal{U}_\Sigma(0, c_0(N), \mathbb{M}) = \left\{ \Sigma_{N \times N} : \max_{1 \leq i \leq N} \Sigma_{(i,i)} \leq \mathbb{M}, \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \{ |\Sigma_{(i,j)}| \neq 0 \} \leq c_0(N) \right\}$$

Making use of the fact that

$$P \left[\max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > (1 - \vartheta) \eta_1 \right\} > 0 \right] = o \left(\frac{1}{N^2} \right) = o(1) \text{ as } N, T \rightarrow \infty$$

we further obtain

$$\begin{aligned} & \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\ & \leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \vartheta \eta_1 \right\} \\ & \quad + \frac{c_0(N)}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \quad (\text{by expression (91)}) \\ & \leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| > (1 - \vartheta) \eta_1 \right\} \\ & \quad + \frac{c_0(N)}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \\ & = o_p \left(\frac{1}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) + O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) \\ & \quad \left(\text{given that } \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| = O_p \left(\sqrt{\frac{\ln N}{T}} \right) \right) \\ & = O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right). \end{aligned} \tag{92}$$

Combining the result given in expression (92) with the fact that

$$\begin{aligned}
& \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}| \mathbb{I} \{ |\bar{\Sigma}_{X,(i,j)}| \leq \eta_1 \} \\
&= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}| \mathbb{I} \{ |\bar{\Sigma}_{X,(i,j)}| \neq 0 \} \mathbb{I} \{ |\bar{\Sigma}_{X,(i,j)}| \leq \eta_1 \} \\
&\leq \frac{\eta_1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \{ |\bar{\Sigma}_{X,(i,j)}| \neq 0 \} \\
&\leq \frac{\eta_1 c_0(N)}{\lambda_1} \\
&= \bar{C} \sqrt{\frac{\ln N}{T_h}} \frac{c_0(N)}{\lambda_1} \\
&= O \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right)
\end{aligned}$$

we have

$$\begin{aligned}
[1] &= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \hat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, |\bar{\Sigma}_{X,(i,j)}| \leq \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, |\bar{\Sigma}_{X,(i,j)}| \leq \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}| \mathbb{I} \{ |\bar{\Sigma}_{X,(i,j)}| \leq \eta_1 \} \\
&= O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) + O \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) \\
&= O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right). \tag{93}
\end{aligned}$$

Next, for term [2], we have

$$\begin{aligned}
[2] &= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N |\bar{\Sigma}_{X,(i,j)}| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \bar{\Sigma}_{X,(i,j)} - \hat{\Sigma}_{X,(i,j)} + \hat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left[\left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| + \left| \hat{\Sigma}_{X,(i,j)} \right| \right] \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i,j \leq N} \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \hat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i,j \leq N} \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\quad + \frac{\eta_1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&= \frac{1}{\lambda_1} \max_{1 \leq i,j \leq N} \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ |\bar{\Sigma}_{X,(i,j)}| \neq 0 \right\} \mathbb{I} \left\{ |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\quad + \frac{\eta_1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ |\bar{\Sigma}_{X,(i,j)}| \neq 0 \right\} \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, |\bar{\Sigma}_{X,(i,j)}| > \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \left[\max_{1 \leq i,j \leq N} \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| + \eta_1 \right] \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ |\bar{\Sigma}_{X,(i,j)}| \neq 0 \right\} \\
&\leq \frac{c_0(N)}{\lambda_1} \max_{1 \leq i,j \leq N} \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| + \frac{\eta_1}{\lambda_1} c_0(N) \quad (\text{since } \bar{\Sigma}_X \in \mathcal{U}_\Sigma(0, c_0(N), \mathbb{M})) \\
&= O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) + O \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) \\
&\quad \left(\text{since } \eta_1 = \bar{C} \sqrt{\frac{\ln N}{T_h}} \asymp \sqrt{\frac{\ln N}{T}} \text{ and } \max_{1 \leq i,j \leq N} \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| = O_p \left(\sqrt{\frac{\ln N}{T}} \right) \right) \\
&= O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) \tag{94}
\end{aligned}$$

We now turn our attention to term [3] in expression (89), which we can rewrite as follow

$$\begin{aligned}
& [3] \\
&= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left[\left(\left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| + \left| s_{\eta_1} \left(\widehat{\Sigma}_{X,(i,j)} \right) - \widehat{\Sigma}_{X,(i,j)} \right| + \left| s_{\eta_1} \left(\overline{\Sigma}_{X,(i,j)} \right) - \overline{\Sigma}_{X,(i,j)} \right| \right) \right. \\
&\quad \left. \times \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \right] \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1} \left(\widehat{\Sigma}_{X,(i,j)} \right) - \widehat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
&\quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1} \left(\overline{\Sigma}_{X,(i,j)} \right) - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \tag{95}
\end{aligned}$$

Focusing initially on the first-term on the right-hand side of expression (95) above, note that by applying part (c) of Lemma C-5, i.e., the result

$$\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| = O_p \left(\sqrt{\frac{\ln N}{T}} \right),$$

we obtain

$$\begin{aligned}
& \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
&= \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \neq 0 \right\} \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \neq 0 \right\} \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \neq 0 \right\} \\
&\leq \frac{1}{\lambda_1} \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| c_0(N) \quad (\text{since } \overline{\Sigma}_X \in \mathcal{U}_\Sigma(0, c_0(N), \mathbb{M})) \\
&= O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) \left(\text{since } \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| = O_p \left(\sqrt{\frac{\ln N}{T}} \right) \right). \tag{96}
\end{aligned}$$

Now, for the second term on the right-hand side of expression (95), note that, by applying condition (iii) of Assumption 6-1, we have

$$\begin{aligned}
& \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1} \left(\widehat{\Sigma}_{X,(i,j)} \right) - \widehat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \eta_1 \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad \left(\text{since, by condition (iii) of Assumption 6-1, } \left| s_{\eta_1} \left(\widehat{\Sigma}_{X,(i,j)} \right) - \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right) \\
& = \frac{\eta_1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \Sigma_{X,(i,j)} \right| \neq 0 \right\} \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \leq \frac{\eta_1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \left\{ \left| \overline{\Sigma}_{X,(i,j)} \right| \neq 0 \right\} \\
& \leq \frac{\eta_1 c_0(N)}{\lambda_1} \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) \quad \left(\text{since } \eta_1 = \overline{C} \sqrt{\frac{\ln N}{T_h}} \asymp \sqrt{\frac{\ln N}{T}} \right) \tag{97}
\end{aligned}$$

Finally, for the third term on the right-hand side of expression (95), note that, similar to the argument given for the second term of expression ((95)),

$$\begin{aligned}
& \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N |s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) - \bar{\Sigma}_{X,(i,j)}| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \eta_1 \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad \text{(since, by condition (iii) of Assumption 6-1, } |s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) - \bar{\Sigma}_{X,(i,j)}| \leq \eta_1 \text{)} \\
& = \frac{\eta_1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \{ |\bar{\Sigma}_{X,(i,j)}| \neq 0 \} \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \leq \frac{\eta_1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \mathbb{I} \{ |\bar{\Sigma}_{X,(i,j)}| \neq 0 \} \\
& \leq \frac{\eta_1 c_0(N)}{\lambda_1} \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) \quad \left(\text{since } \eta_1 = \bar{C} \sqrt{\frac{\ln N}{T_h}} \asymp \sqrt{\frac{\ln N}{T}} \right) \tag{98}
\end{aligned}$$

Combining the results given in expressions (96)-(98), we obtain

$$\begin{aligned}
& [3] \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \hat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\hat{\Sigma}_{X,(i,j)}) - \hat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| s_{\eta_1}(\bar{\Sigma}_{X,(i,j)}) - \bar{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \hat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \bar{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) + O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) + O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) \tag{99}
\end{aligned}$$

It further follows from expressions (89), (93), (94), and (99)

$$\begin{aligned}
& \left\| \frac{\widehat{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \right\|_2 \\
& \leq \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \widehat{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| \leq \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left| \overline{\Sigma}_{X,(i,j)} \right| \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| \leq \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \\
& \quad + \frac{1}{\lambda_1} \max_{1 \leq i \leq N} \sum_{j=1}^N \left[\left(\left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| + \left| s_{\eta_1} \left(\widehat{\Sigma}_{X,(i,j)} \right) - \widehat{\Sigma}_{X,(i,j)} \right| + \left| s_{\eta_1} \left(\overline{\Sigma}_{X,(i,j)} \right) - \overline{\Sigma}_{X,(i,j)} \right| \right) \right. \\
& \quad \left. \times \mathbb{I} \left\{ \left| \widehat{\Sigma}_{X,(i,j)} \right| > \eta_1, \left| \overline{\Sigma}_{X,(i,j)} \right| > \eta_1 \right\} \right] \quad (\text{by the triangle inequality}) \\
& = [1] + [2] + [3] \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) + O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) + O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right). \tag{100}
\end{aligned}$$

Finally, applying the results given in expressions (88) and (100) to expression (75), we obtain, for $q = 0$, the result

$$\begin{aligned}
\left\| \frac{\widehat{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \right\|_2 & \leq \left\| \frac{\overline{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \right\|_2 + \left\| \frac{\widehat{\Sigma}_X^S - \overline{\Sigma}_X^S}{\lambda_1} \right\|_2 \\
& = O \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) + O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right) \\
& = O_p \left(\frac{c_0(N)}{\lambda_1} \sqrt{\frac{\ln N}{T}} \right).
\end{aligned}$$

The lemma then follows immediately upon combining the results we have obtained for Case 1, where $0 < q < 1$, and Case 2, where $q = 0$. $\square$

Lemma C-7: Let $T_h = T - h$, where $h \geq 1$ is a (fixed) finite integer. Suppose that Assumptions 4-2, 4-3, and 4-4 hold and suppose that $(\ln N)^{\frac{2}{\gamma_*} - 1} = o(T)$, where

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < 1$$

and where γ_1 and γ_2 are positive constants as defined in Assumption 4-3. Under these conditions, the following statements are true as $N, T \rightarrow \infty$:

(a) There exists a constant $C_* > 0$ such that

$$P \left\{ \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]' \beta \right| \geq C_* \sqrt{\frac{\ln N}{T_h}} \right\} = o \left(\frac{1}{N^2} \right).$$

(b) There exists a constant $C^\dagger > 0$ such that

$$P \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] [X_t - \mu_X]' - e'_{i,N} \bar{\Sigma}_X) \beta \right| \geq C^\dagger \sqrt{\frac{\ln N}{T_h}} \right\} = o \left(\frac{1}{N^2} \right)$$

(c) There exists a constant $C^\Delta > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]' \beta \right| \geq C^\Delta \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

(d) There exists a constant $B_* > 0$ such that

$$P \left(|\bar{\varepsilon}_{T_h}| \geq B_* \sqrt{\frac{\ln N}{T_h}} \right) = P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} \right| \geq B_* \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

(e) There exists a constant $C^\# > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| |\bar{\varepsilon}_{T_h}| \geq C^\# \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right), \text{ where } \bar{\varepsilon}_{T_h} = \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h}.$$

(f) There exists a constant $B^\# > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \varepsilon_{t+h} \right| \geq B^\# \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right).$$

Proof of Lemma C-7:

To show part (a), note first that $X_t - \mu_X = W_t + u_t$. so we can write

$$\frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]' \beta = \frac{1}{T_h} \sum_{t=1}^{T_h} W_t' \beta + \frac{1}{T_h} \sum_{t=1}^{T_h} u_t' \beta.$$

To show the desired result, we will first show that there exists a constant $C_*^{(1)} > 0$ such that

$$P \left\{ \left| \frac{1}{T_h} \sum_{t=1}^{T_h} W'_t \beta \right| \geq C_*^{(1)} \sqrt{\frac{\ln N}{T_h}} \right\} = o \left(\frac{1}{N^2} \right).$$

To proceed, note first that, by Assumption 4-1, $E(W'_t \beta) = E[W'_t] \beta = 0$. Moreover, by part (b) of Assumption 4-4, we have for all $s > 0$

$$\sup_{t \in \mathbb{Z}} P(|W'_t \beta| > s) \leq \exp \left\{ 1 - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\}$$

Next, let $Z_t^{(1)} = W'_t \beta$, and let $V_t = (\varepsilon_t, W'_t, u'_t)'$ be as defined in part (b) of Assumption 4-3. Note that $\mathcal{F}_{Z^{(1)}, -\infty}^t = \sigma(\dots, Z_{t-2}^{(1)}, Z_{t-1}^{(1)}, Z_t^{(1)}) \subseteq \mathcal{F}_{-\infty}^t = \sigma(\dots, V_{t-2}, V_{t-1}, V_t)$ and $\mathcal{F}_{Z^{(1)}, t+m}^\infty = \sigma(Z_{t+m}^{(1)}, Z_{t+m+1}^{(1)}, Z_{t+m+2}^{(1)}, \dots) \subseteq \mathcal{F}_{t+m}^\infty = \sigma(V_{t+m}, V_{t+m+1}, V_{t+m+2}, \dots)$. Hence, it follows by part (b) of Assumption 4-3 and Theorem 14.1 of Davidson (1994) that $\{Z_t^{(1)}\}_{t \in \mathbb{Z}}$ is an α -mixing process with α coefficient $\alpha_{Z^{(1)}}(m) = \sup_t \alpha(\mathcal{F}_{Z^{(1)}, -\infty}^t, \mathcal{F}_{Z^{(1)}, t+m}^\infty)$ satisfying the rate of decay

$$\begin{aligned} \alpha_{Z^{(1)}}(m) &= \sup_t \alpha(\mathcal{F}_{Z^{(1)}, -\infty}^t, \mathcal{F}_{Z^{(1)}, t+m}^\infty) \\ &\leq \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1 \end{aligned}$$

and for some positive constants γ_2 , a and c . In addition, note that, by Assumption 4-3, there exists a constant $\gamma \in (0, 1)$ such that

$$\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.$$

Hence, all the assumptions needed to obtain the Bernstein inequality given in Merlevède, Peligrad, and Rio (2011) are fulfilled. It follows from applying this inequality that there exist constants $C_{1,1}, C_{1,2}, C_{1,3}, C_{1,4}$ and $C_{1,5}$ that depend on a, b_1, c, γ , and γ_2 such that, for

any $s > 0$,

$$\begin{aligned}
& P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} W'_t \beta \right| \geq s \right) \\
&= P \left(\left| \sum_{t=1}^{T_h} W'_t \beta \right| \geq s T_h \right) \\
&\leq T_h \exp \left\{ -\frac{(s T_h)^\gamma}{C_{1,1}} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{C_{1,2} (1 + T_h C_{1,3})} \right\} + \exp \left\{ -\frac{(s T_h)^2}{C_{1,4} T_h} \exp \left\{ \frac{(s T_h)^{\gamma(1-\gamma)}}{C_{1,5} (\ln s T_h)^\gamma} \right\} \right\}.
\end{aligned}$$

For some $C_*^{(1)} > 0$, let

$$s = C_*^{(1)} \sqrt{\frac{\ln N}{T_h}}.$$

By choosing $C_*^{(1)}$ to be large enough, we can obtain the following three results, which we will show in turn:

(i)

$$T_h \exp \left\{ -\frac{(s T_h)^\gamma}{C_{1,1}} \right\} = T_h \exp \left\{ -\frac{[C_*^{(1)}]^\gamma}{C_{1,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(ii)

$$\exp \left\{ -\frac{(s T_h)^2}{C_{1,4} T_h} \exp \left\{ \frac{(s T_h)^{\gamma(1-\gamma)}}{C_{1,5} (\ln [s T_h])^\gamma} \right\} \right\} = o \left(\frac{1}{N^2} \right).$$

(iii)

$$\exp \left\{ -\frac{s^2 T_h^2}{C_{1,2} (1 + T_h C_{1,3})} \right\} = \exp \left\{ -\frac{[C_*^{(1)}]^2 \ln N}{C_{1,2} \left(\frac{1}{T_h} + C_{1,3} \right)} \right\} = O \left(\frac{1}{N^2} \right).$$

To show result (i), first write

$$\begin{aligned}
N^2 T_h \exp \left\{ -\frac{(s T_h)^\gamma}{C_{1,1}} \right\} &= N^2 T_h \exp \left\{ -\frac{[C_*^{(1)}]^\gamma}{C_{1,1}} T_h^\gamma \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma}{2}} \right\} \\
&= N^2 T_h \exp \left\{ -\frac{[C_*^{(1)}]^\gamma}{C_{1,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\}
\end{aligned}$$

Taking the log transformation, we have

$$\ln \left[N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{1,1}} \right\} \right] = 2 \ln N + \ln T_h - \frac{[C_*^{(1)}]^\gamma}{C_{1,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}$$

Now, by assumption, $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$, where

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} > 0.$$

Since, as shown previously,

$$\gamma = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} > \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} = \gamma_*;$$

it follows that $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$ implies that $(\ln N)^{\frac{2}{\gamma}-1} = o(T)$. Since

$$\frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0 \quad (\text{given that } \gamma > 0)$$

and since

$$\begin{aligned} \frac{2(\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} &= 2 \left(\frac{T}{T_h} \right)^{\frac{\gamma}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \right)^{\frac{\gamma}{2}} \rightarrow 0 \quad \left(\text{given that } \frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \rightarrow 0 \text{ and} \right. \\ &\quad \left. \text{given that } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \text{ since } h \text{ is a fixed finite integer} \right), \end{aligned}$$

we see that

$$\frac{2 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} = \frac{2(\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned} &N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{1,1}} \right\} \\ &= \exp \left\{ 2 \ln N + \ln T_h - \frac{[C_*^{(1)}]^\gamma}{C_{1,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\} \\ &= \exp \left\{ -\frac{[C_*^{(1)}]^\gamma}{C_{1,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \left[1 - \frac{C_{1,1}}{[C_*^{(1)}]^\gamma} \frac{2 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \right] \right\} \\ &= o(1) \end{aligned}$$

It follows that

$$\begin{aligned} T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{1,1}} \right\} &= \frac{1}{N^2} N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{1,1}} \right\} \\ &= O\left(\frac{1}{N^2}\right) o(1) = o\left(\frac{1}{N^2}\right). \end{aligned}$$

Next, to show result (ii), note that

$$\begin{aligned} & N^2 \exp \left\{ -\frac{(sT_h)^2}{C_{1,4}T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_{1,5}(\ln[sT_h])^\gamma} \right\} \right\} \\ &= N^2 \exp \left\{ -\frac{T_h^2}{C_{1,4}T_h} \left(C_*^{(1)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma(1-\gamma)} \left(C_*^{(1)} \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma(1-\gamma)}}{C_{1,5} \left(\ln \left[T_h C_*^{(1)} \sqrt{\frac{\ln N}{T_h}} \right] \right)^\gamma} \right) \right\} \\ &= N^2 \exp \left\{ -\frac{[C_*^{(1)}]^2}{C_{1,4}} \ln N \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} [C_*^{(1)}]^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_{1,5} \left(\ln T_h + \ln C_*^{(1)} + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h \right)^\gamma} \right) \right\} \\ &= N^2 \exp \left\{ -\frac{[C_*^{(1)}]^2}{C_{1,4}} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} [C_*^{(1)}]^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_{1,5} \left(\frac{1}{2} \ln T_h + \ln C_*^{(1)} + \frac{1}{2} \ln \ln N \right)^\gamma} \right) \right\} \\ &= \exp \left\{ -\ln N \left[\frac{[C_*^{(1)}]^2}{C_{1,4}} \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} [C_*^{(1)}]^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_{1,5} \left(\frac{1}{2} \ln T_h + \ln C_*^{(1)} + \frac{1}{2} \ln \ln N \right)^\gamma} \right) - 2 \right] \right\} \\ &= o(1). \end{aligned}$$

This, in turn, implies that

$$\begin{aligned} \exp \left\{ -\frac{(sT_h)^2}{C_{1,4}T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_{1,5}(\ln[T_h s])^\gamma} \right\} \right\} &= \frac{1}{N^2} N^2 \exp \left\{ -\frac{(sT_h)^2}{C_{1,4}T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_{1,5}(\ln[sT_h])^\gamma} \right\} \right\} \\ &= o\left(\frac{1}{N^2}\right). \end{aligned}$$

Turning our attention now to showing result (iii), we first write

$$\begin{aligned}
N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_{1,2} (1 + T_h C_{1,3})} \right\} &= N^2 \exp \left\{ -\frac{T_h^2}{C_{1,2} (1 + T_h C_{1,3})} \left(C_*^{(1)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\
&= N^2 \exp \left\{ -\frac{[C_*^{(1)}]^2 \ln N}{C_{1,2} \left(\frac{1}{T_h} + C_{1,3} \right)} \right\}
\end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned}
\ln \left[N^2 \exp \left\{ -\frac{[C_*^{(1)}]^2 \ln N}{C_{1,2} \left(\frac{1}{T_h} + C_{1,3} \right)} \right\} \right] &= 2 \ln N - \frac{[C_*^{(1)}]^2 \ln N}{C_{1,2} \left(\frac{1}{T_h} + C_{1,3} \right)} \\
&= 2 \ln N - \frac{[C_*^{(1)}]^2}{C_{1,2} C_{1,3}} \ln N \left[1 + C_{1,3}^{-1} \frac{1}{T_h} \right]^{-1}
\end{aligned}$$

Now, let $C_*^{(1)}$ be a positive constant such that $[C_*^{(1)}]^2 > 2C_{1,2} (1 + C_{1,3})$, and set

$$C_{(1)}^* = \frac{[C_*^{(1)}]^2}{C_{1,2} C_{1,3}} - 2.$$

Note that, by this choice of $C_*^{(1)}$ and $C_{(1)}^*$, we have

$$C_{(1)}^* > \frac{2C_{1,2} (1 + C_{1,3})}{C_{1,2} C_{1,3}} - 2 = \frac{2}{C_{1,3}} > 0$$

It, thus, follows that

$$\begin{aligned}
& N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_{1,2} (1 + T_h C_{1,3})} \right\} \\
= & N^2 \exp \left\{ -\frac{[C_*^{(1)}]^2 \ln N}{C_{1,2} \left(\frac{1}{T_h} + C_{1,3} \right)} \right\} \\
= & \exp \left\{ \left(2 \ln N \left[1 + C_{1,3}^{-1} \frac{1}{T_h} \right] - \frac{[C_*^{(1)}]^2}{C_{1,2} C_{1,3}} \ln N \right) \left[1 + C_{1,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(2 \ln N \left[1 + C_{1,3}^{-1} \frac{1}{T_h} \right] - [C_{(1)}^* + 2] \ln N \right) \left[1 + C_{1,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(\ln N \left[2 C_{1,3}^{-1} \frac{1}{T_h} - C_{(1)}^* \right] \right) \left[1 + C_{1,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ -C_{(1)}^* \ln N \left[1 - \frac{2}{C_{(1)}^* C_{1,3}} \frac{1}{T_h} \right] \left[1 + C_{1,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
< & \exp \left\{ -C_{(1)}^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_{1,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{2}{C_{(1)}^* C_{1,3}} < 1 \right) \\
= & o(1)
\end{aligned}$$

This, in turn, implies that

$$\exp \left\{ -\frac{s^2 T_h^2}{C_{1,2} (1 + T_h C_{1,3})} \right\} = \left(\frac{1}{N^2} \right) N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_{1,2} (1 + T_h C_{1,3})} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (iii).

Combining the results given in (i)-(iii) above, we obtain

$$\begin{aligned}
& P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} W'_t \beta \right| \geq C_*^{(1)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{1,1}} \right\} + \exp \left\{ -\frac{(sT_h)^2}{C_{1,4}T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_{1,5}(\ln sT_h)^\gamma} \right\} \right\} + \exp \left\{ -\frac{s^2T_h^2}{C_{1,2}(1+T_hC_{1,3})} \right\} \\
& \quad \left(\text{with } s = C_*^{(1)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right). \tag{101}
\end{aligned}$$

Next, we show that there exists a constant $C_*^{(2)} > 0$ such that

$$P \left\{ \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u'_t \underline{\beta} \right| \geq C_*^{(2)} \sqrt{\frac{\ln N}{T_h}} \right\} = o \left(\frac{1}{N^2} \right), \text{ where } \underline{\beta} = \frac{\beta}{\|\beta\|_2}.$$

To proceed, note that, by Assumption 4-1, $E(u'_t \underline{\beta}) = E[u'_t] \underline{\beta} = 0$. Moreover, since $\|\underline{\beta}\|_2 = \|\beta\|_2 / \|\beta\|_2 = 1$, we can apply part (a) of Assumption 4-3, so that for all $s > 0$

$$\sup_{t \in \mathbb{Z}} P(|u'_t \underline{\beta}| > s) \leq \exp \left\{ 1 - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\}.$$

Let $Z_t^{(2)} = u'_t \underline{\beta}$, and let $V_t = (\varepsilon_t, W'_t, u'_t)'$ be as defined in part (b) of Assumption 4-3. Note that $\mathcal{F}_{Z^{(2)}, -\infty}^t = \sigma(\dots, Z_{t-2}^{(2)}, Z_{t-1}^{(2)}, Z_t^{(2)}) \subseteq \mathcal{F}_{-\infty}^t = \sigma(\dots, V_{t-2}, V_{t-1}, V_t)$ and $\mathcal{F}_{Z^{(2)}, t+m}^\infty = \sigma(Z_{t+m}^{(2)}, Z_{t+m+1}^{(2)}, Z_{t+m+2}^{(2)}, \dots) \subseteq \mathcal{F}_{t+m}^\infty = \sigma(V_{t+m}, V_{t+m+1}, V_{t+m+2}, \dots)$. Hence, it follows by part (b) of Assumption 4-3 and Theorem 14.1 of Davidson (1994) that $\{Z_t^{(2)}\}_{t \in \mathbb{Z}}$ is an α -mixing process with α coefficient $\alpha_{Z^{(2)}}(m) = \sup_t \alpha(\mathcal{F}_{Z^{(2)}, -\infty}^t, \mathcal{F}_{Z^{(2)}, t+m}^\infty)$ satisfying the rate of decay

$$\begin{aligned}
\alpha_{Z^{(2)}}(m) &= \sup_t \alpha(\mathcal{F}_{Z^{(2)}, -\infty}^t, \mathcal{F}_{Z^{(2)}, t+m}^\infty) \\
&\leq \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty) \\
&\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1
\end{aligned}$$

and for some positive constants γ_2 , a and c . In addition, note that, by Assumption 4-3, there exists a constant $\gamma \in (0, 1)$ such that

$$\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.$$

Hence, all the assumptions needed to obtain the Bernstein inequality given in Merlevède, Peligrad, and Rio (2011) are fulfilled. It follows from applying this inequality that there exist constants $C_{2,1}, C_{2,2}, C_{2,3}, C_{2,4}$ and $C_{2,5}$ that depend on a, b_1, c, γ , and γ_2 such that, for any $s > 0$,

$$\begin{aligned} & P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} u'_t \beta \right| \geq s \right) \\ &= P \left(\left| \sum_{t=1}^{T_h} u'_t \beta \right| \geq s T_h \right) \\ &\leq T_h \exp \left\{ -\frac{(s T_h)^\gamma}{C_{2,1}} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{C_{2,2} (1 + T_h C_{2,3})} \right\} + \exp \left\{ -\frac{(s T_h)^2}{C_{2,4} T_h} \exp \left\{ \frac{(s T_h)^{\gamma(1-\gamma)}}{C_{2,5} (\ln s T_h)^\gamma} \right\} \right\}. \end{aligned}$$

For some $C_*^{(2)} > 0$, let

$$s = C_*^{(2)} \sqrt{\frac{\ln N}{T_h}}.$$

By choosing $C_*^{(2)}$ to be large enough, we can obtain the following three results, which we will show in turn:

(i)

$$T_h \exp \left\{ -\frac{(s T_h)^\gamma}{C_{2,1}} \right\} = T_h \exp \left\{ -\frac{[C_*^{(2)}]^\gamma}{C_{2,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(ii)

$$\exp \left\{ -\frac{(s T_h)^2}{C_{2,4} T_h} \exp \left\{ \frac{(s T_h)^{\gamma(1-\gamma)}}{C_{2,5} (\ln [s T_h])^\gamma} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(iii)

$$\exp \left\{ -\frac{s^2 T_h^2}{C_{2,2} (1 + T_h C_{2,3})} \right\} = \exp \left\{ -\frac{[C_*^{(2)}]^2 \ln N}{C_{2,2} \left(\frac{1}{T_h} + C_{2,3} \right)} \right\} = O \left(\frac{1}{N^2} \right).$$

To show result (i), first write

$$\begin{aligned} N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{2,1}} \right\} &= N^2 T_h \exp \left\{ -\frac{[C_*^{(2)}]^\gamma}{C_{2,1}} T_h^\gamma \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma}{2}} \right\} \\ &= N^2 T_h \exp \left\{ -\frac{[C_*^{(2)}]^\gamma}{C_{2,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\} \end{aligned}$$

Taking the log transformation, we have

$$\ln \left[N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{2,1}} \right\} \right] = 2 \ln N + \ln T_h - \frac{[C_*^{(2)}]^\gamma}{C_{2,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}$$

Now, by assumption, $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$, where

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} > 0.$$

Since, as shown previously,

$$\gamma = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} > \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} = \gamma_*;$$

it follows that $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$ implies that $(\ln N)^{\frac{2}{\gamma}-1} = o(T)$. Since

$$\frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0 \quad (\text{given that } \gamma > 0)$$

and since

$$\begin{aligned} \frac{2(\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} &= 2 \left(\frac{T}{T_h} \right)^{\frac{\gamma}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \right)^{\frac{\gamma}{2}} \rightarrow 0 \quad \left(\text{given that } \frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \rightarrow 0 \right. \\ &\quad \left. \text{and } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \text{ since } h \text{ is a fixed finite integer} \right), \end{aligned}$$

we see that

$$\frac{2 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} = \frac{2(\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned}
N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{2,1}} \right\} &= \exp \left\{ 2 \ln N + \ln T_h - \frac{[C_*^{(2)}]^\gamma}{C_{2,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\} \\
&= \exp \left\{ -\frac{[C_*^{(2)}]^\gamma}{C_{2,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \left[1 - \frac{C_{2,1}}{[C_*^{(2)}]^\gamma} \frac{2 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \right] \right\} \\
&= o(1)
\end{aligned}$$

It follows that

$$T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{2,1}} \right\} = \frac{1}{N^2} N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{2,1}} \right\} = O\left(\frac{1}{N^2}\right) o(1) = o\left(\frac{1}{N^2}\right).$$

Next, to show result (ii), note that

$$\begin{aligned}
&N^2 \exp \left\{ -\frac{(sT_h)^2}{C_{2,4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_{2,5} (\ln [sT_h])^\gamma} \right\} \right\} \\
&= N^2 \exp \left\{ -\frac{T_h^2}{C_{2,4} T_h} \left(C_*^{(2)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma(1-\gamma)} \left(C_*^{(2)} \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma(1-\gamma)}}{C_{2,5} \left(\ln \left[T_h C_*^{(2)} \sqrt{\frac{\ln N}{T_h}} \right] \right)^\gamma} \right) \right\} \\
&= N^2 \exp \left\{ -\frac{[C_*^{(2)}]^2}{C_{2,4}} \ln N \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} [C_*^{(2)}]^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_{2,5} \left(\ln T_h + \ln C_*^{(2)} + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h \right)^\gamma} \right) \right\} \\
&= N^2 \exp \left\{ -\frac{[C_*^{(2)}]^2}{C_{2,4}} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} [C_*^{(2)}]^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_{2,5} \left(\frac{1}{2} \ln T_h + \ln C_*^{(2)} + \frac{1}{2} \ln \ln N \right)^\gamma} \right) \right\} \\
&= \exp \left\{ -\ln N \left[\frac{[C_*^{(2)}]^2}{C_{2,4}} \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} [C_*^{(2)}]^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_{2,5} \left(\frac{1}{2} \ln T_h + \ln C_*^{(2)} + \frac{1}{2} \ln \ln N \right)^\gamma} \right) - 2 \right] \right\} \\
&= o(1).
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
& \exp \left\{ -\frac{(sT_h)^2}{C_{2,4}T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_{2,5}(\ln [T_h s])^\gamma} \right\} \right\} \\
&= \frac{1}{N^2} N^2 \exp \left\{ -\frac{(sT_h)^2}{C_{2,4}T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_{2,5}(\ln [sT_h])^\gamma} \right\} \right\} \\
&= o\left(\frac{1}{N^2}\right)
\end{aligned}$$

Turning our attention now to showing result (iii), we first write

$$\begin{aligned}
N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_{2,2}(1 + T_h C_3)} \right\} &= N^2 \exp \left\{ -\frac{T_h^2}{C_{2,2}(1 + T_h C_{2,3})} \left(C_*^{(2)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\
&= N^2 \exp \left\{ -\frac{[C_*^{(2)}]^2 \ln N}{C_{2,2} \left(\frac{1}{T_h} + C_{2,3} \right)} \right\}
\end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned}
\ln \left[N^2 \exp \left\{ -\frac{[C_*^{(2)}]^2 \ln N}{C_{2,2} \left(\frac{1}{T_h} + C_{2,3} \right)} \right\} \right] &= 2 \ln N - \frac{[C_*^{(2)}]^2 \ln N}{C_{2,2} \left(\frac{1}{T_h} + C_3 \right)} \\
&= 2 \ln N - \frac{[C_*^{(2)}]^2}{C_{2,2} C_{2,3}} \ln N \left[1 + C_{2,3}^{-1} \frac{1}{T_h} \right]^{-1}
\end{aligned}$$

Now, let $C_*^{(2)}$ be a positive constant such that $[C_*^{(2)}]^2 > 2C_{2,2}(1 + C_{2,3})$, and set

$$C_{(2)}^* = \frac{[C_*^{(2)}]^2}{C_{2,2} C_{2,3}} - 2.$$

Note that, by this choice of $C_*^{(2)}$ and $C_{(2)}^*$, we have

$$C_{(2)}^* > \frac{2C_{2,2}(1 + C_{2,3})}{C_{2,2} C_{2,3}} - 2 = \frac{2}{C_{2,3}} > 0$$

It, thus, follows that

$$\begin{aligned}
& N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_{2,2} (1 + T_h C_{2,3})} \right\} \\
= & N^2 \exp \left\{ -\frac{[C_*^{(2)}]^2 \ln N}{C_{2,2} \left(\frac{1}{T_h} + C_{2,3} \right)} \right\} \\
= & \exp \left\{ \left(2 \ln N \left[1 + C_{2,3}^{-1} \frac{1}{T_h} \right] - \frac{[C_*^{(2)}]^2}{C_{2,2} C_{2,3}} \ln N \right) \left[1 + C_{2,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(2 \ln N \left[1 + C_{2,3}^{-1} \frac{1}{T_h} \right] - [C_{(2)}^* + 2] \ln N \right) \left[1 + C_{2,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(\ln N \left[2 C_{2,3}^{-1} \frac{1}{T_h} - C_{(2)}^* \right] \right) \left[1 + C_{2,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ -C_{(2)}^* \ln N \left[1 - \frac{2}{C_{(2)}^* C_{2,3}} \frac{1}{T_h} \right] \left[1 + C_{2,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
< & \exp \left\{ -C_{(2)}^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_{2,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{2}{C_{(2)}^* C_{2,3}} < 1 \right) \\
= & o(1)
\end{aligned}$$

This, in turn, implies that

$$\exp \left\{ -\frac{s^2 T_h^2}{C_{2,2} (1 + T_h C_{2,3})} \right\} = \left(\frac{1}{N^2} \right) N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_{2,2} (1 + T_h C_{2,3})} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (iii).

Combining the results given in (i)-(iii) above, we obtain

$$\begin{aligned}
& P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} u'_t \underline{\beta} \right| \geq C_*^{(2)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{2,1}} \right\} + \exp \left\{ -\frac{(sT_h)^2}{C_{2,4}T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_{2,5}(\ln sT_h)^\gamma} \right\} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{C_{2,2}(1 + T_h C_{2,3})} \right\} \\
& \quad \left(\text{with } s = C_*^{(2)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

Moreover, since $0 < C_*^{(2)} < \infty$ and since, by part (a) of Assumption 4-4, there exists a constant $\bar{c} > 0$ such that

$$0 < \sqrt{\kappa} \frac{1}{\bar{c}} \leq \sqrt{\kappa} \min_{1 \leq g \leq \kappa} |\phi_g| \leq \|\phi\|_2 = \sqrt{\phi' \phi} \leq \sqrt{\kappa} \max_{1 \leq g \leq \kappa} |\phi_g| \leq \sqrt{\kappa} \bar{c} < \infty,$$

we let $C_\dagger^{(2)} = C_*^{(2)} \|\phi\|_2$; and note that $0 < C_*^{(2)} \sqrt{\kappa}/\bar{c} \leq C_\dagger^{(2)} \leq C_*^{(2)} \sqrt{\kappa} \bar{c} < \infty$, so that $C_\dagger^{(2)} \in (0, \infty)$. In addition, note that

$$\underline{\beta} = \frac{\beta}{\|\beta\|_2} = \frac{L_1 \phi / \sqrt{\lambda_{1,1}}}{\|\phi\|_2 / \sqrt{\lambda_{1,1}}} = \frac{L_1 \phi}{\|\phi\|_2}, \text{ so that } \|\underline{\beta}\|_2 = \sqrt{\frac{\phi' L_1' L_1 \phi}{\|\phi\|_2^2}} = \frac{\|\phi\|_2}{\|\phi\|_2} = 1.$$

Hence, given that, by definition, $\beta = \underline{\beta} \|\beta\|_2$, we further deduce that

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u'_t \beta \right| \geq C_{\dagger}^{(2)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u'_t \underline{\beta} \right| \|\beta\|_2 \geq C_{\dagger}^{(2)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u'_t \underline{\beta} \right| \geq \frac{C_{\dagger}^{(2)}}{\|\beta\|_2} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u'_t \underline{\beta} \right| \geq \frac{C_{\dagger}^{(2)} \sqrt{\Delta_{1,1}}}{\|\phi\|_2} \sqrt{\frac{\ln N}{T_h}} \right) \quad \left(\text{since } \|\beta\|_2 = \sqrt{\frac{\phi' L'_1 L_1 \phi}{\Delta_{1,1}}} = \frac{\|\phi\|_2}{\sqrt{\Delta_{1,1}}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u'_t \underline{\beta} \right| \geq C_*^{(2)} \sqrt{\Delta_{1,1}} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u'_t \underline{\beta} \right| \geq C_*^{(2)} \sqrt{\frac{\ln N}{T_h}} \right) \quad (\text{since by Assumption 4-2(a), } \Delta_{1,1} \rightarrow \infty) \\
&= o \left(\frac{1}{N^2} \right). \tag{102}
\end{aligned}$$

Finally, let C_* be a positive constant such that $C_* \geq \max \left\{ 2C_*^{(1)}, 2C_{\dagger}^{(2)} \right\}$, and we can combine the results given in expressions (101) and (102) above to obtain

$$\begin{aligned}
& P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]' \beta \right| \geq C_* \sqrt{\frac{\ln N}{T_h}} \right) \\
= & P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [W_t + u_t]' \beta \right| \geq C_* \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} W_t' \beta \right| + \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_t' \beta \right| \geq C_* \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\left\{ \left| \frac{1}{T_h} \sum_{t=1}^{T_h} W_t' \beta \right| \geq \frac{C_*}{2} \sqrt{\frac{\ln N}{T_h}} \right\} \cup \left\{ \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_t' \beta \right| \geq \frac{C_*}{2} \sqrt{\frac{\ln N}{T_h}} \right\} \right) \\
\leq & P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} W_t' \beta \right| \geq \frac{C_*}{2} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_t' \beta \right| \geq \frac{C_*}{2} \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} W_t' \beta \right| \geq C_*^{(1)} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_t' \beta \right| \geq C_{\dagger}^{(2)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \left(\text{since } C_* \geq \max \left\{ 2C_*^{(1)}, 2C_{\dagger}^{(2)} \right\} \text{ implying that } \frac{C_*}{2} \geq C_*^{(1)} \text{ and } \frac{C_*}{2} \geq C_{\dagger}^{(2)} \right) \\
= & o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
= & o \left(\frac{1}{N^2} \right).
\end{aligned}$$

To show part (b), note first that, since $X_t - \mu_X = W_t + u_t$ and $\bar{\Sigma}_X = L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2' + \bar{\Sigma}_u$, we can write

$$\begin{aligned}
& \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] [X_t - \mu_X]' - e'_{i,N} \bar{\Sigma}_X) \beta \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} ([W_{it} + u_{it}] [W_t' + u_t'] - e'_{i,N} \bar{\Sigma}_X) \beta \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} [W_{it} + u_{it}] W_t' \beta + \frac{1}{T_h} \sum_{t=1}^{T_h} [W_{it} + u_{it}] u_t' \beta - \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \bar{\Sigma}_X \beta \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] W_t' - e'_{i,N} (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2') \} \beta \\
&\quad + \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] u_t' - e'_{i,N} \bar{\Sigma}_u \} \beta \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] W_t' - e'_{i,N} (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2') \} \beta \\
&\quad + \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] u_t' - e'_{i,N} \Sigma_{u,t} \} \beta, \quad (\text{where } \Sigma_{u,t} = E[u_t u_t']) \\
&\quad \left(\text{since } \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \bar{\Sigma}_u \beta = \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \frac{1}{T_h} \sum_{t=1}^{T_h} \Sigma_{u,t} \beta = \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Sigma_{u,t} \beta \right)
\end{aligned}$$

To show the desired result, we first show that there exists a constant $C_*^{[1]} > 0$ such that

$$P \left\{ \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] W_t' - e'_{i,N} (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2') \} \beta \right| \geq C_*^{[1]} \sqrt{\frac{\ln N}{T_h}} \right\} = o\left(\frac{1}{N^2}\right).$$

To proceed, note first that, by Assumption 4-1,

$$\begin{aligned}
& E \left[\{ [W_{it} + u_{it}] W_t' - e'_{i,N} (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2') \} \beta \right] \\
&= E \left[[W_{it} + u_{it}] W_t' \beta - e'_{i,N} (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2') \beta \right] \\
&= E \left[e'_{i,N} W_t W_t' \beta + e'_{i,N} u_t W_t' \beta - e'_{i,N} (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2') \beta \right] \\
&\quad (\text{since } W_{it} = e'_{i,N} W_t \text{ and } u_{it} = e'_{i,N} u_t) \\
&= e'_{i,N} E[W_t W_t'] \beta + e'_{i,N} E[u_t W_t'] \beta - e'_{i,N} (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2') \beta \\
&= 0 \quad (\text{since } E[W_t W_t'] = L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2' \text{ and } E[u_t W_t'] = 0)
\end{aligned}$$

from which it follows that the elements of the sum

$$\sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] W'_t \beta - e'_{i,N} (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2) \beta \}$$

given above all have expectation equal to zero. Moreover, let γ_1 and b_2 be the positive constants defined, respectively, in part (a) of Assumption 4-3 and in Lemma C-2. Then, by part (c) of Lemma C-3, there exists a positive constant $\overline{C} \geq \exp\{1\}$ such that, for any $s > 0$, for all integer i with $1 \leq i \leq N$, and for all $t \in \mathbb{Z}$,

$$\begin{aligned} P(|(X_{it} - \mu_{X,i}) W'_t \beta| > s) &= P(|(W_{it} + u_{it}) W'_t \beta| > s) \\ &\leq \exp\left\{1 - \left(\frac{s}{b}\right)^r\right\} \\ &\leq \overline{C} \exp\left\{-\left(\frac{s}{b}\right)^r\right\}. \end{aligned}$$

where $r = \gamma_1/2 > 0$ and $b \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} b_2^2 > 0$. Next, define $\gamma_* = \gamma_1 \gamma_2 / (\gamma_1 + 2\gamma_2)$ with γ_2 being a positive constant as defined in Assumption 4-3, so that the reciprocal of γ_* can be written as

$$\frac{1}{\gamma_*} = \frac{\gamma_1 + 2\gamma_2}{\gamma_1 \gamma_2} = \frac{2}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r} + \frac{1}{\gamma_2}.$$

Now, the condition specified in Assumption 4-3 implies that

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} = \gamma < 1$$

so that, in particular, $\gamma_* \in (0, 1)$. In addition, let $\underline{Z}_{i,t}^{(1)} = (X_{it} - \mu_{X,i}) W'_t \beta = (W_{it} + u_{it}) W'_t \beta$, and let $V_t = (\varepsilon_t, W'_t, u'_t)'$ be as defined in part (b) of Assumption 4-3. Note that $\mathcal{F}_{\underline{Z}^{(1)}, i, -\infty}^t = \sigma(\dots, \underline{Z}_{i,t-2}^{(1)}, \underline{Z}_{i,t-1}^{(1)}, \underline{Z}_{i,t}^{(1)}) \subseteq \mathcal{F}_{-\infty}^t = \sigma(\dots, V_{t-2}, V_{t-1}, V_t)$ and $\mathcal{F}_{\underline{Z}^{(1)}, i, t+m}^\infty = \sigma(\underline{Z}_{i,t+m}^{(1)}, \underline{Z}_{i,t+m+1}^{(1)}, \underline{Z}_{i,t+m+2}^{(1)}, \dots)$. Hence, it follows by part (b) of Assumption 4-3 and Theorem 14.1 of Davidson (1994) that $\{\underline{Z}_{i,t}^{(1)}\}_{t \in \mathbb{Z}}$ is an α -mixing process with α coefficient $\alpha_{\underline{Z}^{(1)}, i}(m) = \sup_t \alpha(\mathcal{F}_{\underline{Z}^{(1)}, i, -\infty}^t, \mathcal{F}_{\underline{Z}^{(1)}, i, t+m}^\infty)$ satisfying the rate of decay

$$\begin{aligned} \alpha_{\underline{Z}^{(1)}, i}(m) &= \sup_t \alpha(\mathcal{F}_{\underline{Z}^{(1)}, i, -\infty}^t, \mathcal{F}_{\underline{Z}^{(1)}, i, t+m}^\infty) \\ &\leq \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1, \text{ for all } i, \end{aligned}$$

and for some positive constants γ_2 , a and c .

Hence, all the assumptions needed to apply the Bernstein inequality given in Lemma C-1 are fulfilled. Making use of this Bernstein inequality then implies that there exist constants

$C_{(1),1}, C_{(1),2}, C_{(1),3}, C_{(1),4}$ and $C_{(1),5}$ depending on a, b, c, γ_* , and γ_2 such that, for any $s > 0$ and for all integer i with $1 \leq i \leq N$,

$$\begin{aligned}
& P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] W'_t - e'_{i,N} (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)) \beta \right| \geq s \right) \\
&= P \left(\left| \sum_{t=1}^{T_h} ([W_{it} + u_{it}] W'_t \beta - e'_{i,N} L_1 \underline{\Delta}_1 L'_1 \beta) \right| \geq T_h s \right) \\
&\quad \left(\text{since } L'_2 \beta = L'_2 L_1 \frac{\phi}{\sqrt{\Delta_{1,1}}} = 0 \text{ and } X_{it} - \mu_{X,i} = W_{it} + u_{it} \right) \\
&\leq T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(1),1}} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{C_{(1),2} (1 + T_h C_{(1),3})} \right\} \\
&\quad + \exp \left\{ -\frac{(sT_h)^2}{C_{(1),4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_{(1),5} (\ln sT_h)^{\gamma_*}} \right\} \right\}
\end{aligned}$$

Using Bonferroni's inequality, we further obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] W'_t - e'_{i,N} (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)) \beta \right| \geq s \right) \\
&\leq N \max_{1 \leq i \leq N} P \left(\left| \sum_{t=1}^{T_h} ([W_{it} + u_{it}] W'_t \beta - e'_{i,N} L_1 \underline{\Delta}_1 L'_1 \beta) \right| \geq sT_h \right) \\
&\leq NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(1),1}} \right\} + N \exp \left\{ -\frac{s^2 T_h^2}{C_{(1),2} (1 + T_h C_{(1),3})} \right\} \\
&\quad + N \exp \left\{ -\frac{(sT_h)^2}{C_{(1),4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_{(1),5} (\ln sT_h)^{\gamma_*}} \right\} \right\}
\end{aligned}$$

Now, for some constant $C_*^{[1]} > 0$, let

$$s = C_*^{[1]} \sqrt{\frac{\ln N}{T_h}}.$$

By choosing $C_*^{[1]}$ to be large enough, we can obtain the following three results, which we will show in turn:

(i)

$$NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(1),1}} \right\} = NT_h \exp \left\{ -\frac{(C_*^{[1]})^{\gamma_*}}{C_{(1),1}} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(ii)

$$N \exp \left\{ -\frac{(sT_h)^2}{C_{(1),4}T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_{(1),5}(\ln[sT_h])^{\gamma_*}} \right\} \right\} = o\left(\frac{1}{N^2}\right)$$

(iii)

$$N \exp \left\{ -\frac{s^2 T_h^2}{C_{(1),2}(1+T_h C_{(1),3})} \right\} = N \exp \left\{ -\frac{(C_*^{[1]})^2 \ln N}{C_{(1),2}\left(\frac{1}{T_h} + C_{(1),3}\right)} \right\} = o\left(\frac{1}{N^2}\right).$$

To show result (i), first write

$$\begin{aligned} N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(1),1}} \right\} &= N^3 T_h \exp \left\{ -\frac{(C_*^{[1]})^{\gamma_*}}{C_{(1),1}} T_h^{\gamma_*} \left(\frac{\ln N}{T_h}\right)^{\frac{\gamma_*}{2}} \right\} \\ &= N^3 T_h \exp \left\{ -\frac{(C_*^{[1]})^{\gamma_*}}{C_{(1),1}} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \end{aligned}$$

Taking the log transformation, we have

$$\ln \left[N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(1),1}} \right\} \right] = 3 \ln N + \ln T_h - \frac{(C_*^{[1]})^{\gamma_*}}{C_{(1),1}} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}$$

Given that

$$\frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0 \quad (\text{since } \gamma_* > 0)$$

and given that

$$\begin{aligned} \frac{3(\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} &= 3 \left(\frac{T}{T_h}\right)^{\frac{\gamma_*}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T}\right)^{\frac{\gamma_*}{2}} \rightarrow 0 \quad \left(\text{given that } \frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0\right. \\ &\quad \left.\text{by assumption, and given that } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty\right); \end{aligned}$$

we see that

$$\frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} = \frac{3(\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned}
& N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(1),1}} \right\} \\
&= \exp \left\{ 3 \ln N + \ln T_h - \frac{(C_*^{[1]})^{\gamma_*}}{C_{(1),1}} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \\
&= \exp \left\{ -\frac{(C_*^{[1]})^{\gamma_*}}{C_{(1),1}} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \left[1 - \frac{C_{(1),1}}{(C_*^{[1]})^{\gamma_*}} \frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \right] \right\} \\
&= o(1)
\end{aligned}$$

It follows that

$$\begin{aligned}
N T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(1),1}} \right\} &= \frac{1}{N^2} N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(1),1}} \right\} \\
&= O\left(\frac{1}{N^2}\right) o(1) = o\left(\frac{1}{N^2}\right).
\end{aligned}$$

Next, to show result (ii), note that

$$\begin{aligned}
& N^3 \exp \left\{ -\frac{(sT_h)^2}{C_{(1),4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_{(1),5} (\ln [sT_h])^{\gamma_*}} \right\} \right\} \\
&= N^3 \exp \left\{ -\frac{T_h^2}{C_{(1),4} T_h} \left(C_*^{[1]} \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma_*(1-\gamma_*)} (C_*^{[1]} \sqrt{\frac{\ln N}{T_h}})^{\gamma_*(1-\gamma_*)}}{C_{(1),5} (\ln [T_h C_*^{[1]} \sqrt{\frac{\ln N}{T_h}}])^{\gamma_*}} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{(C_*^{[1]})^2}{C_{(1),4}} \ln N \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} (C_*^{[1]})^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_{(1),5} (\ln T_h + \ln C_*^{[1]} + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h)^{\gamma_*}} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{(C_*^{[1]})^2}{C_{(1),4}} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} (C_*^{[1]})^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_{(1),5} (\frac{1}{2} \ln T_h + \ln C_*^{[1]} + \frac{1}{2} \ln \ln N)^{\gamma_*}} \right) \right\} \\
&= \exp \left\{ -\ln N \left[\frac{(C_*^{[1]})^2}{C_{(1),4}} \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} (C_*^{[1]})^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_{(1),5} (\frac{1}{2} \ln T_h + \ln C_*^{[1]} + \frac{1}{2} \ln \ln N)^{\gamma_*}} \right) - 3 \right] \right\} \\
&= o(1)
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
& N \exp \left\{ -\frac{(sT_h)^2}{C_{(1),4}T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_{(1),5}(\ln[sT_h])^\gamma} \right\} \right\} \\
&= \left(\frac{1}{N^2} \right) N^3 \exp \left\{ -\frac{(sT_h)^2}{C_{(1),4}T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_{(1),5}(\ln[sT_h])^{\gamma_*}} \right\} \right\} \\
&= O\left(\frac{1}{N^2}\right) o(1) = o\left(\frac{1}{N^2}\right)
\end{aligned}$$

Turning our attention now to showing result (iii), we first write

$$\begin{aligned}
N^3 \exp \left\{ -\frac{s^2 T_h^2}{C_{(1),2}(1 + T_h C_{(1),3})} \right\} &= N^3 \exp \left\{ -\frac{T_h^2}{C_{(1),2}(1 + T_h C_{(1),3})} \left(C_*^{[1]} \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\
&= N^3 \exp \left\{ -\frac{\left(C_*^{[1]} \right)^2 \ln N}{C_{(1),2} \left(\frac{1}{T_h} + C_{(1),3} \right)} \right\}
\end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned}
\ln \left[N^3 \exp \left\{ -\frac{\left(C_*^{[1]} \right)^2 \ln N}{C_{(1),2} \left(\frac{1}{T_h} + C_{(1),3} \right)} \right\} \right] &= 3 \ln N - \frac{\left(C_*^{[1]} \right)^2 \ln N}{C_{(1),2} \left(\frac{1}{T_h} + C_{(1),3} \right)} \\
&= 3 \ln N - \frac{\left(C_*^{[1]} \right)^2}{C_{(1),2} C_{(1),3}} \ln N \left[1 + C_{(1),3}^{-1} \frac{1}{T_h} \right]^{-1}
\end{aligned}$$

Now, let $C_*^{[1]}$ be a positive constant such that $\left(C_*^{[1]} \right)^2 > 3C_{(1),2}(1 + C_{(1),3})$, and set

$$C_{[1]}^* = \frac{\left(C_*^{[1]} \right)^2}{C_{(1),2} C_{(1),3}} - 3.$$

Note that, by this choice of $C_*^{[1]}$ and $C_{[1]}^*$, we have

$$C_{[1]}^* > \frac{3C_{(1),2}(1 + C_{(1),3})}{C_{(1),2} C_{(1),3}} - 3 = \frac{3}{C_{(1),3}} > 0$$

It, thus, follows that

$$\begin{aligned}
& N^3 \exp \left\{ -\frac{s^2 T_h^2}{C_{(1),2} (1 + T_h C_{(1),3})} \right\} \\
= & N^3 \exp \left\{ -\frac{(C_{*}^{[1]})^2 \ln N}{C_{(1),2} \left(\frac{1}{T_h} + C_{(1),3} \right)} \right\} \\
= & \exp \left\{ \left(3 \ln N \left[1 + C_{(1),3}^{-1} \frac{1}{T_h} \right] - \frac{(C_{*}^{[1]})^2}{C_{(1),2} C_{(1),3}} \ln N \right) \left[1 + C_{(1),3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(3 \ln N \left[1 + C_{(1),3}^{-1} \frac{1}{T_h} \right] - [C_{[1]}^{*} + 3] \ln N \right) \left[1 + C_{(1),3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(\ln N \left[3 C_{(1),3}^{-1} \frac{1}{T_h} - C_{[1]}^{*} \right] \right) \left[1 + C_{(1),3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ -C_{[1]}^{*} \ln N \left[1 - \frac{3}{C_{[1]}^{*} C_{(1),3}} \frac{1}{T_h} \right] \left[1 + C_{(1),3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
< & \exp \left\{ -C_{[1]}^{*} \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_{(1),3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{3}{C_{[1]}^{*} C_{(1),3}} < \frac{C_{(1),3} 3}{3 C_{(1),3}} = 1 \right) \\
= & o(1)
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
N \exp \left\{ -\frac{s T_h^2}{C_{(1),2} (1 + T_h C_{(1),3})} \right\} &= \left(\frac{1}{N^2} \right) N^3 \exp \left\{ -\frac{s T_h^2}{C_{(1),2} (1 + T_h C_{(1),3})} \right\} \\
&= O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right)
\end{aligned}$$

which shows result (iii)

Combining the results given in (i)-(iii) above, we obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] W'_t - e'_{i,N} (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)) \beta \right| \geq C_*^{[1]} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([W_{it} + u_{it}] W'_t - e'_{i,N} (L_1 \underline{\Delta}_1 L'_1 + L_2 \underline{\Delta}_2 L'_2)) \beta \right| \geq C_*^{[1]} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\leq N \max_{1 \leq i \leq N} P \left(\left| \sum_{t=1}^{T_h} ([W_{it} + u_{it}] W'_t \beta - e'_{i,N} L_1 \underline{\Delta}_1 L'_1 \beta) \right| \geq C_*^{[1]} T_h \sqrt{\frac{\ln N}{T_h}} \right) \\
&\leq NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(1),1}} \right\} + N \exp \left\{ -\frac{s^2 T_h^2}{C_{(1),2} (1 + T_h C_{(1),3})} \right\} \\
&\quad + N \exp \left\{ -\frac{(sT_h)^2}{C_{(1),4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_{(1),5} (\ln sT_h)^{\gamma_*}} \right\} \right\} \quad \left(\text{with } s = C_*^{[1]} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
&= o \left(\frac{1}{N^2} \right). \tag{103}
\end{aligned}$$

Next, we show that there exists a constant $C_*^{[2]} > 0$ such that

$$P \left\{ \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] u'_t - e'_{i,N} \Sigma_{u,t} \} \underline{\beta} \geq C_*^{[2]} \sqrt{\frac{\ln N}{T_h}} \right\} = o \left(\frac{1}{N^2} \right), \text{ where } \underline{\beta} = \frac{\beta}{\|\beta\|_2}.$$

To proceed, note that, by Assumption 4-1,

$$\begin{aligned}
& E \left(\{ [W_{it} + u_{it}] u'_t - e'_{i,N} \Sigma_{u,t} \} \underline{\beta} \right) \\
&= e'_{i,N} E [W_t u'_t] \underline{\beta} + e'_{i,N} E [u_t u'_t] \underline{\beta} - e'_{i,N} \Sigma_{u,t} \underline{\beta} \\
&\quad (\text{since } W_{it} = e'_{i,N} W_t \text{ and } u_{it} = e'_{i,N} u_t) \\
&= e'_{i,N} \Sigma_{u,t} \underline{\beta} - e'_{i,N} \Sigma_{u,t} \underline{\beta} \quad (\text{since } E [W_t u'_t] = 0 \text{ and } \Sigma_{u,t} = E [u_t u'_t]) \\
&= 0
\end{aligned}$$

from which it follows that the elements of the sum

$$\sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] u'_t \underline{\beta} - e'_{i,N} \Sigma_{u,t} \underline{\beta} \}$$

given above all have expectation equal to zero. Moreover, let γ_1 and b_2 be the positive constants defined, respectively, in part (a) of Assumption 4-3 and in Lemma C-2. Then,

by part (d) of Lemma C-3, there exists a positive constant $\overline{C} \geq \exp\{1\}$ such that, for any $s > 0$, for all integer i with $1 \leq i \leq N$, and for all $t \in \mathbb{Z}$,

$$\begin{aligned} P\left(\left|(X_{it} - \mu_{X,i}) u'_t \underline{\beta}\right| > s\right) &= P\left(\left|(W_{it} + u_{it}) u'_t \underline{\beta}\right| > s\right) \\ &\leq \exp\left\{1 - \left(\frac{s}{b}\right)^r\right\} \\ &\leq \overline{C} \exp\left\{-\left(\frac{s}{b}\right)^r\right\}. \end{aligned}$$

where $r = \gamma_1/2 > 0$ and $b \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} b_2^2 > 0$. Next, define $\gamma_* = \gamma_1 \gamma_2 / (\gamma_1 + 2\gamma_2)$ with γ_2 being a positive constant as defined in Assumption 4-3, so that the reciprocal of γ_* can be written as

$$\frac{1}{\gamma_*} = \frac{\gamma_1 + 2\gamma_2}{\gamma_1 \gamma_2} = \frac{2}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r} + \frac{1}{\gamma_2}.$$

Now, the condition specified in Assumption 4-3 implies that

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} = \gamma < 1$$

so that, in particular, $\gamma_* \in (0, 1)$. In addition, let $\underline{Z}_{i,t}^{(2)} = (X_{it} - \mu_{X,i}) u'_t \underline{\beta} = (W_{it} + u_{it}) u'_t \underline{\beta}$, and let $V_t = (\varepsilon_t, W'_t, u'_t)'$ be as defined in part (b) of Assumption 4-3. Note that $\mathcal{F}_{\underline{Z}^{(2)},i,-\infty}^t = \sigma(\dots, \underline{Z}_{i,t-2}^{(2)}, \underline{Z}_{i,t-1}^{(2)}, \underline{Z}_{i,t}^{(2)}) \subseteq \mathcal{F}_{-\infty}^t = \sigma(\dots, V_{t-2}, V_{t-1}, V_t)$ and $\mathcal{F}_{\underline{Z}^{(2)},i,t+m}^\infty = \sigma(\underline{Z}_{i,t+m}^{(2)}, \underline{Z}_{i,t+m+1}^{(2)}, \underline{Z}_{i,t+m+2}^{(2)}, \dots)$. Hence, it follows by part (b) of Assumption 4-3 and Theorem 14.1 of Davidson (1994) that $\{\underline{Z}_{i,t}^{(2)}\}_{t \in \mathbb{Z}}$ is an α -mixing process with α coefficient $\alpha_{\underline{Z}^{(2)},i}(m) = \sup_t \alpha(\mathcal{F}_{\underline{Z}^{(2)},i,-\infty}^t, \mathcal{F}_{\underline{Z}^{(2)},i,t+m}^\infty)$ satisfying the rate of decay

$$\begin{aligned} \alpha_{\underline{Z}^{(2)},i}(m) &= \sup_t \alpha(\mathcal{F}_{\underline{Z}^{(2)},i,-\infty}^t, \mathcal{F}_{\underline{Z}^{(2)},i,t+m}^\infty) \\ &\leq \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1, \text{ for all } i, \end{aligned}$$

and for some positive constants γ_2 , a and c .

Hence, all the assumptions needed to apply the Bernstein inequality given in Lemma C-1 are fulfilled. Making use of this Bernstein inequality then implies that there exist constants $C_{(2),1}, C_{(2),2}, C_{(2),3}, C_{(2),4}$ and $C_{(2),5}$ depending on a, b, c, γ_* , and γ_2 such that, for any $s > 0$

and for all integer i with $1 \leq i \leq N$,

$$\begin{aligned}
& P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [X_{it} - \mu_{X,i}] u'_t - e'_{i,N} \Sigma_{u,t} \} \underline{\beta} \right| \geq s \right) \\
&= P \left(\left| \sum_{t=1}^{T_h} ([W_{it} + u_{it}] u'_t \underline{\beta} - e'_{i,N} \Sigma_{u,t} \underline{\beta}) \right| \geq T_h s \right) \\
&\leq T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(2),1}} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{C_{(2),2} (1 + T_h C_{(2),3})} \right\} \\
&\quad + \exp \left\{ -\frac{(sT_h)^2}{C_{(2),4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_{(2),5} (\ln sT_h)^{\gamma_*}} \right\} \right\}
\end{aligned}$$

Using Bonferroni's inequality, we further obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] u'_t - e'_{i,N} \Sigma_{u,t}) \underline{\beta} \right| \geq s \right) \\
&\leq N \max_{1 \leq i \leq N} P \left(\left| \sum_{t=1}^{T_h} ([W_{it} + u_{it}] u'_t \underline{\beta} - e'_{i,N} \Sigma_{u,t} \underline{\beta}) \right| \geq sT_h \right) \\
&\leq NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(2),1}} \right\} + N \exp \left\{ -\frac{s^2 T_h^2}{C_{(2),2} (1 + T_h C_{(2),3})} \right\} \\
&\quad + N \exp \left\{ -\frac{(sT_h)^2}{C_{(2),4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_{(2),5} (\ln sT_h)^{\gamma_*}} \right\} \right\}
\end{aligned}$$

Now, for some constant $C_*^{[2]} > 0$, let

$$s = C_*^{[2]} \sqrt{\frac{\ln N}{T_h}}.$$

By choosing $C_*^{[2]}$ to be large enough, we can obtain the following three results, which we will show in turn:

(i)

$$NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(2),1}} \right\} = NT_h \exp \left\{ -\frac{(C_*^{[2]})^{\gamma_*}}{C_{(2),1}} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(ii)

$$N \exp \left\{ -\frac{(sT_h)^2}{C_{(2),4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_{(2),5} (\ln [sT_h])^{\gamma_*}} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(iii)

$$N \exp \left\{ -\frac{s^2 T_h^2}{C_{(2),2} (1 + T_h C_{(2),3})} \right\} = N \exp \left\{ -\frac{(C_*^{[2]})^2 \ln N}{C_{(2),2} \left(\frac{1}{T_h} + C_{(2),3} \right)} \right\} = o \left(\frac{1}{N^2} \right).$$

To show result (i), first write

$$\begin{aligned} N^3 T_h \exp \left\{ -\frac{(s T_h)^{\gamma_*}}{C_{(2),1}} \right\} &= N^3 T_h \exp \left\{ -\frac{(C_*^{[2]})^{\gamma_*}}{C_{(2),1}} T_h^{\gamma_*} \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma_*}{2}} \right\} \\ &= N^3 T_h \exp \left\{ -\frac{(C_*^{[2]})^{\gamma_*}}{C_{(2),1}} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \end{aligned}$$

Taking the log transformation, we have

$$\ln \left[N^3 T_h \exp \left\{ -\frac{(s T_h)^{\gamma_*}}{C_{(2),1}} \right\} \right] = 3 \ln N + \ln T_h - \frac{(C_*^{[2]})^{\gamma_*}}{C_{(2),1}} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}$$

Given that

$$\frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0 \quad (\text{since } \gamma_* > 0)$$

and given that

$$\begin{aligned} \frac{3 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} &= 3 \left(\frac{T}{T_h} \right)^{\frac{\gamma_*}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \right)^{\frac{\gamma_*}{2}} \rightarrow 0 \quad \left(\text{given that } \frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \right. \\ &\quad \left. \text{by assumption and given that } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \right); \end{aligned}$$

we see that

$$\frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} = \frac{3 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned}
& N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(2),1}} \right\} \\
&= \exp \left\{ 3 \ln N + \ln T_h - \frac{\left(C_*^{[2]}\right)^{\gamma_*}}{C_{(2),1}} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \\
&= \exp \left\{ -\frac{\left(C_*^{[2]}\right)^{\gamma_*}}{C_{(2),1}} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \left[1 - \frac{C_{(2),1}}{\left(C_*^{[2]}\right)^{\gamma_*}} \frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \right] \right\} \\
&= o(1)
\end{aligned}$$

It follows that

$$N T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(2),1}} \right\} = \frac{1}{N^2} N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(2),1}} \right\} = O\left(\frac{1}{N^2}\right) o(1) = o\left(\frac{1}{N^2}\right).$$

Next, to show result (ii), note that

$$\begin{aligned}
& N^3 \exp \left\{ -\frac{(sT_h)^2}{C_{(2),4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_{(2),5} (\ln [sT_h])^{\gamma_*}} \right\} \right\} \\
&= N^3 \exp \left\{ -\frac{T_h^2}{C_{(2),4} T_h} \left(C_*^{[2]} \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma_*(1-\gamma_*)} \left(C_*^{[2]} \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma_*(1-\gamma_*)}}{C_{(2),5} \left(\ln \left[T_h C_*^{[2]} \sqrt{\frac{\ln N}{T_h}} \right] \right)^{\gamma_*}} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{\left(C_*^{[2]}\right)^2}{C_{(2),4}} \ln N \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} \left(C_*^{[2]}\right)^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_{(2),5} \left(\ln T_h + \ln C_*^{[2]} + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h \right)^{\gamma_*}} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{\left(C_*^{[2]}\right)^2}{C_{(2),4}} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} \left(C_*^{[2]}\right)^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_{(2),5} \left(\frac{1}{2} \ln T_h + \ln C_*^{[2]} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) \right\} \\
&= \exp \left\{ -\ln N \left[\frac{\left(C_*^{[2]}\right)^2}{C_{(2),4}} \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} \left(C_*^{[2]}\right)^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_{(2),5} \left(\frac{1}{2} \ln T_h + \ln C_*^{[2]} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) - 3 \right] \right\} \\
&= o(1)
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
& N \exp \left\{ -\frac{(sT_h)^2}{C_{(2),4}T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_{(2),5}(\ln[sT_h])^\gamma} \right\} \right\} \\
&= \left(\frac{1}{N^2} \right) N^3 \exp \left\{ -\frac{(sT_h)^2}{C_{(2),4}T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_{(2),5}(\ln[sT_h])^{\gamma_*}} \right\} \right\} \\
&= O\left(\frac{1}{N^2}\right) o(1) = o\left(\frac{1}{N^2}\right)
\end{aligned}$$

Turning our attention now to showing result (iii), we first write

$$\begin{aligned}
N^3 \exp \left\{ -\frac{s^2 T_h^2}{C_{(2),2}(1 + T_h C_{(2),3})} \right\} &= N^3 \exp \left\{ -\frac{T_h^2}{C_{(2),2}(1 + T_h C_{(2),3})} \left(C_*^{[2]} \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\
&= N^3 \exp \left\{ -\frac{\left(C_*^{[2]} \right)^2 \ln N}{C_{(2),2} \left(\frac{1}{T_h} + C_{(2),3} \right)} \right\}
\end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned}
\ln \left[N^3 \exp \left\{ -\frac{\left(C_*^{[2]} \right)^2 \ln N}{C_{(2),2} \left(\frac{1}{T_h} + C_{(2),3} \right)} \right\} \right] &= 3 \ln N - \frac{\left(C_*^{[2]} \right)^2 \ln N}{C_{(2),2} \left(\frac{1}{T_h} + C_{(2),3} \right)} \\
&= 3 \ln N - \frac{\left(C_*^{[2]} \right)^2}{C_{(2),2} C_{(2),3}} \ln N \left[1 + C_{(2),3}^{-1} \frac{1}{T_h} \right]^{-1}
\end{aligned}$$

Now, let $C_*^{[2]}$ be a positive constant such that $\left(C_*^{[2]} \right)^2 > 3C_{(2),2}(1 + C_{(2),3})$, and set

$$C_{[2]}^* = \frac{\left(C_*^{[2]} \right)^2}{C_{(2),2} C_{(2),3}} - 3.$$

Note that, by this choice of $C_*^{[2]}$ and $C_{[2]}^*$, we have

$$C_{[2]}^* > \frac{3C_{(2),2}(1 + C_{(2),3})}{C_{(2),2} C_{(2),3}} - 3 = \frac{3}{C_{(2),3}} > 0$$

It, thus, follows that

$$\begin{aligned}
& N^3 \exp \left\{ -\frac{s^2 T_h^2}{C_{(2),2} (1 + T_h C_{(2),3})} \right\} \\
= & N^3 \exp \left\{ -\frac{(C_{*}^{[2]})^2 \ln N}{C_{(2),2} \left(\frac{1}{T_h} + C_{(2),3} \right)} \right\} \\
= & \exp \left\{ \left(3 \ln N \left[1 + C_{(2),3}^{-1} \frac{1}{T_h} \right] - \frac{(C_{*}^{[2]})^2}{C_{(2),2} C_{(2),3}} \ln N \right) \left[1 + C_{(2),3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(3 \ln N \left[1 + C_{(2),3}^{-1} \frac{1}{T_h} \right] - [C_{[2]}^{*} + 3] \ln N \right) \left[1 + C_{(2),3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(\ln N \left[3 C_{(2),3}^{-1} \frac{1}{T_h} - C_{[2]}^{*} \right] \right) \left[1 + C_{(2),3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ -C_{[2]}^{*} \ln N \left[1 - \frac{3}{C_{[2]}^{*} C_{(2),3}} \frac{1}{T_h} \right] \left[1 + C_{(2),3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
< & \exp \left\{ -C_{[2]}^{*} \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_{(2),3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{3}{C_{[2]}^{*} C_{(2),3}} < \frac{C_{(2),3} 3}{3 C_{(2),3}} = 1 \right) \\
= & o(1)
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
N \exp \left\{ -\frac{s T_h^2}{C_{(1),2} (1 + T_h C_{(1),3})} \right\} &= \left(\frac{1}{N^2} \right) N^3 \exp \left\{ -\frac{s T_h^2}{C_{(1),2} (1 + T_h C_{(1),3})} \right\} \\
&= O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right)
\end{aligned}$$

which shows result (iii)

Combining the results given in (i)-(iii) above, we obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] u'_t - e'_{i,N} \Sigma_{u,t}) \underline{\beta} \right| \geq C_*^{[2]} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq N \max_{1 \leq i \leq N} P \left(\left| \sum_{t=1}^{T_h} ([W_{it} + u_{it}] u'_t \underline{\beta} - e'_{i,N} \Sigma_{u,t} \underline{\beta}) \right| \geq C_*^{[2]} T_h \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_{(2),1}} \right\} + N \exp \left\{ -\frac{s^2 T_h^2}{C_{(2),2} (1 + T_h C_{(2),3})} \right\} \\
& \quad + N \exp \left\{ -\frac{(sT_h)^2}{C_{(2),4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma_* (1-\gamma_*)}}{C_{(2),5} (\ln sT_h)^{\gamma_*}} \right\} \right\} \quad \left(\text{with } s = C_*^{[2]} \sqrt{\frac{\ln N}{T_h}} \right) \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

Moreover, since $0 < C_*^{[2]} < \infty$ and since, by part (a) of Assumption 4-4, there exists a constant $\bar{c} > 0$ such that

$$0 < \sqrt{\kappa} \frac{1}{\bar{c}} \leq \sqrt{\kappa} \min_{1 \leq g \leq \kappa} |\phi_g| \leq \|\phi\|_2 = \sqrt{\phi' \phi} \leq \sqrt{\kappa} \max_{1 \leq g \leq \kappa} |\phi_g| \leq \sqrt{\kappa} \bar{c} < \infty,$$

we let $C_{\dagger}^{[2]} = C_*^{[2]} \|\phi\|_2$; and note that $0 < C_*^{[2]} \sqrt{\kappa} / \bar{c} \leq C_{\dagger}^{[2]} \leq C_*^{[2]} \sqrt{\kappa} \bar{c} < \infty$, so that $C_{\dagger}^{[2]} \in (0, \infty)$. In addition, note that

$$\underline{\beta} = \frac{\beta}{\|\beta\|_2} = \frac{L_1 \phi / \sqrt{\Delta_{1,1}}}{\|\phi\|_2 / \sqrt{\Delta_{1,1}}} = \frac{L_1 \phi}{\|\phi\|_2}, \text{ so that } \|\underline{\beta}\|_2 = \sqrt{\frac{\phi' L_1' L_1 \phi}{\|\phi\|_2^2}} = \frac{\|\phi\|_2}{\|\phi\|_2} = 1.$$

Hence, given that, by definition, $\beta = \underline{\beta} \|\beta\|_2$, we further deduce that

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] u'_t - e'_{i,N} \Sigma_{u,t}) \beta \right| \geq C_{\dagger}^{[2]} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([W_{it} + u_{it}] u'_t \beta - e'_{i,N} \Sigma_{u,t} \beta) \right| \geq C_{\dagger}^{[2]} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([W_{it} + u_{it}] u'_t \underline{\beta} - e'_{i,N} \Sigma_{u,t} \underline{\beta}) \right| \|\beta\|_2 \geq C_{\dagger}^{[2]} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([W_{it} + u_{it}] u'_t \underline{\beta} - e'_{i,N} \Sigma_{u,t} \underline{\beta}) \right| \geq \frac{C_{\dagger}^{[2]}}{\|\beta\|_2} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([W_{it} + u_{it}] u'_t \underline{\beta} - e'_{i,N} \Sigma_{u,t} \underline{\beta}) \right| \geq \frac{C_{\dagger}^{[2]} \sqrt{\underline{\lambda}_{1,1}}}{\|\phi\|_2} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad \left(\text{since } \|\beta\|_2 = \sqrt{\frac{\phi' L'_1 L_1 \phi}{\underline{\lambda}_{1,1}}} = \frac{\|\phi\|_2}{\sqrt{\underline{\lambda}_{1,1}}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([W_{it} + u_{it}] u'_t \underline{\beta} - e'_{i,N} \Sigma_{u,t} \underline{\beta}) \right| \geq C_*^{[2]} \sqrt{\underline{\lambda}_{1,1}} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([W_{it} + u_{it}] u'_t \underline{\beta} - e'_{i,N} \Sigma_{u,t} \underline{\beta}) \right| \geq C_*^{[2]} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad (\text{since by Assumption 4-2(a), } \underline{\lambda}_{1,1} \rightarrow \infty) \\
&= o \left(\frac{1}{N^2} \right). \tag{104}
\end{aligned}$$

Finally, let $C^\dagger$ be a positive constant such that $C^\dagger \geq \max \left\{ 2C_*^{[1]}, 2C_\dagger^{[2]} \right\}$, and we can combine the results given in expressions (103) and (104) above to obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] [X_t - \mu_X]' - e'_{i,N} \bar{\Sigma}_X) \beta \right| \geq C^\dagger \sqrt{\frac{\ln N}{T_h}} \right) \\
= & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([W_{it} + u_{it}] [W_t' + u_t'] - e'_{i,N} \bar{\Sigma}_X) \beta \right| \geq C^\dagger \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] W_t' - e'_{i,N} (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2') \} \beta \right| \right. \\
& \left. + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] u_t' - e'_{i,N} \Sigma_{u,t} \} \beta \right| \geq C^\dagger \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] W_t' - e'_{i,N} (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2') \} \beta \right| \geq \frac{C^\dagger}{2} \sqrt{\frac{\ln N}{T_h}} \right\} \right. \\
& \left. \cup \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] u_t' - e'_{i,N} \Sigma_{u,t} \} \beta \right| \geq \frac{C^\dagger}{2} \sqrt{\frac{\ln N}{T_h}} \right\} \right) \\
\leq & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] W_t' - e'_{i,N} (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2') \} \beta \right| \geq \frac{C^\dagger}{2} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] u_t' - e'_{i,N} \Sigma_{u,t} \} \beta \right| \geq \frac{C^\dagger}{2} \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] W_t' - e'_{i,N} (L_1 \underline{\Delta}_1 L_1' + L_2 \underline{\Delta}_2 L_2') \} \beta \right| \geq C_*^{[1]} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \{ [W_{it} + u_{it}] u_t' - e'_{i,N} \Sigma_{u,t} \} \beta \right| \geq C_\dagger^{[2]} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \left(\text{since } C^\dagger \geq \max \left\{ 2C_*^{[1]}, 2C_\dagger^{[2]} \right\} \text{ implying that } \frac{C^\dagger}{2} \geq C_*^{[1]} \text{ and } \frac{C^\dagger}{2} \geq C_\dagger^{[2]} \right) \\
= & o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
= & o \left(\frac{1}{N^2} \right).
\end{aligned}$$

Turning our attention to part (c), first set

$$C^\Delta \geq \max \{ B_{12}, C_*^2 \}$$

where $B_{12} > 0$ is the constant given in part (a) of Lemma C-5 and where $C_* > 0$ is the constant given in part (a) of this lemma. It then follows by part (a) of these two lemmas that, for all N and T sufficiently large,

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]' \beta \right| \geq C^\Delta \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq P \left(\left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq \sqrt{C^\Delta} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right. \\
& \quad \left. \cup \left\{ \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]' \beta \right| \geq \sqrt{C^\Delta} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq \sqrt{C^\Delta} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \\
& \quad + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]' \beta \right| \geq \sqrt{C^\Delta} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \\
& \quad \text{(by Bonferroni inequality)} \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq \sqrt{C^\Delta} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]' \beta \right| \geq \sqrt{C^\Delta} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad \left(\text{since } \sqrt{\frac{\ln N}{T_h}} < \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \text{ given that } \frac{\ln N}{T_h} = \frac{\ln N}{T-h} < 1 \text{ for all } N \text{ and } T \text{ sufficiently large} \right) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq \sqrt{B_{12}} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]' \beta \right| \geq C_* \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad \left(\text{since setting } C^\Delta \geq \max \{B_{12}, C_*^2\} \text{ ensures that } \sqrt{C^\Delta} \geq \max \{ \sqrt{B_{12}}, C_* \} \right) \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \quad \text{(by part (a) of Lemma C-5 and part (a) of this lemma)} \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

To show part (d), note that, given that $h \geq 1$ is a fixed finite integer, we can let $\tau = t + h$ and, without loss of generality, write $\varepsilon_{t+h} = \varepsilon_\tau$. Now, by part (c) of Assumption 4-3, we have $E[\varepsilon_\tau] = 0$ for all $\tau \in \mathbb{Z}$. Moreover, by part (a) of Assumption 4-3, there exist positive constants γ_1 and b_1 such that for any $s > 0$,

$$\sup_{\tau \in \mathbb{Z}} P(|\varepsilon_\tau| > s) \leq \exp \left\{ 1 - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\},$$

where $\overline{C}$ is a positive constant such that $\overline{C} \geq \exp\{1\}$. Next, let $V_\tau = (\varepsilon_\tau, W'_\tau, u'_\tau)'$, and note that $\mathcal{F}_{\varepsilon, -\infty}^\tau = \sigma(\dots, \varepsilon_{\tau-1}, \varepsilon_\tau) \subseteq \mathcal{F}_{-\infty}^\tau = \sigma(\dots, V_{\tau-2}, V_{\tau-1}, V_\tau)$ and $\mathcal{F}_{\varepsilon, \tau+m}^\infty = \sigma(\varepsilon_{\tau+m}, \varepsilon_{\tau+m+1}, \varepsilon_{\tau+m+2}, \dots) \subseteq \mathcal{F}_{\tau+m}^\infty = \sigma(V_{\tau+m}, V_{\tau+m+1}, V_{\tau+m+2}, \dots)$. Hence, it follows by part (b) of Assumption 4-3 and Theorem 14.1 of Davidson (1994) that $\{\varepsilon_{t+h}\}_{t \in \mathbb{Z}} \equiv \{\varepsilon_\tau\}_{\tau \in \mathbb{Z}}$ is an α -mixing process with α coefficient $\alpha_\varepsilon(m) = \sup_\tau \alpha(\mathcal{F}_{\varepsilon, -\infty}^\tau, \mathcal{F}_{\varepsilon, \tau+m}^\infty)$ satisfying the rate of decay

$$\begin{aligned} \alpha_\varepsilon(m) &= \sup_\tau \alpha(\mathcal{F}_{\varepsilon, -\infty}^\tau, \mathcal{F}_{\varepsilon, \tau+m}^\infty) \\ &\leq \sup_\tau \alpha(\mathcal{F}_{-\infty}^\tau, \mathcal{F}_{\tau+m}^\infty) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1. \end{aligned}$$

and for some positive constants γ_2 , a and c . Furthermore, note that, by Assumption 4-3, there exists a constant $\gamma \in (0, 1)$ such that

$$\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.$$

Therefore, all the assumptions needed to obtain the Bernstein-type inequality given in Merlevède, Peligrad, and Rio (2011) (restated here as Lemma C-1) are fulfilled. Applying this Bernstein-type inequality then implies that there exist constants $C_{*,1}, C_{*,2}, C_{*,3}, C_{*,4}$ and $C_{*,5}$ depending on a, b_1, c, γ , and γ_2 such that, for any $s > 0$,

$$\begin{aligned} &P(|\bar{\varepsilon}_{T_h}| \geq s) \\ &= P\left(\left|\frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h}\right| \geq s\right) \\ &\leq T_h \exp\left\{-\frac{(sT_h)^\gamma}{C_{*,1}}\right\} + \exp\left\{-\frac{s^2 T_h^2}{C_{*,2}(1 + T_h C_{*,3})}\right\} + \exp\left\{-\frac{(sT_h)^2}{C_{*,4} T_h} \exp\left\{\frac{(sT_h)^{\gamma(1-\gamma)}}{C_{*,5} (\ln sT_h)^\gamma}\right\}\right\}. \end{aligned}$$

For some $B_* > 0$, let

$$s = B_* \sqrt{\frac{\ln N}{T_h}}.$$

By choosing B_* to be large enough, we can obtain the following three results, which we will show in turn:

(i)

$$T_h \exp\left\{-\frac{(sT_h)^\gamma}{C_{*,1}}\right\} = T_h \exp\left\{-\frac{B_*^\gamma}{C_{*,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}\right\} = o\left(\frac{1}{N^2}\right).$$

(ii)

$$\exp\left\{-\frac{(sT_h)^2}{C_{*,4} T_h} \exp\left\{\frac{(sT_h)^{\gamma(1-\gamma)}}{C_{*,5} (\ln [sT_h])^\gamma}\right\}\right\} = o\left(\frac{1}{N^2}\right)$$

(iii)

$$\exp \left\{ -\frac{s^2 T_h^2}{C_{*,2}(1 + T_h C_{*,3})} \right\} = \exp \left\{ -\frac{B_*^2 \ln N}{C_{*,2} \left(\frac{1}{T_h} + C_{*,3} \right)} \right\} = O \left(\frac{1}{N^2} \right).$$

To show result (i), first write

$$N^2 T_h \exp \left\{ -\frac{(s T_h)^\gamma}{C_{*,1}} \right\} = N^2 T_h \exp \left\{ -\frac{B_*^\gamma}{C_{*,1}} T_h^\gamma \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma}{2}} \right\} = N^2 T_h \exp \left\{ -\frac{B_*^\gamma}{C_{*,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\}$$

Taking the log transformation, we have

$$\ln \left[N^2 T_h \exp \left\{ -\frac{(s T_h)^\gamma}{C_{*,1}} \right\} \right] = 2 \ln N + \ln T_h - \frac{B_*^\gamma}{C_{*,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}$$

Now, by assumption, $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$, where

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} > 0.$$

Since

$$\gamma = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} > \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} = \gamma_*,$$

it follows that $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$ implies that $(\ln N)^{\frac{2}{\gamma}-1} = o(T)$. Making use of the fact that

$$\frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0 \quad (\text{given that } \gamma > 0)$$

and

$$\begin{aligned} \frac{2(\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} &= 2 \left(\frac{T}{T_h} \right)^{\frac{\gamma}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \right)^{\frac{\gamma}{2}} \rightarrow 0 \quad \left(\text{given that } \frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \rightarrow 0 \text{ and} \right. \\ &\quad \left. \text{given that } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \text{ since } h \text{ is a fixed finite integer} \right) \end{aligned}$$

we see that

$$\frac{2 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} = \frac{2(\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned}
N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{*,1}} \right\} &= \exp \left\{ 2 \ln N + \ln T_h - \frac{B_*^\gamma}{C_{*,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\} \\
&= \exp \left\{ -\frac{B_*^\gamma}{C_{*,1}} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \left[1 - \frac{C_{*,1}}{B_*^\gamma} \frac{2 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \right] \right\} \\
&= o(1)
\end{aligned}$$

It further follows that

$$T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{*,1}} \right\} = \frac{1}{N^2} N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{*,1}} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (ii), note that

$$\begin{aligned}
&N^2 \exp \left\{ -\frac{(sT_h)^2}{C_{*,4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_{*,5} (\ln [sT_h])^\gamma} \right\} \right\} \\
&= N^2 \exp \left\{ -\frac{T_h^2}{C_{*,4} T_h} \left(B_* \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma(1-\gamma)} \left(B_* \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma(1-\gamma)}}{C_{*,5} \left(\ln \left[T_h B_* \sqrt{\frac{\ln N}{T_h}} \right] \right)^\gamma} \right) \right\} \\
&= N^2 \exp \left\{ -\frac{B_*^2}{C_{*,4}} \ln N \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} B_*^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_{*,5} \left(\ln T_h + \ln B_* + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h \right)^\gamma} \right) \right\} \\
&= N^2 \exp \left\{ -\frac{B_*^2}{C_{*,4}} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} B_*^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_{*,5} \left(\frac{1}{2} \ln T_h + \ln B_* + \frac{1}{2} \ln \ln N \right)^\gamma} \right) \right\} \\
&= \exp \left\{ -\ln N \left[\frac{B_*^2}{C_{*,4}} \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} B_*^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_{*,5} \left(\frac{1}{2} \ln T_h + \ln B_* + \frac{1}{2} \ln \ln N \right)^\gamma} \right) - 2 \right] \right\} \\
&= o(1)
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
&\exp \left\{ -\frac{(sT_h)^2}{C_{*,4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_{*,5} (\ln [T_h s])^\gamma} \right\} \right\} \\
&= \left(\frac{1}{N^2} \right) N^2 \exp \left\{ -\frac{(sT_h)^2}{C_{*,4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_{*,5} (\ln [sT_h])^\gamma} \right\} \right\} \\
&= O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

Turning our attention now to showing result (iii), we first write

$$\begin{aligned}
N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_{*,2} (1 + T_h C_{*,3})} \right\} &= N^2 \exp \left\{ -\frac{T_h^2}{C_{*,2} (1 + T_h C_{*,3})} \left(B_* \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\
&= N^2 \exp \left\{ -\frac{B_*^2 \ln N}{C_{*,2} \left(\frac{1}{T_h} + C_{*,3} \right)} \right\}
\end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned}
\ln \left[N^2 \exp \left\{ -\frac{B_*^2 \ln N}{C_{*,2} \left(\frac{1}{T_h} + C_{*,3} \right)} \right\} \right] &= 2 \ln N - \frac{B_*^2 \ln N}{C_{*,2} \left(\frac{1}{T_h} + C_{*,3} \right)} \\
&= 2 \ln N - \frac{B_*^2}{C_{*,2} C_{*,3}} \ln N \left[1 + C_{*,3}^{-1} \frac{1}{T_h} \right]^{-1}
\end{aligned}$$

Now, let B_* be a positive constant such that $B_*^2 > 2C_{*,2} (1 + C_{*,3})$, and set

$$C^* = \frac{B_*^2}{C_{*,2} C_{*,3}} - 2.$$

Note that, by this choice of B_* and C^* , we have

$$C^* > \frac{2C_{*,2} (1 + C_{*,3})}{C_{*,2} C_{*,3}} - 2 = \frac{2}{C_{*,3}} > 0$$

It, thus, follows that

$$\begin{aligned}
& N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_{*,2} (1 + T_h C_{*,3})} \right\} \\
= & N^2 \exp \left\{ -\frac{B_*^2 \ln N}{C_{*,2} \left(\frac{1}{T_h} + C_{*,3} \right)} \right\} \\
= & \exp \left\{ \left(2 \ln N \left[1 + C_{*,3}^{-1} \frac{1}{T_h} \right] - \frac{B_*^2}{C_{*,2} C_{*,3}} \ln N \right) \left[1 + C_{*,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(2 \ln N \left[1 + C_{*,3}^{-1} \frac{1}{T_h} \right] - [C^* + 2] \ln N \right) \left[1 + C_{*,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(\ln N \left[2 C_{*,3}^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_{*,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ -C^* \ln N \left[1 - \frac{2}{C^* C_{*,3}} \frac{1}{T_h} \right] \left[1 + C_{*,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
< & \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_{*,3}^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{2}{C^* C_{*,3}} < \frac{C_{*,3} 2}{2 C_{*,3}} = 1 \right) \\
= & o(1)
\end{aligned}$$

This, in turn, implies that

$$\exp \left\{ -\frac{s^2 T_h^2}{C_{*,2} (1 + T_h C_{*,3})} \right\} = \left(\frac{1}{N^2} \right) N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_{*,2} (1 + T_h C_{*,3})} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (iii).

Combining the results given in (i)-(iii) above, we obtain

$$\begin{aligned}
& P \left(|\bar{\varepsilon}_{T_h}| \geq B_* \sqrt{\frac{\ln N}{T_h}} \right) \\
= & P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} \right| \geq B_* \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_{*,1}} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{C_{*,2}(1+T_h C_{*,3})} \right\} + \exp \left\{ -\frac{(sT_h)^2}{C_{*,4} T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_{*,5} (\ln sT_h)^\gamma} \right\} \right\} \\
= & o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
= & o \left(\frac{1}{N^2} \right).
\end{aligned}$$

Next, to show part (e), first set $C^\# = \max \{C_*^2, B_*^2\}$, where $C_* > 0$ is the constant given in part (a) of Lemma C-5 and where $B_* > 0$ is the constant given in part (d) of this lemma.

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| |\bar{\varepsilon}_{T_h}| \geq C^\# \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq \sqrt{C^\#} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \cup \left\{ |\bar{\varepsilon}_{T_h}| \geq \sqrt{C^\#} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right) \\
\leq & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq \sqrt{C^\#} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) + P \left(|\bar{\varepsilon}_{T_h}| \geq \sqrt{C^\#} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \\
& \text{(by Bonferroni inequality)} \\
\leq & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq \sqrt{C^\#} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(|\bar{\varepsilon}_{T_h}| \geq \sqrt{C^\#} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \left(\text{since } \sqrt{\frac{\ln N}{T_h}} < \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \text{ given that } \frac{\ln N}{T_h} = \frac{\ln N}{T-h} < 1 \text{ for all } N, T \text{ sufficiently large} \right) \\
\leq & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq C_* \sqrt{\frac{\ln N}{T_h}} \right) + P \left(|\bar{\varepsilon}_{T_h}| \geq B_* \sqrt{\frac{\ln N}{T_h}} \right) \\
& \left(\text{since } C^\# = \max \{C_*^2, B_*^2\} \text{ implies that } \sqrt{C^\#} \geq C_* \text{ and } \sqrt{C^\#} \geq B_* \right).
\end{aligned}$$

It then follows, by part (a) of Lemma C-5 and and part (d) of this lemma, that for all N and T sufficiently large,

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| |\bar{\varepsilon}_{T_h}| \geq C^\# \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \geq C_* \sqrt{\frac{\ln N}{T_h}} \right) + P \left(|\bar{\varepsilon}_{T_h}| \geq B_* \sqrt{\frac{\ln N}{T_h}} \right) \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

Finally, to show part (f), first note that, by part (c) of Assumption 4-3, $E[(X_{it} - \mu_{X,i}) \varepsilon_{t+h}] = E[(W_{it} + u_{it}) \varepsilon_{t+h}] = E[W_{it} \varepsilon_{t+h}] + E[u_{it} \varepsilon_{t+h}] = 0$ for all i and t . In addition, by part (b) of Lemma C-3 above, there exists a positive constant $\bar{C} \geq \exp\{1\}$ such that $\forall s > 0$, for all integer i with $1 \leq i \leq N$, and for all $t \in \mathbb{Z}$,

$$P(|(X_{it} - \mu_{X,i}) \varepsilon_{t+h}| > s) \leq \exp \left\{ 1 - \left(\frac{s}{b} \right)^r \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{b} \right)^r \right\},$$

where $r = \gamma_1/2 > 0$ and where $b \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} b_1^2 > 0$. Moreover, let $\gamma_* = \gamma_1 \gamma_2 / (\gamma_1 + 2\gamma_2)$, so that

$$\frac{1}{\gamma_*} = \frac{\gamma_1 + 2\gamma_2}{\gamma_1 \gamma_2} = \frac{2}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r} + \frac{1}{\gamma_2}.$$

Now, Assumption 4-3 implies that

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} = \gamma < 1$$

which, in turn, implies that $\gamma_* \in (0, 1)$. Let $\tau = t + h$, and we can rewrite $(X_{it} - \mu_{X,i}) \varepsilon_{t+h} = (W_{it} + u_{it}) \varepsilon_{t+h} = (W_{i,\tau-h} + u_{i,\tau-h}) \varepsilon_\tau = \underline{U}_{i,\tau}$. Next, let $V_t = (\varepsilon_t, W'_t, u'_t)'$ be as defined in part (b) of Assumption 4-3, and note that $\mathcal{F}_{\underline{U},i,-\infty}^\tau = \sigma(\dots, \underline{U}_{i,\tau-2}, \underline{U}_{i,\tau-1}, \underline{U}_{i,\tau}) \subseteq \mathcal{F}_{-\infty}^{t+h} = \sigma(\dots, V_{t+h-2}, V_{t+h-1}, V_{t+h})$ and $\mathcal{F}_{\underline{U},i,\tau+m}^\infty = \sigma(\underline{U}_{i,\tau+m}, \underline{U}_{i,\tau+m+1}, \underline{U}_{i,\tau+m+2}, \dots) \subseteq \mathcal{F}_{t+m}^\infty = \sigma(V_{t+m}, V_{t+m+1}, V_{t+m+2}, \dots)$. Hence, given that $h \geq 1$ is a fixed (finite) integer, it follows by part (b) of Assumption 4-3 and Theorem 14.1 of Davidson (1994) that $\{\underline{U}_{i,\tau}\}_{\tau \in \mathbb{Z}}$ is an α -mixing process with α coefficient given by $\alpha_{\underline{U},i}(m) = \sup_\tau \alpha(\mathcal{F}_{\underline{U},i,-\infty}^\tau, \mathcal{F}_{\underline{U},i,\tau+m}^\infty)$. To analyze the rate of decay of $\alpha_{\underline{U},i}(m)$, we first consider the case where $m \geq h + 1$. In this case, note

that

$$\begin{aligned}
& \alpha_{\underline{U},i}(m) \\
&= \sup_{\tau} \alpha(\mathcal{F}_{\underline{U},i,-\infty}^{\tau}, \mathcal{F}_{\underline{U},i,\tau+m}^{\infty}) \\
&\leq \sup_t \alpha(\mathcal{F}_{-\infty}^{t+h}, \mathcal{F}_{t+m}^{\infty}) \\
&\leq a \exp\{-c(m-h)^{\gamma_2}\} \text{ for all integers } m \geq h+1 \text{ and for all } i \\
&\quad \text{(by part (b) of Assumption 4-3)} \\
&= a \exp\left\{-cm^{\gamma_2} \left(1 - \frac{h}{m}\right)^{\gamma_2}\right\} \\
&\leq a \exp\left\{-cm^{\gamma_2} \left(1 - \frac{h}{h+1}\right)^{\gamma_2}\right\} \quad (\text{since } m \geq h+1) \\
&= a \exp\left\{-cm^{\gamma_2} \left(\frac{1}{h+1}\right)^{\gamma_2}\right\} \\
&\leq a \exp\{-c^{\#}m^{\gamma_2}\} \text{ where } c^{\#} = c \left(\frac{1}{h+1}\right)^{\gamma_2} > 0 \text{ given that } h \text{ is a finite positive int.} \quad (105)
\end{aligned}$$

Next, consider the case where $m \leq h$. In this case, we have

$$\begin{aligned}
\alpha_{\underline{U},i}(m) &= \sup_{\tau} \alpha(\mathcal{F}_{\underline{U},i,-\infty}^{\tau}, \mathcal{F}_{\underline{U},i,\tau+m}^{\infty}) \\
&\leq 1 \\
&= \exp\{c^{\#}m^{\gamma_2}\} \exp\{-c^{\#}m^{\gamma_2}\} \\
&\leq \exp\{c^{\#}h^{\gamma_2}\} \exp\{-c^{\#}m^{\gamma_2}\} \\
&= a_* \exp\{-c^{\#}m^{\gamma_2}\} \quad (106)
\end{aligned}$$

where $a_* = \exp\{c^{\#}h^{\gamma_2}\} > 0$ since $c^{\#} > 0$ and $h \geq 1$. Moreover, note that $a_* = \exp\{c^{\#}h^{\gamma_2}\} < \infty$ since $h < \infty$ and since

$$c^{\#} = c \left(\frac{1}{h+1}\right)^{\gamma_2} < \infty$$

Let $a^{\#} = \max\{a, a_*\}$, and it follows by combining expressions (105) and (106) that

$$\alpha_{\underline{U},i}(m) \leq a^{\#} \exp\{-c^{\#}m^{\gamma_2}\} \text{ for all } m \geq 1 \text{ and for all } i.$$

Therefore, all the assumptions needed to apply the Bernstein inequality given in Merlevède, Peligrad, and Rio (2011) (restated here as Lemma C-1) are fulfilled. Application of this Bernstein inequality then implies that there exist positive constants $\underline{C}_1, \underline{C}_2, \underline{C}_3, \underline{C}_4$ and $\underline{C}_5$ depending on $a^{\#}, b, c^{\#}, \gamma_*$, and γ_2 such that for any $s > 0$ and for all integer i with

$$1 \leq i \leq N$$

$$\begin{aligned}
& P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \varepsilon_{t+h} \right| \geq s \right) \\
&= P \left(\left| \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \varepsilon_{t+h} \right| \geq T_h s \right) \\
&\leq T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{\underline{C}_1} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{\underline{C}_2 (1 + T_h \underline{C}_3)} \right\} + \exp \left\{ -\frac{(sT_h)^2}{\underline{C}_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_* (1-\gamma_*)}}{\underline{C}_5 (\ln sT_h)^{\gamma_*}} \right\} \right\}
\end{aligned}$$

Using Bonferroni's inequality, we further obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \varepsilon_{t+h} \right| \geq s \right) \\
&\leq N \max_{1 \leq i \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \varepsilon_{t+h} \right| \geq s \right) \\
&\leq NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{\underline{C}_1} \right\} + N \exp \left\{ -\frac{s^2 T_h^2}{\underline{C}_2 (1 + T_h \underline{C}_3)} \right\} \\
&\quad + N \exp \left\{ -\frac{(sT_h)^2}{\underline{C}_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_* (1-\gamma_*)}}{\underline{C}_5 (\ln sT_h)^{\gamma_*}} \right\} \right\}
\end{aligned}$$

For some $B^\# > 0$, let

$$s = B^\# \sqrt{\frac{\ln N}{T_h}}.$$

By choosing $B^\#$ to be large enough, we can obtain the following three results, which we will show in turn:

(i)

$$NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{\underline{C}_1} \right\} = NT_h \exp \left\{ -\frac{[B^\#]^{\gamma_*}}{\underline{C}_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(ii)

$$N \exp \left\{ -\frac{(sT_h)^2}{\underline{C}_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_* (1-\gamma_*)}}{\underline{C}_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(iii)

$$N \exp \left\{ -\frac{s^2 T_h^2}{\underline{C}_2 (1 + T_h \underline{C}_3)} \right\} = N \exp \left\{ -\frac{[B^\#]^2 \ln N}{\underline{C}_2 \left(\frac{1}{T_h} + \underline{C}_3 \right)} \right\} = O \left(\frac{1}{N^2} \right).$$

To show result (i), first write

$$\begin{aligned} N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{\underline{C}_1} \right\} &= N^3 T_h \exp \left\{ -\frac{[B^\#]^{\gamma_*}}{\underline{C}_1} T_h^{\gamma_*} \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma_*}{2}} \right\} \\ &= N^3 T_h \exp \left\{ -\frac{[B^\#]^{\gamma_*}}{\underline{C}_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \end{aligned}$$

Taking the log transformation, we have

$$\ln \left[N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{\underline{C}_1} \right\} \right] = 3 \ln N + \ln T_h - \frac{[B^\#]^{\gamma_*}}{\underline{C}_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}$$

Since

$$\frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0 \quad (\text{given that } \gamma_* > 0)$$

and since

$$\begin{aligned} \frac{3 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} &= 3 \left(\frac{T}{T_h} \right)^{\frac{\gamma_*}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \right)^{\frac{\gamma_*}{2}} \rightarrow 0 \quad \left(\text{given that } \frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \right. \\ &\quad \left. \text{and given that } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \right), \end{aligned}$$

we see that

$$\frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} = \frac{3 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned} N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{\underline{C}_1} \right\} &= \exp \left\{ 3 \ln N + \ln T_h - \frac{[B^\#]^{\gamma_*}}{\underline{C}_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \\ &= \exp \left\{ -\frac{[B^\#]^{\gamma_*}}{\underline{C}_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \left[1 - \frac{\underline{C}_1}{[B^\#]^{\gamma_*}} \frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \right] \right\} \\ &= o(1) \end{aligned}$$

It follows that

$$N T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{\underline{C}_1} \right\} = \frac{1}{N^2} N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{\underline{C}_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (ii), note that

$$\begin{aligned}
& N^3 \exp \left\{ -\frac{(sT_h)^2}{\underline{C}_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{\underline{C}_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} \\
&= N^3 \exp \left\{ -\frac{T_h^2}{\underline{C}_4 T_h} \left(B^\# \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma_*(1-\gamma_*)} \left(B^\# \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma_*(1-\gamma_*)}}{\underline{C}_5 \left(\ln \left[T_h B^\# \sqrt{\frac{\ln N}{T_h}} \right] \right)^{\gamma_*}} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{[B^\#]^2}{\underline{C}_4} \ln N \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} [B^\#]^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{\underline{C}_5 \left(\ln T_h + \ln B^\# + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h \right)^{\gamma_*}} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{[B^\#]^2}{\underline{C}_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} [B^\#]^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{\underline{C}_5 \left(\frac{1}{2} \ln T_h + \ln B^\# + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) \right\} \\
&= \exp \left\{ -\ln N \left[\frac{[B^\#]^2}{\underline{C}_4} \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} [B^\#]^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{\underline{C}_5 \left(\frac{1}{2} \ln T_h + \ln B^\# + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) - 3 \right] \right\} \\
&= o(1)
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
& N \exp \left\{ -\frac{(sT_h)^2}{\underline{C}_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{\underline{C}_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} \\
&= \frac{1}{N^2} N^3 \exp \left\{ -\frac{(sT_h)^2}{\underline{C}_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{\underline{C}_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} \\
&= o\left(\frac{1}{N^2}\right)
\end{aligned}$$

Turning our attention now to showing result (iii), we first write

$$\begin{aligned}
N^3 \exp \left\{ -\frac{s^2 T_h^2}{\underline{C}_2 (1 + T_h \underline{C}_3)} \right\} &= N^3 \exp \left\{ -\frac{T_h^2}{\underline{C}_2 (1 + T_h \underline{C}_3)} \left(B^\# \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\
&= N^3 \exp \left\{ -\frac{[B^\#]^2 \ln N}{\underline{C}_2 \left(\frac{1}{T_h} + \underline{C}_3 \right)} \right\}
\end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned} \ln \left[N^3 \exp \left\{ -\frac{[B^\#]^2 \ln N}{\underline{C}_2 \left(\frac{1}{T_h} + \underline{C}_3 \right)} \right\} \right] &= 3 \ln N - \frac{[B^\#]^2 \ln N}{\underline{C}_2 \left(\frac{1}{T_h} + \underline{C}_3 \right)} \\ &= 3 \ln N - \frac{[B^\#]^2}{\underline{C}_2 \underline{C}_3} \ln N \left[1 + \underline{C}_3^{-1} \frac{1}{T_h} \right]^{-1} \end{aligned}$$

Now, let $B^\#$ be a positive constant such that $[B^\#]^2 > 3\underline{C}_2(1 + \underline{C}_3)$, and set

$$C^* = \frac{[B^\#]^2}{\underline{C}_2 \underline{C}_3} - 3.$$

Note that, by this choice of $B^\#$ and C^* , we have

$$C^* > \frac{3\underline{C}_2(1 + \underline{C}_3)}{\underline{C}_2 \underline{C}_3} - 3 = \frac{3}{\underline{C}_3} > 0$$

It, thus, follows that

$$\begin{aligned} & N^3 \exp \left\{ -\frac{s^2 T_h^2}{\underline{C}_2 (1 + T_h \underline{C}_3)} \right\} \\ &= N^3 \exp \left\{ -\frac{[B^\#]^2 \ln N}{\underline{C}_2 \left(\frac{1}{T_h} + \underline{C}_3 \right)} \right\} \\ &= \exp \left\{ \left(3 \ln N \left[1 + \underline{C}_3^{-1} \frac{1}{T_h} \right] - \frac{[B^\#]^2}{\underline{C}_2 \underline{C}_3} \ln N \right) \left[1 + \underline{C}_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ \left(3 \ln N \left[1 + \underline{C}_3^{-1} \frac{1}{T_h} \right] - [C^* + 3] \ln N \right) \left[1 + \underline{C}_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ \left(\ln N \left[3\underline{C}_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + \underline{C}_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ -C^* \ln N \left[1 - \frac{3}{C^* \underline{C}_3} \frac{1}{T_h} \right] \left[1 + \underline{C}_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &< \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + \underline{C}_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{3}{C^* \underline{C}_3} < \frac{\underline{C}_3 3}{3\underline{C}_3} = 1 \right) \\ &= o(1) \end{aligned}$$

This, in turn, implies that

$$N \exp \left\{ -\frac{s^2 T_h^2}{\underline{C}_2 (1 + T_h \underline{C}_3)} \right\} = \left(\frac{1}{N^2} \right) N^3 \exp \left\{ -\frac{T_h^2 s^2}{\underline{C}_2 (1 + T_h \underline{C}_3)} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (iii).

Combining the results given in (i)-(iii) above, we obtain

$$\begin{aligned} & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \varepsilon_{t+h} \right| \geq B^\# \sqrt{\frac{\ln N}{T_h}} \right) \\ & \leq N \max_{1 \leq i \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \varepsilon_{t+h} \right| \geq B^\# \sqrt{\frac{\ln N}{T_h}} \right) \\ & \leq N T_h \exp \left\{ -\frac{(s T_h)^{\gamma_*}}{\underline{C}_1} \right\} + N \exp \left\{ -\frac{(s T_h)^2}{\underline{C}_4 T_h} \exp \left\{ \frac{(s T_h)^{\gamma_* (1-\gamma_*)}}{\underline{C}_5 (\ln s T_h)^{\gamma_*}} \right\} \right\} \\ & \quad + N \exp \left\{ -\frac{s^2 T_h^2}{\underline{C}_2 (1 + T_h \underline{C}_3)} \right\} \\ & = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\ & = o \left(\frac{1}{N^2} \right). \quad \square \end{aligned}$$

Lemma C-8: Let

$$\widehat{\sigma}_{Xy,i} = \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \bar{X}_{i,T_h}] y_{t+h} \text{ where } T_h = T - h.$$

Under Assumptions 4-2, 4-3, and 4-4; there exists a constant $\bar{B} > 0$ such that, for all N and T sufficiently large,

$$P \left(\max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \geq \bar{B} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right).$$

or

$$\max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| = O_p \left(\sqrt{\frac{\ln N}{T}} \right).$$

Proof of Lemma C-8:

For $i = 1, \dots, N$; write

$$\begin{aligned}
& \widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \overline{X}_{i,T_h}] y_{t+h} - \overline{\sigma}_{Xy,i} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \overline{X}_{i,T_h}] \{ \alpha + [X_t - \mu_X]' \beta + \varepsilon_{t+h} \} - \overline{\sigma}_{Xy,i} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \overline{X}_{i,T_h}] [X_t - \mu_X]' \beta - \overline{\sigma}_{Xy,i} + \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \overline{X}_{i,T_h}] \varepsilon_{t+h} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \overline{X}_{i,T_h}] [X_t - \mu_X]' - e'_{i,N} \overline{\Sigma}_X) \beta + \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \overline{X}_{i,T_h}] \varepsilon_{t+h} \\
&\quad (\text{since } \overline{\sigma}_{Xy,i} = e'_{i,N} \overline{\Sigma}_X \beta) \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i} + \mu_{X,i} - \overline{X}_{i,T_h}] [X_t - \mu_X]' - e'_{i,N} \overline{\Sigma}_X) \beta \\
&\quad + \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i} + \mu_{X,i} - \overline{X}_{i,T_h}] \varepsilon_{t+h} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] [X_t - \mu_X]' - e'_{i,N} \overline{\Sigma}_X) \beta + \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \varepsilon_{t+h} \\
&\quad - (\overline{X}_{i,T_h} - \mu_{X,i}) [\overline{X}_{T_h} - \mu_X]' \beta - (\overline{X}_{i,T_h} - \mu_{X,i}) \overline{\varepsilon}_{T_h}
\end{aligned}$$

where $\overline{\varepsilon}_{T_h} = T_h^{-1} \sum_{t=1}^{T_h} \varepsilon_{t+h}$. Hence, applying the triangle inequality and taking the max over all $i \in \{1, \dots, N\}$; we obtain

$$\begin{aligned}
& \max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i}| \\
&\leq \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] [X_t - \mu_X]' - e'_{i,N} \overline{\Sigma}_X) \beta \right| + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \varepsilon_{t+h} \right| \\
&\quad + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]' \beta \right| + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| |\overline{\varepsilon}_{T_h}|
\end{aligned}$$

Now, let $\overline{B} > 0$ be a constant such that

$$\overline{B} \geq 4 \max \{ C^\dagger, C^\Delta, C^\#, B^\# \}$$

where $C^\dagger$, C^Δ , $C^\#$, and $B^\#$ are positive constants as defined in parts (b), (c), (e), and (f) of Lemma C-7. It then follows from applying the results in parts (b), (c), (e), and (f) of

Lemma C-7 that

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \geq \bar{B} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] [X_t - \mu_X]' - e'_{i,N} \bar{\Sigma}_X) \beta \right| + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \varepsilon_{t+h} \right| \right. \\
& \quad + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]' \beta \right| \\
& \quad \left. + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| |\bar{\varepsilon}_{T_h}| \geq \bar{B} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq P \left(\left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] [X_t - \mu_X]' - e'_{i,N} \bar{\Sigma}_X) \beta \right| \geq \frac{\bar{B}}{4} \sqrt{\frac{\ln N}{T_h}} \right\} \right. \\
& \quad \cup \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]' \beta \right| \geq \frac{\bar{B}}{4} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \quad \cup \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| |\bar{\varepsilon}_{T_h}| \geq \frac{\bar{B}}{4} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \quad \left. \cup \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \varepsilon_{t+h} \right| \geq \frac{\bar{B}}{4} \sqrt{\frac{\ln N}{T_h}} \right\} \right) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} ([X_{it} - \mu_{X,i}] [X_t - \mu_X]' - e'_{i,N} \bar{\Sigma}_X) \beta \right| \geq C^\dagger \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]' \beta \right| \geq C^\Delta \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \right| |\bar{\varepsilon}_{T_h}| \geq C^\# \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i}] \varepsilon_{t+h} \right| \geq B^\# \sqrt{\frac{\ln N}{T_h}} \right) \quad (\text{since } \bar{B} \geq 4 \max \{C^\dagger, C^\Delta, C^\#, B^\#\}) \\
& \quad \text{implies that } \frac{\bar{B}}{4} \geq C^\dagger, \frac{\bar{B}}{4} \geq C^\Delta, \frac{\bar{B}}{4} \geq C^\#, \text{ and } \frac{\bar{B}}{4} \geq B^\# \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right). \quad \square
\end{aligned}$$

Lemma C-9: Let

$$\eta_2 = B_2 \sqrt{\frac{\ln N}{T_h}}$$

for some sufficiently large constant $B_2 > 0$. Suppose that Assumptions 4-2, 4-3, 6-1, and 6-2 hold. Then, for $0 \leq q < 1$

$$\left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 = O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right),$$

where

$$\bar{\sigma}_{Xy} = \frac{1}{T_h} \sum_{t=1}^{T_h} E[(X_t - \mu_i)(y_{t+h} - \alpha)] = \frac{1}{T_h} \sum_{t=1}^{T_h} \bar{\sigma}_{Xy}$$

Proof of Lemma C-9: To proceed, note first that

$$\begin{aligned} \left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 &= \left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}^S + \bar{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\ &\leq \left\| \frac{\bar{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 + \left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2 \end{aligned} \quad (107)$$

where $\bar{\sigma}_{Xy}^S$ is the $N \times 1$ vector whose i^{th} element $\bar{\sigma}_{Xy,i}^S = s_{\eta_2}(\bar{\sigma}_{Xy,i})$. We shall consider the case where $0 < q < 1$ (Case 1) and the case where $q = 0$ (Case 2) separately.

Case 1 $0 < q < 1$:

For this case, consider the first term on the right-hand side of expression (107) above. By conditions (ii) and (iii) of Assumptions 6-1, we can write

$$\begin{aligned}
& \left\| \frac{\bar{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}]^2 \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\bar{\sigma}_{Xy,i}) \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} + s_{\eta_2}(\bar{\sigma}_{Xy,i}) \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\} - \bar{\sigma}_{Xy,i} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad - \bar{\sigma}_{Xy,i} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\}]^2 \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\bar{\sigma}_{Xy,i}) \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} - \bar{\sigma}_{Xy,i} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} - \bar{\sigma}_{Xy,i} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\}]^2 \\
&\quad (\text{since } s_{\eta_2}(\bar{\sigma}_{Xy,i}) = 0 \text{ for } |\bar{\sigma}_{Xy,i}| \leq \eta_2 \text{ by condition (ii) of Assumption 6-1}) \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N [|s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}| \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} + |\bar{\sigma}_{Xy,i}| \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\}]^2 \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N [\eta_2 \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} + |\bar{\sigma}_{Xy,i}| \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\}]^2 \\
&\quad (\text{since } |s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}| \leq \eta_2 \text{ by condition (iii) of Assumption 6-1}) \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N \left[\eta_2^{\frac{q}{2}} \eta_2^{1-\frac{q}{2}} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} + |\bar{\sigma}_{Xy,i}|^{\frac{q}{2}} |\bar{\sigma}_{Xy,i}|^{1-\frac{q}{2}} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\} \right]^2 \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N \left[|\bar{\sigma}_{Xy,i}|^{\frac{q}{2}} \eta_2^{1-\frac{q}{2}} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} + |\bar{\sigma}_{Xy,i}|^{\frac{q}{2}} \eta_2^{1-\frac{q}{2}} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\} \right]^2 \\
&= \frac{\eta_2^{2-q}}{\lambda_1} \sum_{i=1}^N |\bar{\sigma}_{Xy,i}|^q
\end{aligned}$$

Since, in this case,

$$\bar{\sigma}_{Xy} \in \mathcal{U}_{\sigma}(q, a(N), \mathbb{M}) = \left\{ \sigma_{N \times 1} : \max_{1 \leq i \leq N} |\sigma_i| \leq \mathbb{M}, \sum_{i=1}^N |\sigma_i|^q \leq a(N) \right\}$$

with $0 < q < 1$, it follows that

$$\begin{aligned}
& \left\| \frac{\bar{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 \\
& \leq \frac{\eta_2^{2-q}}{\lambda_1} \sum_{i=1}^N |\bar{\sigma}_{Xy,i}|^q \\
& \leq \frac{\eta_2^{2-q}}{\lambda_1} a(N) \quad (\text{given that } \bar{\sigma}_{Xy} \in \mathcal{U}_\sigma(q, a(N), \mathbb{M}) \text{ by part (b) of Assumption 6-2}) \\
& = O\left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right) \quad \left(\text{given that } \eta_2^{2-q} = B_2^{2-q} \left(\frac{\ln N}{T_h}\right)^{\frac{2-q}{2}} \asymp \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right).
\end{aligned}$$

from which we further deduce that

$$\left\| \frac{\bar{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \leq \eta_2^{1-\frac{q}{2}} \sqrt{\frac{a(N)}{\lambda_1}} = O\left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T}\right]^{\frac{1}{2}(1-\frac{q}{2})}\right) \quad (108)$$

Now, consider the second term on the right-hand side of expression (107) above, i.e., the term

$$\left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2.$$

Here, note that, by conditions (i) and (ii) of Assumption 6-1, we have

$$\begin{aligned}
& \left\| \frac{\widehat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2^2 \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
&\quad + \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
&\quad + \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \quad (\text{by condition (ii) of} \\
&\quad \text{Assumption 6-1, } s_{\eta_2}(\bar{\sigma}_{Xy,i}) = 0 \text{ if } |\bar{\sigma}_{Xy,i}| \leq \eta_2 \text{ and } s_{\eta_2}(\widehat{\sigma}_{Xy,i}) = 0 \text{ if } |\widehat{\sigma}_{Xy,i}| \leq \eta_2) \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N \widehat{\sigma}_{Xy,i}^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} + \frac{1}{\lambda_1} \sum_{i=1}^N \bar{\sigma}_{Xy,i}^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad (\text{since, by condition (i) of Assumption 6-1, } |s_{\eta_2}(\widehat{\sigma}_{Xy,i})| \leq |\widehat{\sigma}_{Xy,i}| \text{ and } |s_{\eta_2}(\bar{\sigma}_{Xy,i})| \leq |\bar{\sigma}_{Xy,i}|) \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N \widehat{\sigma}_{Xy,i}^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} + \frac{1}{\lambda_1} \sum_{i=1}^N \bar{\sigma}_{Xy,i}^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{3}{\lambda_1} \sum_{i=1}^N \left[([\widehat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 + [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - \widehat{\sigma}_{Xy,i}]^2 + [s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}]^2) \right. \\
&\quad \quad \left. \times \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \right] \left(\text{since } \left| \sum_{i=1}^m a_i \right|^r \leq m^{r-1} \sum_{i=1}^m |a_i|^r \text{ for } r \geq 1 \right) \\
&= (1) + (2) + (3)
\end{aligned} \tag{109}$$

To bound term (1), write

$$\begin{aligned}
(1) &= \frac{1}{\lambda_1} \sum_{i=1}^N \hat{\sigma}_{Xy,i}^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i} + \bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&\leq \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&\quad + \frac{2}{\lambda_1} \sum_{i=1}^N [\bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&\quad \left(\text{again using the inequality } \left| \sum_{i=1}^m a_i \right|^r \leq m^{r-1} \sum_{i=1}^m |a_i|^r \text{ for } r \geq 1 \right) \\
&\leq \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&\quad + \frac{2}{\lambda_1} \sum_{i=1}^N [\bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \tag{110}
\end{aligned}$$

To bound the second term in the expression above, note that

$$\begin{aligned}
\frac{2}{\lambda_1} \sum_{i=1}^N [\bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} &= \frac{2}{\lambda_1} \sum_{i=1}^N |\bar{\sigma}_{Xy,i}|^{2-q} |\bar{\sigma}_{Xy,i}|^q \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&< 2\eta_2^{2-q} \frac{1}{\lambda_1} \sum_{i=1}^N |\bar{\sigma}_{Xy,i}|^q \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&\leq \frac{2\eta_2^{2-q}}{\lambda_1} \sum_{i=1}^N |\bar{\sigma}_{Xy,i}|^q \\
&\leq \frac{2\eta_2^{2-q}}{\lambda_1} a(N) \\
&\quad (\text{since, in this case, } \bar{\sigma}_{Xy} \in \mathcal{U}_\sigma(q, a(N), \mathbb{M}) \text{ with } 0 < q < 1).
\end{aligned}$$

Next, take $\vartheta \in (0, 1)$, and we can bound the first term in expression (110) as follows

$$\begin{aligned}
& \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
= & \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2\} \\
& + \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, \vartheta \eta_2 < |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
\leq & \frac{2}{\lambda_1} \max_{1 \leq i \leq N} [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \sum_{i=1}^N \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2\} \\
& + \frac{2}{\lambda_1} \max_{1 \leq i \leq N} [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \sum_{i=1}^N \frac{|\bar{\sigma}_{Xy,i}|^q}{|\bar{\sigma}_{Xy,i}|^q} \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, \vartheta \eta_2 < |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
\leq & \frac{2}{\lambda_1} \max_{1 \leq i \leq N} [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \sum_{i=1}^N \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2\} \\
& + \frac{2}{\lambda_1 (\vartheta \eta_2)^q} \max_{1 \leq i \leq N} [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \sum_{i=1}^N |\bar{\sigma}_{Xy,i}|^q \\
\leq & \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2\} \\
& + \frac{2a(N)}{\lambda_1 (\vartheta \eta_2)^q} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \\
& \left(\text{since } \bar{\sigma}_{Xy} \in \mathcal{U}_\sigma(q, a(N), \mathbb{M}) = \left\{ \sigma_{N \times 1} : \max_{1 \leq i \leq N} |\sigma_i| \leq \mathbb{M}, \sum_{i=1}^N |\sigma_i|^q \leq a(N) \right\} \right)
\end{aligned}$$

Moreover, observe that

$$\begin{aligned}
& \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2 \} \\
& \leq \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| + |\bar{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2 \} \\
& = \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > \eta_2 - |\bar{\sigma}_{Xy,i}|, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2 \} \\
& \leq \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > \eta_2 - \vartheta \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2 \} \\
& \leq \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > (1 - \vartheta) \eta_2 \}.
\end{aligned}$$

Note that, by the result of Lemma C-8, if we choose

$$\eta_2 = B_2 \sqrt{\frac{\ln N}{T_h}} \text{ where } B_2 = \frac{\bar{B}}{1 - \vartheta}$$

and where $\bar{B}$ is the positive constant given in Lemma C-8; then,

$$\begin{aligned}
P \left[\sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > (1 - \vartheta) \eta_2 \} > 0 \right] &= P \left[\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > (1 - \vartheta) \eta_2 \right] \\
&= o \left(\frac{1}{N^2} \right),
\end{aligned}$$

uniformly on

$$\mathcal{U}_\sigma(q, a(N), \mathbb{M}) = \left\{ \sigma_{N \times 1} : \max_{1 \leq i \leq N} |\sigma_i| \leq \mathbb{M}, \sum_{i=1}^N |\sigma_i|^q \leq a(N) \right\}$$

Making use of the fact that

$$P \left[\sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > (1 - \vartheta) \eta_2 \} > 0 \right] = o \left(\frac{1}{N^2} \right) = o(1) \text{ as } N, T \rightarrow \infty,$$

we further obtain

$$\begin{aligned}
& \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
& \leq \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2 \} \\
& \quad + \frac{2a(N)}{\lambda_1 (\vartheta \eta_2)^q} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \\
& \leq \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > (1 - \vartheta) \eta_2 \} \\
& \quad + \frac{2a(N)}{\lambda_1 (\vartheta \eta_2)^q} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \\
& = \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > (1 - \vartheta) \eta_2 \} \\
& \quad + \frac{2a(N)}{\lambda_1 \vartheta^q} \left(\frac{1}{B_2} \right)^q \left(\frac{T_h}{\ln N} \right)^{\frac{q}{2}} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \\
& = o_p \left(\frac{1}{\lambda_1} \frac{\ln N}{T} \right) + O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{1-\frac{q}{2}} \right) \\
& \quad \left(\text{given that } \max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| = O_p \left(\sqrt{\frac{\ln N}{T}} \right) \right) \\
& = O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{1-\frac{q}{2}} \right) \tag{111}
\end{aligned}$$

where first equality above follows from the fact that $\eta_2 = B_2 \sqrt{\ln N / T_h}$ and where the second equality follows in part from the fact that $T_h = T - h \asymp T$. Combining the result given in expression (111) with the fact that

$$\begin{aligned}
\frac{1}{\lambda_1} \sum_{i=1}^N [\bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} & \leq \frac{2\eta_2^{2-q}}{\lambda_1} a(N) \\
& = \frac{2a(N)}{\lambda_1} B_2^{2-q} \left(\frac{\ln N}{T_h} \right)^{1-\frac{q}{2}} \\
& = O \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{1-\frac{q}{2}} \right)
\end{aligned}$$

we have

$$\begin{aligned}
(1) &= \frac{1}{\lambda_1} \sum_{i=1}^N \hat{\sigma}_{Xy,i}^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
&\leq \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} + \frac{2}{\lambda_1} \sum_{i=1}^N [\bar{\sigma}_{Xy,i}]^2 \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
&= O_p\left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right) + O\left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right) \\
&= O_p\left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right). \tag{112}
\end{aligned}$$

Next, for term (2), we have

$$\begin{aligned}
(2) &= \frac{1}{\lambda_1} \sum_{i=1}^N \bar{\sigma}_{Xy,i}^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N |\bar{\sigma}_{Xy,i} - \hat{\sigma}_{Xy,i} + \hat{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\leq \frac{2}{\lambda_1} \sum_{i=1}^N [|\bar{\sigma}_{Xy,i} - \hat{\sigma}_{Xy,i}|^2 + |\hat{\sigma}_{Xy,i}|^2] \mathbb{I}\{|\hat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\leq \frac{2}{\lambda_1} \max_{1 \leq i \leq N} |\bar{\sigma}_{Xy,i} - \hat{\sigma}_{Xy,i}|^2 \sum_{i=1}^N \mathbb{I}\{|\hat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{2}{\lambda_1} \sum_{i=1}^N |\hat{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\leq \frac{2}{\lambda_1} \max_{1 \leq i \leq N} |\bar{\sigma}_{Xy,i} - \hat{\sigma}_{Xy,i}|^2 \sum_{i=1}^N \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} + \frac{2\eta_2^2}{\lambda_1} \sum_{i=1}^N \mathbb{I}\{|\hat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\leq \frac{2}{\lambda_1} \left[\left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 + \eta_2^2 \right] \sum_{i=1}^N \frac{|\bar{\sigma}_{Xy,i}|^q}{|\bar{\sigma}_{Xy,i}|^q} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\leq \frac{2}{\lambda_1} \left[\left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 + \eta_2^2 \right] \frac{1}{\eta_2^q} \sum_{i=1}^N |\bar{\sigma}_{Xy,i}|^q \\
&\leq \frac{2a(N)}{\lambda_1 \eta_2^q} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 + \frac{2}{\lambda_1} \eta_2^{2-q} a(N) \\
&\quad \left(\text{since } \bar{\sigma}_{Xy} \in \mathcal{U}_\sigma(q, a(N), \mathbb{M}) = \left\{ \sigma_{N \times 1} : \max_{1 \leq i \leq N} |\sigma_i| \leq \mathbb{M}, \sum_{i=1}^N |\sigma_i|^q \leq a(N) \right\} \right) \\
&= O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{1-\frac{q}{2}} \right) + O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{1-\frac{q}{2}} \right) \\
&\quad \left(\text{since } \eta_2^q = (B_2)^q \left(\frac{\ln N}{T_h} \right)^{\frac{q}{2}} \asymp \left[\frac{\ln N}{T} \right]^{\frac{q}{2}} \text{ and } \max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| = O_p \left(\sqrt{\frac{\ln N}{T}} \right) \right) \\
&= O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{1-\frac{q}{2}} \right) \tag{113}
\end{aligned}$$

We now turn our attention to term (3) in expression (109), which we can rewrite as follow

$$\begin{aligned}
& (3) \\
&= \frac{3}{\lambda_1} \sum_{i=1}^N \left\{ \left([\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 + [s_{\eta_2}(\hat{\sigma}_{Xy,i}) - \hat{\sigma}_{Xy,i}]^2 + [s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}]^2 \right) \right. \\
&\quad \left. \times \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \right\} \\
&= \frac{3}{\lambda_1} \sum_{i=1}^N |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{3}{\lambda_1} \sum_{i=1}^N |s_{\eta_2}(\hat{\sigma}_{Xy,i}) - \hat{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{3}{\lambda_1} \sum_{i=1}^N |s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \tag{114}
\end{aligned}$$

Focusing initially on the first term on the right-hand side of expression (114) above, note that by applying the result of Lemma C-8, i.e.,

$$\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| = O_p \left(\sqrt{\frac{\ln N}{T}} \right),$$

we obtain

$$\begin{aligned}
& \frac{3}{\lambda_1} \sum_{i=1}^N |\widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i}|^2 \mathbb{I} \{ |\widehat{\sigma}_{Xy,i}| > \eta_2, |\overline{\sigma}_{Xy,i}| > \eta_2 \} \\
& \leq \frac{3}{\lambda_1} \left(\max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \frac{|\overline{\sigma}_{Xy,i}|^q}{|\overline{\sigma}_{Xy,i}|^q} \mathbb{I} \{ |\widehat{\sigma}_{Xy,i}| > \eta_2, |\overline{\sigma}_{Xy,i}| > \eta_2 \} \\
& \leq \frac{3}{\lambda_1} \left(\max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \frac{|\overline{\sigma}_{Xy,i}|^q}{\eta_2^q} \\
& \leq \frac{3}{\lambda_1} \left(\max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i}| \right)^2 \frac{a(N)}{\eta_2^q} \\
& \quad \left(\text{since } \overline{\sigma}_{Xy} \in \mathcal{U}_\sigma(q, a(N), \mathbb{M}) = \left\{ \sigma_{N \times 1} : \max_{1 \leq i \leq N} |\sigma_i| \leq \mathbb{M}, \sum_{i=1}^N |\sigma_i|^q \leq a(N) \right\} \right) \\
& = O\left(\frac{1}{\lambda_1}\right) O_p\left(\frac{\ln N}{T}\right) O\left(a(N) \left[\frac{\ln N}{T}\right]^{-\frac{q}{2}}\right) \\
& \quad \left(\text{since } \eta_2^{-q} = (B_2)^{-q} \left(\frac{\ln N}{T_h}\right)^{-\frac{q}{2}} \asymp \left[\frac{\ln N}{T}\right]^{-\frac{q}{2}} \right) \\
& = O_p\left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right). \tag{115}
\end{aligned}$$

Now, for the second term on the right-hand side of expression (114), note that, by applying condition (iii) of Assumption 6-1, we have

$$\begin{aligned}
& \frac{3}{\lambda_1} \sum_{i=1}^N |s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - \widehat{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\overline{\sigma}_{Xy,i}| > \eta_2\} \\
& \leq \frac{3}{\lambda_1} \sum_{i=1}^N \eta_2^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\overline{\sigma}_{Xy,i}| > \eta_2\} \\
& \quad (\text{since, by condition (iii) of Assumption 6-1, } |s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - \widehat{\sigma}_{Xy,i}| \leq \eta_2) \\
& = \frac{3}{\lambda_1} \sum_{i=1}^N \eta_2^{2-q} \eta_2^q \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\overline{\sigma}_{Xy,i}| > \eta_2\} \\
& \leq \frac{3\eta_2^{2-q}}{\lambda_1} \sum_{i=1}^N |\overline{\sigma}_{Xy,i}|^q \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\overline{\sigma}_{Xy,i}| > \eta_2\} \\
& \leq \frac{3\eta_2^{2-q}}{\lambda_1} \sum_{i=1}^N |\overline{\sigma}_{Xy,i}|^q \\
& \leq \frac{3\eta_2^{2-q}}{\lambda_1} a(N) \\
& = O_p\left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right) \left(\text{since } \eta_2^{2-q} = B_2^{2-q} \left(\frac{\ln N}{T_h}\right)^{1-\frac{q}{2}} \asymp \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right) \quad (116)
\end{aligned}$$

Finally, for the third term on the right-hand side of expression (114), note that, similar to the argument given for the second term of expression (114),

$$\begin{aligned}
& \frac{3}{\lambda_1} \sum_{i=1}^N |s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
& \leq \frac{3}{\lambda_1} \sum_{i=1}^N \eta_2^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
& \quad (\text{since, by condition (iii) of Assumption 6-1, } |s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}| \leq \eta_2) \\
& = \frac{3}{\lambda_1} \sum_{i=1}^N \eta_2^{2-q} \eta_2^q \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
& \leq \frac{3\eta_2^{2-q}}{\lambda_1} \sum_{i=1}^N |\bar{\sigma}_{Xy,i}|^q \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
& \leq \frac{3\eta_2^{2-q}}{\lambda_1} \sum_{i=1}^N |\bar{\sigma}_{Xy,i}|^q \\
& \leq \frac{3\eta_2^{2-q}}{\lambda_1} a(N) \\
& = O_p\left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right) \left(\text{since } \eta_2^{2-q} = B_2^{2-q} \left(\frac{\ln N}{T_h}\right)^{1-\frac{q}{2}} \asymp \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right) \quad (117)
\end{aligned}$$

Combining the results given in expressions (115)-(117), we obtain

$$\begin{aligned}
& (3) \\
& \leq \frac{3}{\lambda_1} \sum_{i=1}^N |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
& \quad + \frac{3}{\lambda_1} \sum_{i=1}^N |s_{\eta_2}(\hat{\sigma}_{Xy,i}) - \hat{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
& \quad + \frac{3}{\lambda_1} \sum_{i=1}^N |s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
& = O_p\left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right) + O_p\left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right) + O_p\left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right) \\
& = O_p\left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T}\right]^{1-\frac{q}{2}}\right) \quad (118)
\end{aligned}$$

It further follows from expressions (109), (112), (113), and (118)

$$\begin{aligned}
& \left\| \frac{\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2^2 \\
& \leq \frac{1}{\lambda_1} \sum_{i=1}^N \widehat{\sigma}_{Xy,i}^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\overline{\sigma}_{Xy,i}| \leq \eta_2\} + \frac{1}{\lambda_1} \sum_{i=1}^N \overline{\sigma}_{Xy,i}^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| \leq \eta_2, |\overline{\sigma}_{Xy,i}| > \eta_2\} \\
& \quad + \frac{3}{\lambda_1} \sum_{i=1}^N \left\{ \left([\widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i}]^2 + [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - \widehat{\sigma}_{Xy,i}]^2 + [s_{\eta_2}(\overline{\sigma}_{Xy,i}) - \overline{\sigma}_{Xy,i}]^2 \right) \right. \\
& \quad \left. \times \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\overline{\sigma}_{Xy,i}| > \eta_2\} \right\} \\
& = (1) + (2) + (3) \\
& = O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{1-\frac{q}{2}} \right) + O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{1-\frac{q}{2}} \right) + O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{1-\frac{q}{2}} \right) \\
& = O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{1-\frac{q}{2}} \right)
\end{aligned}$$

so that

$$\left\| \frac{\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2 = O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right) \quad (119)$$

Finally, applying the results given in expressions (108) and (119) to expression (107), we obtain the result that, under Case 1 where $0 < q < 1$,

$$\begin{aligned}
& \left\| \frac{\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2 \\
& \leq \left\| \frac{\overline{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2 + \left\| \frac{\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2 \\
& = O \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right) + O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right) \\
& = O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right).
\end{aligned}$$

Case 2: $q = 0$:

Under this case, consider again the first term on the right-hand side of expression (107) above. By conditions (ii) and (iii) of Assumption 6-1, we can write

$$\begin{aligned}
& \left\| \frac{\bar{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}]^2 \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\bar{\sigma}_{Xy,i}) \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} + s_{\eta_2}(\bar{\sigma}_{Xy,i}) \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\} - \bar{\sigma}_{Xy,i} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad - \bar{\sigma}_{Xy,i} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\}]^2 \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\bar{\sigma}_{Xy,i}) \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} - \bar{\sigma}_{Xy,i} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} - \bar{\sigma}_{Xy,i} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\}]^2 \\
&\quad (\text{since } s_{\eta_2}(\bar{\sigma}_{Xy,i}) = 0 \text{ for } |\bar{\sigma}_{Xy,i}| \leq \eta_2 \text{ by condition (ii) of Assumption 6-1}) \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N [|s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}| \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} + |\bar{\sigma}_{Xy,i}| \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\}]^2 \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N [\eta_2 \mathbb{I}\{|\bar{\sigma}_{Xy,i}| > \eta_2\} + |\bar{\sigma}_{Xy,i}| \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\}]^2 \\
&\quad (\text{since } |s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}| \leq \eta_2 \text{ by condition (iii) of Assumption 6-1}) \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N [\eta_2 \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} + |\bar{\sigma}_{Xy,i}| \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\}]^2 \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N [\eta_2 \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} + \eta_2 \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\}]^2 \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N [2\eta_2 \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\}]^2 \\
&= \frac{4\eta_2^2}{\lambda_1} \sum_{i=1}^N \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\}
\end{aligned}$$

Moreover, since, in this case,

$$\bar{\sigma}_{Xy} \in \mathcal{U}_\sigma(0, a(N), \mathbb{M}) = \left\{ \sigma_{N \times 1} : \max_{1 \leq i \leq N} |\sigma_i| \leq \mathbb{M}, \sum_{i=1}^N \mathbb{I}\{|\sigma_i| \neq 0\} \leq a(N) \right\},$$

it follows that

$$\begin{aligned}
\left\| \frac{\bar{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 &\leq \frac{4\eta_2^2}{\lambda_1} \sum_{i=1}^N \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} \\
&\leq \frac{4\eta_2^2}{\lambda_1} a(N) \quad (\text{given that } \bar{\sigma}_{Xy} \in \mathcal{U}_\sigma(0, a(N), \mathbb{M})) \\
&= O\left(\frac{a(N)}{\lambda_1} \left\lceil \frac{\ln N}{T} \right\rceil\right) \quad \left(\text{given that } \eta_2^2 = B_2^2 \left(\frac{\ln N}{T_h}\right) \asymp \frac{\ln N}{T}\right).
\end{aligned}$$

from which we further deduce that

$$\left\| \frac{\bar{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \leq 2\eta_2 \sqrt{\frac{a(N)}{\lambda_1}} = O\left(\sqrt{\frac{a(N)}{\lambda_1}} \sqrt{\frac{\ln N}{T}}\right) \quad (120)$$

Now, consider the second term on the right-hand side of expression (107) above, i.e., the term

$$\left\| \frac{\hat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2,$$

in the $q = 0$ case. Here, note that, by conditions (i) and (ii) of Assumption 6-1, we have

$$\begin{aligned}
& \left\| \frac{\widehat{\sigma}_{Xy}^S - \bar{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2^2 \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
&\quad + \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
&\quad + \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \quad (\text{by condition (ii) of} \\
&\quad \text{Assumption 6-1, } s_{\eta_2}(\bar{\sigma}_{Xy,i}) = 0 \text{ if } |\bar{\sigma}_{Xy,i}| \leq \eta \text{ and } s_{\eta_2}(\widehat{\sigma}_{Xy,i}) = 0 \text{ if } |\widehat{\sigma}_{Xy,i}| \leq \eta_2) \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N \widehat{\sigma}_{Xy,i}^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} + \frac{1}{\lambda_1} \sum_{i=1}^N \bar{\sigma}_{Xy,i}^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{1}{\lambda_1} \sum_{i=1}^N [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - s_{\eta_2}(\bar{\sigma}_{Xy,i})]^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad (\text{since, by condition (i) of Assumption 6-1, } |s_{\eta_2}(\widehat{\sigma}_{Xy,i})| \leq |\widehat{\sigma}_{Xy,i}| \text{ and } |s_{\eta_2}(\bar{\sigma}_{Xy,i})| \leq |\bar{\sigma}_{Xy,i}|) \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^N \widehat{\sigma}_{Xy,i}^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} + \frac{1}{\lambda_1} \sum_{i=1}^N \bar{\sigma}_{Xy,i}^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{3}{\lambda_1} \sum_{i=1}^N \left[([\widehat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 + [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - \widehat{\sigma}_{Xy,i}]^2 + [s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}]^2) \right. \\
&\quad \quad \left. \times \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \right] \left(\text{since } \left| \sum_{i=1}^m a_i \right|^r \leq m^{r-1} \sum_{i=1}^m |a_i|^r \text{ for } r \geq 1 \right) \\
&= [1] + [2] + [3] \tag{121}
\end{aligned}$$

To bound term [1], write

$$\begin{aligned}
[1] &= \frac{1}{\lambda_1} \sum_{i=1}^N \hat{\sigma}_{Xy,i}^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i} + \bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&\leq \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&\quad + \frac{2}{\lambda_1} \sum_{i=1}^N [\bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&\quad \left(\text{again using the inequality } \left| \sum_{i=1}^m a_i \right|^r \leq m^{r-1} \sum_{i=1}^m |a_i|^r \text{ for } r \geq 1 \right) \\
&\leq \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&\quad + \frac{2}{\lambda_1} \sum_{i=1}^N [\bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \tag{122}
\end{aligned}$$

To bound the second term in the expression above, note that

$$\begin{aligned}
\frac{2}{\lambda_1} \sum_{i=1}^N [\bar{\sigma}_{Xy,i}]^2 \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} &= \frac{2}{\lambda_1} \sum_{i=1}^N |\bar{\sigma}_{Xy,i}|^2 \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \neq 0 \} \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&\leq \frac{2\eta_2^2}{\lambda_1} \sum_{i=1}^N \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \neq 0 \} \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \leq \eta_2 \} \\
&\leq \frac{2\eta_2^2}{\lambda_1} \sum_{i=1}^N \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \neq 0 \} \\
&\leq \frac{2\eta_2^2 a(N)}{\lambda_1} \quad (\text{since } \bar{\sigma}_{Xy} \in \mathcal{U}_\sigma(0, a(N), \mathbb{M})).
\end{aligned}$$

Next, take $\vartheta \in (0, 1)$, and we can bound the first term in expression (122) as follows

$$\begin{aligned}
& \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
= & \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2\} \\
& + \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, \vartheta \eta_2 < |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
\leq & \frac{2}{\lambda_1} \max_{1 \leq i \leq N} [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \sum_{i=1}^N \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2\} \\
& + \frac{2}{\lambda_1} \max_{1 \leq i \leq N} [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \sum_{i=1}^N \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, \vartheta \eta_2 < |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
\leq & \frac{2}{\lambda_1} \max_{1 \leq i \leq N} [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \sum_{i=1}^N \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2\} \\
& + \frac{2}{\lambda_1} \max_{1 \leq i \leq N} [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \sum_{i=1}^N \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} \\
\leq & \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2\} \\
& + \frac{2a(N)}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \\
& \left(\text{since } \bar{\sigma}_{Xy} \in \mathcal{U}_\sigma(0, a(N), \mathbb{M}) = \left\{ \sigma_{N \times 1} : \max_{1 \leq i \leq N} |\sigma_i| \leq \mathbb{M}, \sum_{i=1}^N \mathbb{I}\{|\sigma_i| \neq 0\} \leq a(N) \right\} \right)
\end{aligned}$$

Moreover, observe that

$$\begin{aligned}
& \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2 \} \\
& \leq \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| + |\bar{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2 \} \\
& = \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > \eta_2 - |\bar{\sigma}_{Xy,i}|, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2 \} \\
& \leq \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > \eta_2 - \vartheta \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2 \} \\
& \leq \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > (1 - \vartheta) \eta_2 \}.
\end{aligned}$$

Note that, by the result given in Lemma C-8, if we choose

$$\eta_2 = B_2 \sqrt{\frac{\ln N}{T_h}}, \text{ where } B_2 = \frac{\bar{B}}{1 - \vartheta}$$

and where $\bar{B}$ is the positive constant given in Lemma C-8; then,

$$\begin{aligned}
P \left[\sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > (1 - \vartheta) \eta_2 \} > 0 \right] &= P \left[\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > (1 - \vartheta) \eta_2 \right] \\
&= o \left(\frac{1}{N^2} \right),
\end{aligned}$$

uniformly on

$$\mathcal{U}_\sigma(0, a(N), \mathbb{M}) = \left\{ \sigma_{N \times 1} : \max_{1 \leq i \leq N} |\sigma_i| \leq \mathbb{M}, \sum_{i=1}^N \mathbb{I} \{ |\sigma_i| \neq 0 \} \leq a(N) \right\}$$

Making use of the fact that

$$P \left[\sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > (1 - \vartheta) \eta_2 \} > 0 \right] = o \left(\frac{1}{N^2} \right) = o(1) \text{ as } N, T \rightarrow \infty,$$

we further obtain

$$\begin{aligned}
& \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
& \leq \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \vartheta \eta_2\} \\
& \quad + \frac{2a(N)}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \\
& \leq \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I}\{|\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| > (1 - \vartheta) \eta_2\} \\
& \quad + \frac{2a(N)}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \\
& = o_p \left(\frac{1}{\lambda_1} \left[\frac{\ln N}{T} \right] \right) + O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right] \right) \\
& \quad \left(\text{given that } \max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| = O_p \left(\sqrt{\frac{\ln N}{T}} \right) \right) \\
& = O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right] \right) \tag{123}
\end{aligned}$$

Combining the result given in expression (123) with the fact that

$$\begin{aligned}
\frac{2}{\lambda_1} \sum_{i=1}^N [\bar{\sigma}_{Xy,i}]^2 \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\} & \leq \frac{2\eta_2^2 a(N)}{\lambda_1} \\
& = \frac{2a(N)}{\lambda_1} B_2^2 \left(\frac{\ln N}{T_h} \right) \\
& = O \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right] \right)
\end{aligned}$$

we have

$$\begin{aligned}
[1] &= \frac{1}{\lambda_1} \sum_{i=1}^N \hat{\sigma}_{Xy,i}^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
&\leq \frac{2}{\lambda_1} \sum_{i=1}^N [\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| \leq \eta_2\} + \frac{2}{\lambda_1} \sum_{i=1}^N [\bar{\sigma}_{Xy,i}]^2 \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \leq \eta_2\} \\
&= O_p\left(\frac{a(N)}{\lambda_1} \left\lfloor \frac{\ln N}{T} \right\rfloor\right) + O\left(\frac{a(N)}{\lambda_1} \left\lfloor \frac{\ln N}{T} \right\rfloor\right) \\
&= O_p\left(\frac{a(N)}{\lambda_1} \left\lfloor \frac{\ln N}{T} \right\rfloor\right). \tag{124}
\end{aligned}$$

Next, for term [2], we have

$$\begin{aligned}
[2] &= \frac{1}{\lambda_1} \sum_{i=1}^N \bar{\sigma}_{Xy,i}^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2 \} \\
&= \frac{1}{\lambda_1} \sum_{i=1}^N |\bar{\sigma}_{Xy,i} - \hat{\sigma}_{Xy,i} + \hat{\sigma}_{Xy,i}|^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2 \} \\
&\leq \frac{2}{\lambda_1} \sum_{i=1}^N [|\bar{\sigma}_{Xy,i} - \hat{\sigma}_{Xy,i}|^2 + |\hat{\sigma}_{Xy,i}|^2] \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2 \} \\
&\leq \frac{2}{\lambda_1} \max_{1 \leq i \leq N} |\bar{\sigma}_{Xy,i} - \hat{\sigma}_{Xy,i}|^2 \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2 \} \\
&\quad + \frac{2}{\lambda_1} \sum_{i=1}^N |\hat{\sigma}_{Xy,i}|^2 \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2 \} \\
&\leq \frac{2}{\lambda_1} \max_{1 \leq i \leq N} |\bar{\sigma}_{Xy,i} - \hat{\sigma}_{Xy,i}|^2 \sum_{i=1}^N \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| > \eta_2 \} + \frac{2\eta_2^2}{\lambda_1} \sum_{i=1}^N \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2 \} \\
&= \frac{2}{\lambda_1} \left(\max_{1 \leq i \leq N} |\bar{\sigma}_{Xy,i} - \hat{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \neq 0 \} \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| > \eta_2 \} \\
&\quad + \frac{2\eta_2^2}{\lambda_1} \sum_{i=1}^N \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \neq 0 \} \mathbb{I} \{ |\hat{\sigma}_{Xy,i}| \leq \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2 \} \\
&\leq \frac{2}{\lambda_1} \left[\left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 + \eta_2^2 \right] \sum_{i=1}^N \mathbb{I} \{ |\bar{\sigma}_{Xy,i}| \neq 0 \} \\
&\leq \frac{2a(N)}{\lambda_1} \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 + \frac{2\eta_2^2 a(N)}{\lambda_1} \\
&\quad \left(\text{since } \bar{\sigma}_{Xy} \in \mathcal{U}_\sigma(0, a(N), \mathbb{M}) = \left\{ \sigma_{N \times 1} : \max_{1 \leq i \leq N} |\sigma_i| \leq \mathbb{M}, \sum_{i=1}^N \mathbb{I} \{ |\sigma_i| \neq 0 \} \leq a(N) \right\} \right) \\
&= O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right] \right) + O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right] \right) \\
&\quad \left(\text{since } \max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| = O_p \left(\sqrt{\frac{\ln N}{T}} \right) \right) \\
&= O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right] \right) \tag{125}
\end{aligned}$$

We now turn our attention to term [3] in expression (121), which we can rewrite as follow

$$\begin{aligned}
& [3] \\
&= \frac{3}{\lambda_1} \sum_{i=1}^N \left[\left([\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}]^2 + [s_{\eta_2}(\hat{\sigma}_{Xy,i}) - \hat{\sigma}_{Xy,i}]^2 + [s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}]^2 \right) \right. \\
&\quad \left. \times \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \right] \\
&= \frac{3}{\lambda_1} \sum_{i=1}^N |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{3}{\lambda_1} \sum_{i=1}^N |s_{\eta_2}(\hat{\sigma}_{Xy,i}) - \hat{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad + \frac{3}{\lambda_1} \sum_{i=1}^N |s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\}. \tag{126}
\end{aligned}$$

Focusing initially on the first-term on the right-hand side of expression (126) above, note that by applying the result of Lemma C-8, i.e.,

$$\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| = O_p \left(\sqrt{\frac{\ln N}{T}} \right),$$

we obtain

$$\begin{aligned}
& \frac{3}{\lambda_1} \sum_{i=1}^N |\widehat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&= \frac{3}{\lambda_1} \sum_{i=1}^N |\widehat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\leq \frac{3}{\lambda_1} \left(\max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\leq \frac{3}{\lambda_1} \left(\max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 \sum_{i=1}^N \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} \\
&\leq \frac{3}{\lambda_1} \left(\max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \right)^2 a(N) \\
&\quad \left(\text{since } \bar{\sigma}_{Xy} \in \mathcal{U}_\sigma(0, a(N), \mathbb{M}) = \left\{ \sigma_{N \times 1} : \max_{1 \leq i \leq N} |\sigma_i| \leq \mathbb{M}, \sum_{i=1}^N \mathbb{I}\{|\sigma_i| \neq 0\} \leq a(N) \right\} \right) \\
&= O\left(\frac{1}{\lambda_1}\right) O_p\left(\frac{\ln N}{T}\right) O(a(N)) \\
&= O_p\left(\frac{a(N)}{\lambda_1} \left\lceil \frac{\ln N}{T} \right\rceil\right). \tag{127}
\end{aligned}$$

Now, for the second term on the right-hand side of expression (126), note that, by applying condition (iii) of Assumption 6-1, we have

$$\begin{aligned}
& \frac{3}{\lambda_1} \sum_{i=1}^N |s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - \widehat{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\leq \frac{3\eta_2^2}{\lambda_1} \sum_{i=1}^N \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\quad (\text{since, by condition (iii) of Assumption 6-1, } |s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - \widehat{\sigma}_{Xy,i}| \leq \eta_2) \\
&= \frac{3\eta_2^2}{\lambda_1} \sum_{i=1}^N \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
&\leq \frac{3\eta_2^2}{\lambda_1} \sum_{i=1}^N \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} \\
&\leq \frac{3\eta_2^2}{\lambda_1} a(N) \\
&= O_p\left(\frac{a(N)}{\lambda_1} \left\lceil \frac{\ln N}{T} \right\rceil\right) \quad \left(\text{since } \eta_2^2 = B_2^2 \left(\frac{\ln N}{T_h} \right) \asymp \frac{\ln N}{T} \right) \tag{128}
\end{aligned}$$

Finally, for the third term on the right-hand side of expression (126), note that, similar to the argument given for the second term of expression (126),

$$\begin{aligned}
& \frac{3}{\lambda_1} \sum_{i=1}^N |s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
& \leq \frac{3\eta_2^2}{\lambda_1} \sum_{i=1}^N \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
& \quad (\text{since, by condition (iii) of Assumption 6-1, } |s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}| \leq \eta_2) \\
& = \frac{3\eta_2^2}{\lambda_1} \sum_{i=1}^N \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
& \leq \frac{3\eta_2^2}{\lambda_1} \sum_{i=1}^N \mathbb{I}\{|\bar{\sigma}_{Xy,i}| \neq 0\} \\
& \leq \frac{3\eta_2^2}{\lambda_1} a(N) \\
& = O_p\left(\frac{a(N)}{\lambda_1} \left\lceil \frac{\ln N}{T} \right\rceil\right) \quad \left(\text{since } \eta_2^2 = B_2^2 \left(\frac{\ln N}{T_h}\right) \asymp \frac{\ln N}{T}\right)
\end{aligned} \tag{129}$$

Combining the results given in expressions (127)-(129), we obtain

$$\begin{aligned}
& [3] \\
& \leq \frac{3}{\lambda_1} \sum_{i=1}^N |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
& \quad + \frac{3}{\lambda_1} \sum_{i=1}^N |s_{\eta_2}(\hat{\sigma}_{Xy,i}) - \hat{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
& \quad + \frac{3}{\lambda_1} \sum_{i=1}^N |s_{\eta_2}(\bar{\sigma}_{Xy,i}) - \bar{\sigma}_{Xy,i}|^2 \mathbb{I}\{|\hat{\sigma}_{Xy,i}| > \eta_2, |\bar{\sigma}_{Xy,i}| > \eta_2\} \\
& = O_p\left(\frac{a(N)}{\lambda_1} \left\lceil \frac{\ln N}{T} \right\rceil\right) + O_p\left(\frac{a(N)}{\lambda_1} \left\lceil \frac{\ln N}{T} \right\rceil\right) + O_p\left(\frac{a(N)}{\lambda_1} \left\lceil \frac{\ln N}{T} \right\rceil\right) \\
& = O_p\left(\frac{a(N)}{\lambda_1} \left\lceil \frac{\ln N}{T} \right\rceil\right)
\end{aligned} \tag{130}$$

It further follows from expressions (121), (124), (125), and (130)

$$\begin{aligned}
& \left\| \frac{\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2^2 \\
& \leq \frac{1}{\lambda_1} \sum_{i=1}^N \widehat{\sigma}_{Xy,i}^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\overline{\sigma}_{Xy,i}| \leq \eta_2\} + \frac{1}{\lambda_1} \sum_{i=1}^N \overline{\sigma}_{Xy,i}^2 \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| \leq \eta_2, |\overline{\sigma}_{Xy,i}| > \eta_2\} \\
& \quad + \frac{3}{\lambda_1} \sum_{i=1}^N \left[\left([\widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i}]^2 + [s_{\eta_2}(\widehat{\sigma}_{Xy,i}) - \widehat{\sigma}_{Xy,i}]^2 + [s_{\eta_2}(\overline{\sigma}_{Xy,i}) - \overline{\sigma}_{Xy,i}]^2 \right) \right. \\
& \quad \quad \left. \times \mathbb{I}\{|\widehat{\sigma}_{Xy,i}| > \eta_2, |\overline{\sigma}_{Xy,i}| > \eta_2\} \right] \\
& = [1] + [2] + [3] \\
& = O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right] \right) + O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right] \right) + O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right] \right) \\
& = O_p \left(\frac{a(N)}{\lambda_1} \left[\frac{\ln N}{T} \right] \right)
\end{aligned}$$

so that

$$\left\| \frac{\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2 = O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \sqrt{\frac{\ln N}{T}} \right) \quad (131)$$

Finally, applying the results given in expressions (120) and (131) to expression (107), we obtain the result that, under Case 2 where $q = 0$,

$$\begin{aligned}
& \left\| \frac{\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2 \\
& \leq \left\| \frac{\overline{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2 + \left\| \frac{\widehat{\sigma}_{Xy}^S - \overline{\sigma}_{Xy}^S}{\sqrt{\lambda_1}} \right\|_2 \\
& = O \left(\sqrt{\frac{a(N)}{\lambda_1}} \sqrt{\frac{\ln N}{T}} \right) + O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \sqrt{\frac{\ln N}{T}} \right) \\
& = O_p \left(\sqrt{\frac{a(N)}{\lambda_1}} \sqrt{\frac{\ln N}{T}} \right).
\end{aligned}$$

The lemma follows immediately upon combining the results we have obtained for Case 1, where $0 < q < 1$, and Case 2, where $q = 0$. $\square$

Lemma D: Proof of Proposition 1 and Some Additional Supporting Lemmas

Proof of Proposition 1:

To proceed, first rewrite the forecast equation given in expression (??) as follows

$$\begin{aligned} y_{t+h} &= \alpha + (X_t - \mu_X)' \beta_{\text{env}} + \varepsilon_{t+h} \\ &= \bar{\alpha} + X_t' \beta_{\text{env}} + \varepsilon_{t+h} \end{aligned}$$

where $\bar{\alpha} = \alpha - \mu_X' \beta_{\text{env}}$. Note that

$$\begin{aligned} E[y_{t+h} | L_1' X_t] &= E[\bar{\alpha} + X_t' \beta_{\text{env}} + \varepsilon_{t+h} | L_1' X_t] \\ &= \bar{\alpha} + E[X_t' \beta_{\text{env}} | L_1' X_t] + E[\varepsilon_{t+h} | L_1' X_t] \\ &= \bar{\alpha} + E[X_t' \beta_{\text{env}} | L_1' X_t] \end{aligned}$$

and

$$\begin{aligned} y_{t+h} - E[y_{t+h} | L_1' X_t] &= \bar{\alpha} + X_t' \beta_{\text{env}} + \varepsilon_{t+h} - (\bar{\alpha} + E[X_t' \beta_{\text{env}} | L_1' X_t]) \\ &= X_t' \beta_{\text{env}} - E[X_t' \beta_{\text{env}} | L_1' X_t] + \varepsilon_{t+h} \end{aligned}$$

so that we can write

$$\begin{aligned} & \text{cov}(y_{t+h}, L_2' X_t \mid L_1' X_t) \\ &= E\{(y_{t+h} - E[y_{t+h} | L_1' X_t])(L_2' X_t - E[L_2' X_t | L_1' X_t])' \mid L_1' X_t\} \\ &= E\{(X_t' \beta_{\text{env}} - E[X_t' \beta_{\text{env}} | L_1' X_t])(L_2' X_t - E[L_2' X_t | L_1' X_t])' \mid L_1' X_t\} \\ &\quad + E\{\varepsilon_{t+h} (L_2' X_t - E[L_2' X_t | L_1' X_t])' \mid L_1' X_t\} \\ &= E\{(X_t' \beta_{\text{env}} - E[X_t' \beta_{\text{env}} | L_1' X_t])(L_2' X_t - E[L_2' X_t | L_1' X_t])' \mid L_1' X_t\} \\ &\quad + E\{E[\varepsilon_{t+h} | X_t] (L_2' X_t - E[L_2' X_t | L_1' X_t])' \mid L_1' X_t\} \\ &= E\{(X_t' \beta_{\text{env}} - E[X_t' \beta_{\text{env}} | L_1' X_t])(L_2' X_t - E[L_2' X_t | L_1' X_t])' \mid L_1' X_t\} \\ &= \text{cov}(\beta_{\text{env}}' X_t, L_2' X_t \mid L_1' X_t) \\ &= \text{cov}(\beta_{\text{env}}' L_1 L_1' X_t + \beta_{\text{env}}' L_2 L_2' X_t, L_2' X_t \mid L_1' X_t) \\ &= E\{(\beta_{\text{env}}' L_1 L_1' X_t - E[\beta_{\text{env}}' L_1 L_1' X_t | L_1' X_t])(L_2' X_t - E[L_2' X_t | L_1' X_t])' \mid L_1' X_t\} \\ &\quad + \text{cov}(\beta_{\text{env}}' L_2 L_2' X_t, L_2' X_t \mid L_1' X_t) \\ &= (\beta_{\text{env}}' L_1 L_1' X_t - E[\beta_{\text{env}}' L_1 L_1' X_t | L_1' X_t])(E[L_2' X_t | L_1' X_t] - E[L_2' X_t | L_1' X_t])' \\ &\quad + \text{cov}(\beta_{\text{env}}' L_2 L_2' X_t, L_2' X_t \mid L_1' X_t) \\ &= \text{cov}(\beta_{\text{env}}' L_2 L_2' X_t, L_2' X_t \mid L_1' X_t) \\ &= \beta_{\text{env}}' L_2 E\{(L_2' X_t - E[L_2' X_t | L_1' X_t])(L_2' X_t - E[L_2' X_t | L_1' X_t])' \mid L_1' X_t\} \end{aligned}$$

Next, note that the condition $\beta_{\text{env}} \in \text{span}(L_1)$ implies that there exists $\varphi \in \mathbb{R}^{\kappa \times 1}$ such that

$\beta_{\text{env}} = L_1\varphi$. It follows that

$$L_2'\beta_{\text{env}} = L_2'L_1\varphi = 0$$

from which we further deduce that

$$\begin{aligned} & \text{cov}(y_{t+h}, L_2'X_t \mid L_1'X_t) \\ &= \beta_{\text{env}}' L_2 E \{ (L_2'X_t - E[L_2'X_t \mid L_1'X_t]) (L_2'X_t - E[L_2'X_t \mid L_1'X_t])' \mid L_1'X_t \} \\ &= 0. \quad \square \end{aligned}$$

Lemma D-1: Consider the model

$$\begin{aligned} y_{t+h} &= \alpha + \left(\underline{F}_t - \underline{\mu}_F \right)' \underline{\varphi} + \varepsilon_{t+h} \\ X_t &= \mu_X + \underline{\Gamma} \left(\underline{F}_t - \underline{\mu}_F \right) + u_t. \end{aligned}$$

where $\underline{F}_t$ is a $K \times 1$ vector of latent factors. Let

$$\overline{\Sigma}_{\underline{F}} = \frac{1}{T_h} \sum_{t=1}^{T_h} E \left[\left(\underline{F}_t - \underline{\mu}_F \right) \left(\underline{F}_t - \underline{\mu}_F \right)' \right]$$

where $T_h = T - h$, with h being a fixed finite integer such that $1 \leq h < T$. Let the real, symmetric matrix $(\underline{\Gamma}'\underline{\Gamma})^{\frac{1}{2}} \overline{\Sigma}_{\underline{F}} (\underline{\Gamma}'\underline{\Gamma})^{\frac{1}{2}}$ have the spectral decomposition

$$(\underline{\Gamma}'\underline{\Gamma})^{\frac{1}{2}} \overline{\Sigma}_{\underline{F}} (\underline{\Gamma}'\underline{\Gamma})^{\frac{1}{2}} = Q \underline{\Lambda} Q'$$

where $\underline{\Lambda}$ is a diagonal matrix whose diagonal elements

$$\lambda_1 \geq \dots \geq \lambda_K > 0$$

are the eigenvalues of the matrix $(\underline{\Gamma}'\underline{\Gamma})^{\frac{1}{2}} \overline{\Sigma}_{\underline{F}} (\underline{\Gamma}'\underline{\Gamma})^{\frac{1}{2}}$ and where Q is a $K \times K$ matrix whose columns are the eigenvectors of $(\underline{\Gamma}'\underline{\Gamma})^{\frac{1}{2}} \overline{\Sigma}_{\underline{F}} (\underline{\Gamma}'\underline{\Gamma})^{\frac{1}{2}}$. Suppose that

$$\lambda_1 \asymp \dots \asymp \lambda_K$$

and suppose that there exists a constant $\underline{c} > 0$ such that

$$\lambda_{\min} \left(\frac{\underline{\Gamma}'\underline{\Gamma}}{\lambda_1} \right) \geq \underline{c} > 0 \tag{132}$$

Then, there exist alternative transformations

$$\begin{aligned}\Gamma^{(g)} &= \underline{\Gamma} \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{-\frac{1}{2}} Q, \quad \varphi^{(g)} = Q' \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{-\frac{1}{2}} \underline{\varphi}, \quad F_t^{(g)} = Q' \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{\frac{1}{2}} \underline{F}_t, \text{ and} \\ \mu_F^{(g)} &= Q' \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{\frac{1}{2}} \underline{\mu}_F\end{aligned}\tag{133}$$

for every $g \in \{1, \dots, K\}$ such that

$$(a) \quad \Gamma^{(g)} \left(F_t^{(g)} - \mu_F^{(g)} \right) = \underline{\Gamma} \left(\underline{F}_t - \underline{\mu}_F \right)$$

$$(b) \quad \left(F_t^{(g)} - \mu_F^{(g)} \right)' \varphi^{(g)} = \left(\underline{F}_t - \underline{\mu}_F \right)' \underline{\varphi}$$

$$(c) \quad \frac{\Gamma^{(g)'} \Gamma^{(g)}}{\underline{\lambda}_g} = I_K$$

$$(d) \quad \overline{\Sigma}_F^{(g)} = \frac{1}{T_h} \sum_{t=1}^{T_h} E \left[\left(F_t^{(g)} - \mu_F^{(g)} \right) \left(F_t^{(g)} - \mu_F^{(g)} \right)' \right] = \frac{\underline{\Lambda}}{\underline{\lambda}_g}$$

$$(e) \quad \Gamma^{(g)} \overline{\Sigma}_F^{(g)} \Gamma^{(g)'} = \underline{\Gamma} \overline{\Sigma}_F \underline{\Gamma}'.$$

Hence, for any $g \in \{1, \dots, K\}$, the set of non-zero eigenvalues of the (transformed) matrix $\Gamma^{(g)} \overline{\Sigma}_F^{(g)} \Gamma^{(g)'}$ is given by $\{\underline{\lambda}_1, \dots, \underline{\lambda}_K\}$ which is the same as the set of non-zero eigenvalues of the (original) common component covariance matrix $\underline{\Gamma} \overline{\Sigma}_F \underline{\Gamma}'$.

Proof of Lemma D-1: To show parts (a)-(c), note first that, for every $g \in \{1, \dots, K\}$, the inverse of the matrix $\underline{\Gamma}' \underline{\Gamma} / \underline{\lambda}_g$ exists since, under the condition given in expression (132) above, we have

$$\lambda_{\min} \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right) = \frac{\underline{\lambda}_1}{\underline{\lambda}_g} \lambda_{\min} \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_1} \right) \geq \underline{c} > 0$$

in light of the fact that $\underline{\lambda}_1 \geq \underline{\lambda}_g > 0$ and $\underline{\lambda}_1 \asymp \underline{\lambda}_g$. It is then easy to see that, for every

$g \in \{1, \dots, K\}$,

$$\begin{aligned}
\Gamma^{(g)} \left(F_t^{(g)} - \mu_F^{(g)} \right) &= \underline{\Gamma} \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{-\frac{1}{2}} Q Q' \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{\frac{1}{2}} \left(\underline{F}_t - \underline{\mu}_F \right) \\
&= \underline{\Gamma} \left(\underline{F}_t - \underline{\mu}_F \right) \quad (\text{since } Q \text{ is an orthogonal matrix so that } Q Q' = I_K), \\
\left(F_t^{(g)} - \mu_F^{(g)} \right)' \varphi^{(g)} &= \left(\underline{F}_t - \underline{\mu}_F \right)' \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{\frac{1}{2}} Q Q' \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{-\frac{1}{2}} \underline{\varphi} \\
&= \left(\underline{F}_t - \underline{\mu}_F \right)' \underline{\varphi} \\
\frac{\Gamma^{(g)'} \Gamma^{(g)}}{\underline{\lambda}_g} &= Q' \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{-\frac{1}{2}} \frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{-\frac{1}{2}} Q \\
&= Q' Q \\
&= I_K \quad (\text{again since } Q \text{ is an orthogonal matrix})
\end{aligned}$$

which shows parts (a)-(c).

Now to show part (d), note that, for every $g \in \{1, \dots, K\}$,

$$\begin{aligned}
\overline{\Sigma}_F^{(g)} &= \frac{1}{T_h} \sum_{t=1}^{T_h} E \left[\left(F_t^{(g)} - \mu_F^{(g)} \right) \left(F_t^{(g)} - \mu_F^{(g)} \right)' \right] \\
&= Q' \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{\frac{1}{2}} \frac{1}{T_h} \sum_{t=1}^{T_h} E \left[\left(\underline{F}_t - \underline{\mu}_F \right) \left(\underline{F}_t - \underline{\mu}_F \right)' \right] \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{\frac{1}{2}} Q \\
&= \frac{1}{\underline{\lambda}_g} Q' (\underline{\Gamma}' \underline{\Gamma})^{\frac{1}{2}} \overline{\Sigma}_F (\underline{\Gamma}' \underline{\Gamma})^{\frac{1}{2}} Q \\
&= \frac{1}{\underline{\lambda}_g} Q' Q \underline{\Lambda} Q' Q \quad \left(\text{since } (\underline{\Gamma}' \underline{\Gamma})^{\frac{1}{2}} \overline{\Sigma}_F (\underline{\Gamma}' \underline{\Gamma})^{\frac{1}{2}} \text{ is a real, symmetric matrix} \right. \\
&\quad \left. \text{and has a spectral decomposition given by } (\underline{\Gamma}' \underline{\Gamma})^{\frac{1}{2}} \overline{\Sigma}_F (\underline{\Gamma}' \underline{\Gamma})^{\frac{1}{2}} = Q \underline{\Lambda} Q' \right) \\
&= \frac{\underline{\Lambda}}{\underline{\lambda}_g}
\end{aligned}$$

Finally, to show part (e), note that, for every $g \in \{1, \dots, K\}$,

$$\begin{aligned}
& \Gamma^{(g)} \bar{\Sigma}_F^{(g)} \Gamma^{(g)'} \\
&= \Gamma^{(g)} \left(\frac{1}{T_h} \sum_{t=1}^{T_h} E \left[\left(F_t^{(g)} - \mu_F^{(g)} \right) \left(F_t^{(g)} - \mu_F^{(g)} \right)' \right] \right) \Gamma^{(g)'} \\
&= \underline{\Gamma} \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{-\frac{1}{2}} Q Q' \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{\frac{1}{2}} \frac{1}{T_h} \sum_{t=1}^{T_h} E \left[\left(\underline{F}_t - \underline{\mu}_F \right) \left(\underline{F}_t - \underline{\mu}_F \right)' \right] \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{\frac{1}{2}} Q \\
&\quad \times Q' \left(\frac{\underline{\Gamma}' \underline{\Gamma}}{\underline{\lambda}_g} \right)^{-\frac{1}{2}} \underline{\Gamma}' \\
&= \underline{\Gamma} \frac{1}{T_h} \sum_{t=1}^{T_h} E \left[\left(\underline{F}_t - \underline{\mu}_F \right) \left(\underline{F}_t - \underline{\mu}_F \right)' \right] \underline{\Gamma}' \\
&= \underline{\Gamma} \bar{\Sigma}_F \underline{\Gamma}' \left(\text{since, by definition, } \bar{\Sigma}_F = \frac{1}{T_h} \sum_{t=1}^{T_h} E \left[\left(\underline{F}_t - \underline{\mu}_F \right) \left(\underline{F}_t - \underline{\mu}_F \right)' \right] \right)
\end{aligned}$$

Next, note that, by definition, $\underline{\lambda}_1, \dots, \underline{\lambda}_K$ are the eigenvalues of the real, symmetric matrix $(\underline{\Gamma}' \underline{\Gamma})^{\frac{1}{2}} \bar{\Sigma}_F (\underline{\Gamma}' \underline{\Gamma})^{\frac{1}{2}}$. Hence, by applying the well-known linear algebra result which states that, for a $m \times n$ matrix A and an $n \times m$ matrix G , the nonzero eigenvalues of AG and GA are identical⁸; we see that $\underline{\lambda}_1, \dots, \underline{\lambda}_K$ are also the eigenvalues of the matrix $(\underline{\Gamma}' \underline{\Gamma}) \bar{\Sigma}_F$ and are, thus, the non-zero eigenvalues of the matrix $\underline{\Gamma} \bar{\Sigma}_F \underline{\Gamma}'$. Since $\Gamma^{(g)} \bar{\Sigma}_F^{(g)} \Gamma^{(g)'} = \underline{\Gamma} \bar{\Sigma}_F \underline{\Gamma}'$ for every $g \in \{1, \dots, K\}$, it follows that $\underline{\lambda}_1, \dots, \underline{\lambda}_K$ are also the nonzero eigenvalues of $\Gamma^{(g)} \bar{\Sigma}_F^{(g)} \Gamma^{(g)'}$ for any $g \in \{1, \dots, K\}$. $\square$

Lemma D-2: Let $\Sigma_{u,t}$ be the symmetric, positive semi-definite $N \times N$ error covariance matrix at time t , and let

$$\lambda_1(\Sigma_{u,t}) \geq \dots \geq \lambda_N(\Sigma_{u,t}) \geq 0$$

denote the eigenvalues of $\Sigma_{u,t}$ and let

$$\lambda_1(\bar{\Sigma}_u) \geq \dots \geq \lambda_N(\bar{\Sigma}_u) \geq 0$$

denote the eigenvalues of

$$\bar{\Sigma}_u = \frac{1}{T_h} \sum_{t=1}^{T_h} \Sigma_{u,t}$$

where T_h is a positive integer. Suppose that there is constant $C > 0$ such that

$$0 \leq \sup_t \lambda_1(\Sigma_{u,t}) \leq C < \infty \tag{134}$$

⁸This result follows from the Weinstein-Aronszajn identity.

Then,

$$0 \leq \lambda_k(\bar{\Sigma}_u) \leq C < \infty \text{ for any } k \in \{1, \dots, N\}.$$

Proof of Lemma D-2:

Since the result is trivial, given the assumption stated in expression (134), if $T_h = 1$, we will focus our proof on the case where $T_h \geq 2$. To proceed, we first prove by mathematical induction the inequality

$$\frac{1}{T_h} \sum_{t=1}^{T_h} \lambda_N(\Sigma_{u,t}) \leq \lambda_k \left(\frac{1}{T_h} \sum_{t=1}^{T_h} \Sigma_{u,t} \right) = \lambda_k(\bar{\Sigma}_u) \leq \frac{1}{T_h} \sum_{t=1}^{T_h} \lambda_1(\Sigma_{u,t})$$

. Consider first the case where $T_h = 2$. By making use of the Weyl inequality, we can show that, for any $k \in \{1, \dots, N\}$,

$$\begin{aligned} \frac{1}{2} \sum_{t=1}^2 \lambda_N(\Sigma_{u,t}) &\leq \frac{\lambda_k(\Sigma_{u,2}) + \lambda_N(\Sigma_{u,1})}{2} \\ &\leq \lambda_k \left(\frac{\Sigma_{u,1} + \Sigma_{u,2}}{2} \right) \\ &\leq \frac{\lambda_k(\Sigma_{u,2}) + \lambda_1(\Sigma_{u,1})}{2} \\ &\leq \frac{1}{2} \sum_{t=1}^2 \lambda_1(\Sigma_{u,t}) \end{aligned}$$

Next, suppose that, for $k \in \{1, \dots, N\}$;

$$\frac{1}{T_h - 1} \sum_{t=1}^{T_h-1} \lambda_N(\Sigma_{u,t}) \leq \lambda_k \left(\frac{1}{T_h - 1} \sum_{t=1}^{T_h-1} \Sigma_{u,t} \right) \leq \frac{1}{T_h - 1} \sum_{t=1}^{T_h-1} \lambda_1(\Sigma_{u,t})$$

Then, multiplying across by $(T_h - 1)/T_h$, we get

$$\frac{1}{T_h} \sum_{t=1}^{T_h-1} \lambda_N(\Sigma_{u,t}) \leq \frac{T_h - 1}{T_h} \lambda_k \left(\frac{1}{T_h - 1} \sum_{t=1}^{T_h-1} \Sigma_{u,t} \right) \leq \frac{1}{T_h} \sum_{t=1}^{T_h-1} \lambda_1(\Sigma_{u,t})$$

or

$$\frac{1}{T_h} \sum_{t=1}^{T_h-1} \lambda_N(\Sigma_{u,t}) \leq \lambda_k \left(\frac{1}{T_h} \sum_{t=1}^{T_h-1} \Sigma_{u,t} \right) \leq \frac{1}{T_h} \sum_{t=1}^{T_h-1} \lambda_1(\Sigma_{u,t})$$

Next, applying the Weyl inequality and using the fact that

$$\lambda_k \left(\frac{\Sigma_{u,T_h}}{T_h} \right) = \frac{1}{T_h} \lambda_k(\Sigma_{u,T_h}) \text{ for } k = 1, \dots, N;$$

we further obtain

$$\begin{aligned}
& \frac{1}{T_h} \sum_{t=1}^{T_h-1} \lambda_N(\Sigma_{u,t}) + \frac{1}{T_h} \lambda_N(\Sigma_{u,T_h}) \\
& \leq \lambda_k \left(\frac{1}{T_h} \sum_{t=1}^{T_h-1} \Sigma_{u,t} \right) + \frac{1}{T_h} \lambda_N(\Sigma_{u,T_h}) \\
& \leq \lambda_k \left(\frac{1}{T_h} \sum_{t=1}^{T_h-1} \Sigma_{u,t} + \frac{1}{T_h} \Sigma_{u,T_h} \right) = \lambda_k \left(\frac{1}{T_h} \sum_{t=1}^{T_h} \Sigma_{u,t} \right) \\
& \leq \lambda_k \left(\frac{1}{T_h} \sum_{t=1}^{T_h-1} \Sigma_{u,t} \right) + \frac{1}{T_h} \lambda_1(\Sigma_{u,T_h}) \\
& \leq \frac{1}{T_h} \sum_{t=1}^{T_h-1} \lambda_1(\Sigma_{u,t}) + \frac{1}{T_h} \lambda_1(\Sigma_{u,T_h})
\end{aligned}$$

Hence, the inequality

$$\frac{1}{T_h} \sum_{t=1}^{T_h} \lambda_N(\Sigma_{u,t}) \leq \lambda_k \left(\frac{1}{T_h} \sum_{t=1}^{T_h} \Sigma_{u,t} \right) \leq \frac{1}{T_h} \sum_{t=1}^{T_h} \lambda_1(\Sigma_{u,t})$$

holds for all positive integer T_h and for any $k \in \{1, \dots, N\}$. Finally, since

$$\inf_t \lambda_N(\Sigma_{u,t}) \leq \frac{1}{T_h} \sum_{t=1}^{T_h} \lambda_N(\Sigma_{u,t}) \quad \text{and} \quad \sup_t \lambda_1(\Sigma_{u,t}) \geq \frac{1}{T_h} \sum_{t=1}^{T_h} \lambda_1(\Sigma_{u,t}),$$

and, since $\Sigma_{u,t}$ is positive semidefinite for all t , it follows that

$$0 \leq \lambda_k \left(\frac{1}{T_h} \sum_{t=1}^{T_h} \Sigma_{u,t} \right) \leq \sup_t \lambda_1(\Sigma_{u,t}) \quad \text{for any } k \in \{1, \dots, N\}$$

which, in turn, implies that

$$0 \leq \lambda_k \left(\frac{1}{T_h} \sum_{t=1}^{T_h} \Sigma_{u,t} \right) \leq C < \infty \quad \text{for any } k \in \{1, \dots, N\}. \quad \square$$

Lemma D-3: Suppose that the $N \times N$ covariance matrix $\bar{\Sigma}_X$ has the decomposition

$$\bar{\Sigma}_X = \bar{\Sigma}_c + \bar{\Sigma}_u$$

where

$$\begin{aligned}\bar{\Sigma}_X &= \frac{1}{T_h} \sum_{t=1}^{T_h} \Sigma_{X,t} = \frac{1}{T_h} \sum_{t=1}^{T_h} E \left[(X_t - \mu_X) (X_t - \mu_X)' \right], \\ \bar{\Sigma}_c &= \frac{1}{T_h} \sum_{t=1}^{T_h} \Sigma_{c,t}, \text{ and } \bar{\Sigma}_u = \frac{1}{T_h} \sum_{t=1}^{T_h} \Sigma_{u,t}, \text{ with } T_h = T - h.\end{aligned}$$

and where both $\bar{\Sigma}_c$ and $\bar{\Sigma}_u$ are positive semi-definite. Let

$$\lambda_1 \geq \dots \geq \lambda_N \geq 0$$

denote the eigenvalues of the matrix $\bar{\Sigma}_X$, and, similarly, let

$$\underline{\lambda}_1 \geq \dots \geq \underline{\lambda}_N \geq 0$$

denote the eigenvalues of $\bar{\Sigma}_c$. Suppose further that the following conditions hold.

- (i) There exists a constant $C > 0$ such that

$$0 \leq \sup_t \lambda_{\max}(\Sigma_{u,t}) \leq C < \infty$$

- (ii) Assume that $\underline{\lambda}_1 \rightarrow \infty$ as $T, N \rightarrow \infty$ and suppose that there exists some (fixed) positive integer J such that, for any $j \in \{1, \dots, J\}$, there exists an absolute constant $c_1^* > 1$ such that

$$1 \leq \frac{\lambda_1}{\underline{\lambda}_j} \leq c_1^* < \infty \text{ for all } N, T \text{ sufficiently large}$$

and for all $j \in \{J+1, J+2, \dots, N\}$

$$\frac{\lambda_j}{\underline{\lambda}_1} \rightarrow 0 \text{ as } N, T \rightarrow \infty$$

Under these conditions, the following statements are true

- (a) For any $j \in \{1, \dots, J\}$

$$0 < \frac{1}{c^*} \leq \frac{\lambda_1}{\underline{\lambda}_j} \leq c^* < \infty \text{ for all } N \text{ and } T \text{ sufficiently large,}$$

where $c^* = c_1^* + c_2^*$ with c_1^* being the positive constant defined in Assumption (ii) above and where c_2^* can be any strictly positive constant.

(b) For any $j \in \{J+1, J+2, \dots, N\}$

$$\frac{\lambda_j}{\underline{\lambda}_1} \rightarrow 0 \text{ as } T, N \rightarrow \infty.$$

Proof of Lemma D-3:

To show part (a), note first that in light of Assumption (i) of this lemma, we can deduce from Lemma D-2 above that

$$0 \leq \lambda_{\max}(\bar{\Sigma}_u) \leq C < \infty.$$

It follows that, for $j \in \{1, \dots, J\}$,

$$\frac{\lambda_{\min}(\bar{\Sigma}_u)}{\underline{\lambda}_j} \rightarrow 0 \text{ and } \frac{\lambda_{\max}(\bar{\Sigma}_u)}{\underline{\lambda}_j} \rightarrow 0 \text{ as } T, N \rightarrow \infty$$

since $\lambda_{\min}(\bar{\Sigma}_u) \leq \lambda_{\max}(\bar{\Sigma}_u) \leq C < \infty$ and since $\underline{\lambda}_j \rightarrow \infty$ as $T, N \rightarrow \infty$ for $j \in \{1, \dots, J\}$ by Assumption (ii) of this lemma. It follows that

$$\frac{\lambda_{\min}(\bar{\Sigma}_u)}{\underline{\lambda}_j} < c_2^* \text{ and } \frac{\lambda_{\max}(\bar{\Sigma}_u)}{\underline{\lambda}_j} < c_2^* \text{ for all } N, T \text{ sufficiently large,}$$

given that c_2^* is a fixed positive constant. Now, by the Weyl inequality, we have, for any $j \in \{1, \dots, J\}$

$$\frac{\underline{\lambda}_1}{\underline{\lambda}_j} + \frac{\lambda_{\min}(\bar{\Sigma}_u)}{\underline{\lambda}_j} \leq \frac{\lambda_1}{\underline{\lambda}_j} \leq \frac{\underline{\lambda}_1}{\underline{\lambda}_j} + \frac{\lambda_{\max}(\bar{\Sigma}_u)}{\underline{\lambda}_j}$$

Making use of Assumptions (i) and (ii), the upper bound of the Weyl inequality implies

$$\begin{aligned} \frac{\lambda_1}{\underline{\lambda}_j} &\leq \frac{\underline{\lambda}_1}{\underline{\lambda}_j} + \frac{\lambda_{\max}(\bar{\Sigma}_u)}{\underline{\lambda}_j} \\ &\leq c_1^* + c_2^* \text{ for all } N, T \text{ sufficiently large} \\ &= c^* \end{aligned}$$

On the other hand, the lower bound of the Weyl inequality implies that

$$\begin{aligned} \frac{\lambda_1}{\underline{\lambda}_j} &\geq \frac{\underline{\lambda}_1}{\underline{\lambda}_j} + \frac{\lambda_{\min}(\bar{\Sigma}_u)}{\underline{\lambda}_j} \\ &\geq \frac{\underline{\lambda}_1}{\underline{\lambda}_j} \left(\text{since } \frac{\lambda_{\min}(\bar{\Sigma}_u)}{\underline{\lambda}_j} \geq 0 \text{ given that } \bar{\Sigma}_u \text{ is positive semi-definite and } \underline{\lambda}_j > 0 \right) \\ &\geq 1. \end{aligned}$$

Combining both the upper and the lower bounds, we deduce that

$$0 < \frac{1}{c^*} < 1 \leq \frac{\lambda_1}{\underline{\lambda}_j} \leq c^* < \infty \text{ for all } N \text{ and } T \text{ sufficiently large,}$$

since $c^* = c_1^* + c_2^* > 1$ given that $c_1^* > 1$ and $c_2^* > 0$. This shows the desired result for part (a).

To show part (b), note that here, again by the Weyl inequality, we have, for $j \in \{J+1, J+2, \dots, N\}$,

$$\frac{\lambda_j}{\underline{\lambda}_1} + \frac{\lambda_{\min}(\overline{\Sigma}_u)}{\underline{\lambda}_1} \leq \frac{\lambda_j}{\underline{\lambda}_1} \leq \frac{\lambda_j}{\underline{\lambda}_1} + \frac{\lambda_{\max}(\overline{\Sigma}_u)}{\underline{\lambda}_1}.$$

Now, since, by Assumption (ii),

$$\frac{\lambda_j}{\underline{\lambda}_1} \rightarrow 0 \text{ for } j \in \{J+1, J+2, \dots, N\}$$

and since, as shown in part (a),

$$0 \leq \frac{\lambda_{\min}(\overline{\Sigma}_u)}{\underline{\lambda}_1} \leq \frac{\lambda_{\max}(\overline{\Sigma}_u)}{\underline{\lambda}_1} \rightarrow 0 \text{ as } T, N \rightarrow \infty;$$

it follows that, for $j \in J\{+1, J+2, \dots, N\}$, $\frac{\lambda_j}{\underline{\lambda}_1} \rightarrow 0$ as $T, N \rightarrow \infty$. $\square$

Lemma D-4: Let $\underline{A}$ and $\underline{E}$ be $m \times m$ symmetric real-valued matrices for some positive integer m , and let $\tilde{A} = \underline{A} + \underline{E}$. Moreover, let

$$\lambda_1 \geq \dots \geq \lambda_m$$

be the set of eigenvalues of $\underline{A}$ and let

$$\tilde{\lambda}_1 \geq \dots \geq \tilde{\lambda}_m$$

be the set of eigenvalues of $\tilde{A}$. Then,

$$\max_{1 \leq i \leq m} |\tilde{\lambda}_i - \lambda_i| \leq \|\underline{E}\|_2.$$

Proof of Lemma D-4: This result is a special case of Theorem 4.8 given on page 202 of Stewart and Sun (1990) and is closely related to Corollary 4.10 of that book. A proof of this lemma can be obtained by specializing the proof of Theorem 4.8 of Stewart and Sun (1990).

Lemma D-5: Let G be a finite non-negative integer. Then,

$$\int_1^\infty z^G \exp\{-z\} dz < \infty$$

Proof of Lemma D-5:

We will first consider the case where $G = 0$. In this case,

$$\int_1^\infty z^0 \exp \{-z\} dz = \int_1^\infty \exp \{-z\} dz$$

Let

$$u = -z \text{ and } du = -dz$$

and note that

$$\begin{aligned} \int_1^\infty \exp \{-z\} dz &= - \int_{-1}^{-\infty} \exp \{u\} du \\ &= \int_{-\infty}^{-1} \exp \{u\} du \\ &= \exp \{-1\} \\ &< \infty. \end{aligned} \tag{135}$$

Next, consider the case where G is a fixed integer such that $G \geq 1$. Here, we will show that

$$\int_1^\infty z^G \exp \{-z\} dz = \left[1 + \sum_{k=1}^G \left(\prod_{j=0}^{k-1} [G-j] \right) \right] \exp \{-1\} < \infty \text{ (given that } G \text{ is finite)}$$

using mathematical induction. To proceed, first consider the case where $G = 1$. Let

$$\begin{aligned} u &= z, \quad du = dz \\ dv &= \exp \{-z\} dz, \quad v = -\exp \{-z\}; \end{aligned}$$

and making use of integration-by-parts, we have

$$\begin{aligned} \int_1^\infty z \exp \{-z\} dz &= -z \exp \{-z\} \Big|_1^\infty + \int_1^\infty \exp \{-z\} dz \\ &= \exp \{-1\} - \exp \{-z\} \Big|_1^\infty \\ &= \exp \{-1\} + \exp \{-1\} \\ &= 2 \exp \{-1\} \\ &= \left\{ 1 + \sum_{k=1}^1 \left(\prod_{j=0}^{k-1} [1-j] \right) \right\} \exp \{-1\} < \infty \end{aligned}$$

Next, for $G = 2$, let

$$\begin{aligned} u &= z^2, \quad du = 2zdz \\ dv &= \exp\{-z\} dz, \quad v = -\exp\{-z\}; \end{aligned}$$

and we again make use of integration-by-parts to obtain

$$\begin{aligned} \int_1^\infty z^2 \exp\{-z\} dz &= -z^2 \exp\{-z\} \Big|_1^\infty + 2 \int_1^\infty z \exp\{-z\} dz \\ &= \exp\{-1\} + 4 \exp\{-1\} \\ &= 5 \exp\{-1\} \\ &= \left[1 + \sum_{k=1}^2 \left(\prod_{j=0}^{k-1} [2-j] \right) \right] \exp\{-1\} \\ &< \infty \end{aligned}$$

To proceed with the induction step, suppose that, for some $G \geq 2$,

$$\int_1^\infty z^{G-1} \exp\{-z\} dz = \left[1 + \sum_{k=1}^{G-1} \left(\prod_{j=0}^{k-1} [G-1-j] \right) \right] \exp\{-1\}.$$

Now, let

$$\begin{aligned} u &= z^G, \quad du = Gz^{G-1}dz \\ dv &= \exp\{-z\} dz, \quad v = -\exp\{-z\}; \end{aligned}$$

and, using integration-by-parts, we have

$$\begin{aligned} \int_1^\infty z^G \exp\{-z\} dz &= -z^G \exp\{-z\} \Big|_1^\infty + G \int_1^\infty z^{G-1} \exp\{-z\} dz \\ &= \exp\{-1\} + G \left[1 + \sum_{k=1}^{G-1} \left(\prod_{j=0}^{k-1} [G-1-j] \right) \right] \exp\{-1\} \\ &= \exp\{-1\} + \left[G + \sum_{k=1}^{G-1} G \left(\prod_{j=0}^{k-1} [G-(j+1)] \right) \right] \exp\{-1\} \\ &= \{1 + G + G(G-1) + G(G-1)(G-2) \\ &\quad + \cdots + G(G-1)(G-2) \times \cdots \times (1)\} \exp\{-1\} \\ &= \left\{ 1 + \sum_{k=1}^G \left(\prod_{j=0}^{k-1} [G-j] \right) \right\} \exp\{-1\} \\ &< \infty \quad (\text{since } G \text{ is assumed to be finite}) \end{aligned} \tag{136}$$

which completes our proof. $\square$

Lemma D-6: Let γ , γ_1 , and γ_2 be positive constants satisfying the relationship $\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$ or $\gamma = \frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}$, such that $0 < \gamma < 1$. Suppose that there exists a constant $0 < b_1 < \infty$, such that for any $s > 0$ and for any $\delta \in \mathbb{R}^N$ with $\|\delta\|_2 = 1$

$$\begin{aligned} \sup_{t \in \mathbb{Z}} P(|\varepsilon_t| > s) &\leq \exp \left\{ 1 - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\} \leq \overline{C} \exp \left\{ - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\}, \\ \sup_{t \in \mathbb{Z}} P(|u'_t \delta| > s) &\leq \exp \left\{ 1 - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\} \leq \overline{C} \exp \left\{ - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\}, \end{aligned}$$

where $\overline{C}$ is a positive constant such that $\overline{C} \geq \exp\{1\}$. Then, the following statements are true:

(a) For any $s > 0$, for all $t \in \mathbb{Z}$, and for any $\delta \in \mathbb{R}^N$, with $\|\delta\|_2 = 1$,

$$\begin{aligned} P(|\varepsilon_t|^4 > s) &\leq \exp \left\{ 1 - \left(\frac{s}{b^*} \right)^{r^*} \right\} \leq \overline{C} \exp \left\{ - \left(\frac{s}{b^*} \right)^{r^*} \right\}, \\ P(|u'_t \delta|^4 > s) &\leq \exp \left\{ 1 - \left(\frac{s}{b^*} \right)^{r^*} \right\} \leq \overline{C} \exp \left\{ - \left(\frac{s}{b^*} \right)^{r^*} \right\}, \end{aligned}$$

where r^* and b^* are fixed constants such that $0 < r^* = \gamma_1/4 < \infty$ and $0 < b^* = b_1^4 < \infty$.

(b) For given constants $\gamma_1 > 0$ and $b_1 > 0$, there exists a constant $C^\bullet > 0$ such that, for all $t \in \mathbb{Z}$,

$$\begin{aligned} E|\varepsilon_t|^4 &\leq C^\bullet < \infty, \\ E|u'_t \delta|^4 &\leq C^\bullet < \infty. \end{aligned}$$

Proof of Lemma D-6: To show the first result of part (a), note first that, for any $s > 0$ and for all $t \in \mathbb{Z}$,

$$\begin{aligned} P(|\varepsilon_t|^4 > s) &= P(|\varepsilon_t| > s^{\frac{1}{4}}) \\ &\leq \exp \left\{ 1 - \left(\frac{s^{\frac{1}{4}}}{b_1} \right)^{\gamma_1} \right\} \\ &= \exp \left\{ 1 - \left(\frac{s}{b_1^4} \right)^{\frac{\gamma_1}{4}} \right\} \\ &= \exp \left\{ 1 - \left(\frac{s}{b^*} \right)^{r^*} \right\} \end{aligned}$$

where

$$r^* = \frac{\gamma_1}{4} > 0 \text{ and } 0 < b^* = b_1^4 < \infty \text{ given that } 0 < b_1 < \infty.$$

Moreover, letting $\overline{C} \geq \exp \{1\}$, we also have, for any $s > 0$ and for all $t \in \mathbb{Z}$,

$$\begin{aligned} P(|\varepsilon_t|^4 > s) &\leq \exp \left\{ 1 - \left(\frac{s}{b^*} \right)^{r^*} \right\} \\ &\leq \overline{C} \exp \left\{ - \left(\frac{s}{b^*} \right)^{r^*} \right\}. \end{aligned}$$

To show the second result of part (a), note that, similar to the proof of the first result; we have, for any $s > 0$, for all $t \in \mathbb{Z}$, and for any $\delta \in \mathbb{R}^N$, with $\|\delta\|_2 = 1$

$$\begin{aligned} P(|u'_t \delta|^4 > s) &= P(|u'_t \delta| > s^{\frac{1}{4}}) \\ &\leq \exp \left\{ 1 - \left(\frac{s^{\frac{1}{4}}}{b_1} \right)^{\gamma_1} \right\} \\ &= \exp \left\{ 1 - \left(\frac{s}{b_1^4} \right)^{\frac{\gamma_1}{4}} \right\} \\ &= \exp \left\{ 1 - \left(\frac{s}{b^*} \right)^{r^*} \right\} \end{aligned}$$

where

$$r^* = \frac{\gamma_1}{4} > 0 \text{ and } 0 < b^* = b_1^4 < \infty \text{ given that } 0 < b_1 < \infty.$$

Moreover, letting $\overline{C} \geq \exp \{1\}$, we also have, for any $s > 0$ and for all $t \in \mathbb{Z}$,

$$\begin{aligned} P(|u'_t \delta|^4 > s) &\leq \exp \left\{ 1 - \left(\frac{s}{b^*} \right)^{r^*} \right\} \\ &\leq \overline{C} \exp \left\{ - \left(\frac{s}{b^*} \right)^{r^*} \right\}. \end{aligned}$$

To show first result given in part (b), note that, for all integer t ,

$$\begin{aligned} E[|\varepsilon_t|^4] &= \int_0^\infty P(|\varepsilon_t|^4 > s) ds \text{ (by equation (21.9) of Billingsley, 1995)} \\ &\leq \int_0^\infty \exp \left\{ 1 - \left(\frac{s}{b^*} \right)^{r^*} \right\} ds \text{ (by the first result of part (a) of this lemma)} \\ &\leq \overline{C} \int_0^\infty \exp \left\{ - \left(\frac{s}{b^*} \right)^{r^*} \right\} ds \end{aligned}$$

Consider first the case where $0 < r^* < 1$ which implies that

$$\frac{1 - r^*}{r^*} > 0$$

For this case, let

$$z = \left(\frac{s}{b^*}\right)^{r^*} \iff s = b^* z^{\frac{1}{r^*}} \text{ and } ds = \frac{b^*}{r^*} z^{\frac{1-r^*}{r^*}} dz$$

and let

$$G_{r^*} = \left\lceil \frac{1 - r^*}{r^*} \right\rceil, \text{ so that } G_{r^*} \text{ is the smallest integer greater than or equal to } \frac{1 - r^*}{r^*}.$$

Changing variables using these notations, we obtain

$$\begin{aligned} E[|\varepsilon_t|^4] &\leq \overline{C} \int_0^\infty \exp\left\{-\left(\frac{s}{b^*}\right)^{r^*}\right\} ds \\ &= \overline{C} \frac{b^*}{r^*} \int_0^\infty z^{\frac{1-r^*}{r^*}} \exp\{-z\} dz \\ &= \overline{C} \frac{b^*}{r^*} \int_0^1 z^{\frac{1-r^*}{r^*}} \exp\{-z\} dz + \overline{C} \frac{b^*}{r^*} \int_1^\infty z^{\frac{1-r^*}{r^*}} \exp\{-z\} dz \\ &\leq \overline{C} \frac{b^*}{r^*} + \overline{C} \frac{b^*}{r^*} \int_1^\infty z^{G_{r^*}} \exp\{-z\} dz \\ &\leq \overline{C} \frac{b^*}{r^*} \left[1 + \left\{ 1 + \sum_{k=1}^{G_{r^*}} \left(\prod_{j=0}^{k-1} [G_{r^*} - j] \right) \right\} \exp\{-1\} \right] \quad (\text{by Lemma D-5}) \\ &\leq C_1^\bullet < \infty \\ &\quad (\text{for some positive constant } C_1^\bullet \text{ given that } r^* \text{ and } b^* \text{ are both fixed constants} \\ &\quad \text{such that } 0 < r^* < 1 \text{ and } 0 < b^* < \infty \text{ and } G_{r^*} \text{ is, thus, also a fixed finite constant}) \end{aligned}$$

Next, consider the case where $r^* \geq 1$. For this case, note that

$$\begin{aligned} E[|\varepsilon_t|^4] &\leq \overline{C} \int_0^\infty \exp\left\{-\left(\frac{s}{b^*}\right)^{r^*}\right\} ds \\ &= \overline{C} \int_0^{b^*} \exp\left\{-\left(\frac{s}{b^*}\right)^{r^*}\right\} ds + \overline{C} \int_{b^*}^\infty \exp\left\{-\left(\frac{s}{b^*}\right)^{r^*}\right\} ds \\ &\leq \overline{C} b^* + \overline{C} \int_{b^*}^\infty \exp\left\{-\left(\frac{s}{b^*}\right)^{r^*}\right\} ds \\ &\quad \left(\text{since } \exp\left\{-\left(\frac{s}{b^*}\right)^{r^*}\right\} \leq \exp\left\{-\left(\frac{s}{b^*}\right)\right\} \text{ for } b^* \leq s < \infty \text{ and } r^* \geq 1 \right). \end{aligned}$$

Next, let

$$u = -\frac{s}{b^*} \iff -b^*u = s \text{ and } ds = -b^*du$$

from which it follows that

$$\begin{aligned} \int_{b^*}^{\infty} \exp \left\{ -\left(\frac{s}{b^*} \right) \right\} ds &= -b^* \int_{-1}^{-\infty} \exp \{u\} du \\ &= b^* \int_{-\infty}^{-1} \exp \{u\} du \\ &= b^* \exp \{-1\} \end{aligned}$$

Hence, for the case where $r^* \geq 1$, we have

$$\begin{aligned} E[|\varepsilon_t|^4] &\leq \overline{C}b^* + \overline{C} \int_{b^*}^{\infty} \exp \left\{ -\left(\frac{s}{b^*} \right) \right\} ds \\ &\leq \overline{C}b^* + \overline{C}b^* \exp \{-1\} \\ &= \overline{C}b^* (1 + \exp \{-1\}) \\ &\leq C_2^\bullet < \infty \quad (\text{for some positive constant } C_2^\bullet \text{ given that } 0 < b^* < \infty) \end{aligned}$$

Putting everything together, we see that, for all t ,

$$\begin{aligned} E[|\varepsilon_t|^4] &\leq \overline{C} \frac{b^*}{r^*} \left[1 + \left\{ 1 + \sum_{k=1}^{G_{r^*}} \left(\prod_{j=0}^{k-1} [G_{r^*} - j] \right) \right\} \exp \{-1\} \right] \mathbb{I}\{0 < r^* < 1\} \\ &\quad + \overline{C}b^* (1 + \exp \{-1\}) \mathbb{I}\{r^* \geq 1\} \\ &\leq C_1^\bullet + C_2^\bullet \\ &\leq C^\bullet \quad (\text{for some positive constant } C^\bullet \geq C_1^\bullet + C_2^\bullet) \\ &< \infty. \end{aligned}$$

To show the second result given in part (b), note that, similar to the proof of the first result; we have, for all $t \in \mathbb{Z}$ and for any $\delta \in \mathbb{R}^N$, with $\|\delta\|_2 = 1$,

$$\begin{aligned} E[|u'_t \delta|^4] &= \int_0^\infty P(|u'_t \delta|^4 > s) ds \quad (\text{by equation (21.9) of Billingsley, 1995}) \\ &\leq \int_0^\infty \exp \left\{ 1 - \left(\frac{s}{b^*} \right)^{r^*} \right\} ds \quad (\text{by the second result of part (a) of this lemma}) \\ &\leq \overline{C} \int_0^\infty \exp \left\{ -\left(\frac{s}{b^*} \right)^{r^*} \right\} ds \end{aligned}$$

Consider first the case where $0 < r^* < 1$ which implies that

$$\frac{1 - r^*}{r^*} > 0$$

For this case, let

$$z = \left(\frac{s}{b^*}\right)^{r^*} \iff s = b^* z^{\frac{1}{r^*}} \text{ and } ds = \frac{b^*}{r^*} z^{\frac{1-r^*}{r^*}} dz$$

and let

$$G_{r^*} = \left\lceil \frac{1 - r^*}{r^*} \right\rceil, \text{ so that } G_{r^*} \text{ is the smallest integer greater than or equal to } \frac{1 - r^*}{r^*}.$$

Changing variables using these notations, we obtain

$$\begin{aligned} E \left[|u'_t \delta|^4 \right] &\leq \overline{C} \int_0^\infty \exp \left\{ - \left(\frac{s}{b^*} \right)^{r^*} \right\} ds \\ &= \overline{C} \frac{b^*}{r^*} \int_0^\infty z^{\frac{1-r^*}{r^*}} \exp \{-z\} dz \\ &= \overline{C} \frac{b^*}{r^*} \int_0^1 z^{\frac{1-r^*}{r^*}} \exp \{-z\} dz + \overline{C} \frac{b^*}{r^*} \int_1^\infty z^{\frac{1-r^*}{r^*}} \exp \{-z\} dz \\ &\leq \overline{C} \frac{b^*}{r^*} + \overline{C} \frac{b^*}{r^*} \int_1^\infty z^{G_{r^*}} \exp \{-z\} dz \\ &\leq \overline{C} \frac{b^*}{r^*} \left[1 + \left\{ 1 + \sum_{k=1}^{G_{r^*}} \left(\prod_{j=0}^{k-1} [G_{r^*} - j] \right) \right\} \exp \{-1\} \right] \quad (\text{by Lemma D-5}) \\ &\leq C_1^\bullet < \infty \\ &\quad (\text{for some positive constant } C_1^\bullet \text{ given that } r^* \text{ and } b^* \text{ are both fixed constants} \\ &\quad \text{such that } 0 < r^* < 1 \text{ and } 0 < b^* < \infty \text{ and } G_{r^*} \text{ is also a fixed finite constant}) \end{aligned}$$

Next, consider the case where $r^* \geq 1$. For this case, note that

$$\begin{aligned} E \left[|u'_t \delta|^4 \right] &\leq \overline{C} \int_0^\infty \exp \left\{ - \left(\frac{s}{b^*} \right)^{r^*} \right\} ds \\ &= \overline{C} \int_0^{b^*} \exp \left\{ - \left(\frac{s}{b^*} \right)^{r^*} \right\} ds + \overline{C} \int_{b^*}^\infty \exp \left\{ - \left(\frac{s}{b^*} \right)^{r^*} \right\} ds \\ &\leq \overline{C} b^* + \overline{C} \int_{b^*}^\infty \exp \left\{ - \left(\frac{s}{b^*} \right)^{r^*} \right\} ds \\ &\quad \left(\text{since } \exp \left\{ - \left(\frac{s}{b^*} \right)^{r^*} \right\} \leq \exp \left\{ - \left(\frac{s}{b^*} \right) \right\} \text{ for } b^* \leq s < \infty \text{ and } r^* \geq 1 \right) \end{aligned}$$

Next, let

$$u = -\frac{s}{b^*} \iff -b^*u = s \text{ and } ds = -b^*du$$

from which it follows that

$$\begin{aligned} \int_{b^*}^{\infty} \exp \left\{ -\left(\frac{s}{b^*} \right) \right\} ds &= -b^* \int_{-1}^{-\infty} \exp \{u\} du \\ &= b^* \int_{-\infty}^{-1} \exp \{u\} du \\ &= b^* \exp \{-1\} \end{aligned}$$

Hence, for the case where $r^* \geq 1$, we have

$$\begin{aligned} E \left[|u'_t \delta|^4 \right] &\leq \overline{C} b^* + \overline{C} \int_{b^*}^{\infty} \exp \left\{ -\left(\frac{s}{b^*} \right) \right\} ds \\ &\leq \overline{C} b^* + \overline{C} b^* \exp \{-1\} \\ &= \overline{C} b^* (1 + \exp \{-1\}) \\ &\leq C_2^\bullet < \infty \quad (\text{for some positive constant } C_2^\bullet \text{ given that } 0 < b^* < \infty) \end{aligned}$$

Putting everything together, we see that, for all $t \in \mathbb{Z}$,

$$\begin{aligned} E \left[|u'_t \delta|^4 \right] &\leq \overline{C} \frac{b^*}{r^*} \left[1 + \left\{ 1 + \sum_{k=1}^{G_{r^*}} \left(\prod_{j=0}^{k-1} [G_{r^*} - j] \right) \right\} \exp \{-1\} \right] \mathbb{I} \{0 < r^* < 1\} \\ &\quad + \overline{C} b^* (1 + \exp \{-1\}) \mathbb{I} \{r^* \geq 1\} \\ &\leq C_1^\bullet + C_2^\bullet \\ &\leq C^\bullet \quad (\text{for some positive constant } C^\bullet \geq C_1^\bullet + C_2^\bullet) \\ &< \infty. \quad \square \end{aligned}$$

The following lemma is a slightly modified version of Theorem 3.9 of Stewart and Sun (1990), which we present here for completeness.

Lemma D-7: Let $\mathcal{X}$ be an invariant subspace of the (symmetric) covariance matrix Σ , so that $\Sigma \mathcal{X} \subseteq \mathcal{X}$, and let L be a semi-orthogonal basis matrix for $\mathcal{X}$. Then, there is a unique matrix A such that

$$\Sigma L = LA$$

The matrix A is the representation of Σ on $\mathcal{X}$ with respect to the basis L . In particular, (v, λ) is an eigenpair of A if and only if (Lv, λ) is an eigenpair of Σ . Moreover, A is symmetric in this case.

Proof of Lemma D-7: Write

$$L = (l_1, \dots, l_k) \text{ and, by assumption, } L' L = I_k$$

Since

$$\Sigma l_i \in \mathcal{X},$$

Σl_i can be expressed as a unique linear combination of the columns of L ; that is,

$$\Sigma l_i = L a_i \text{ for some unique vector } a_i$$

and this is true for every $i \in \{1, \dots, k\}$. The matrix

$$A = (a_1, \dots, a_k)$$

is the required matrix. Moreover, since

$$L' \Sigma L = L' L A = A,$$

A is symmetric given that $L' \Sigma L$ is symmetric. The fact that A is the representation of Σ on L follows from the discussion on Jordan canonical form. Now, let v be an eigenvector of A associated with the eigenvalue λ , so that

$$A v = \lambda v$$

It follows from this representation that

$$\Sigma L v = L A v = \lambda L v$$

from which it follows that $L v$ is an eigenvector of Σ associated with the eigenvalue λ . On the other hand, if $L v$ is an eigenvector of Σ associated with the eigenvalue λ , so that

$$\Sigma L v = \lambda L v;$$

then,

$$L A v = \lambda L v$$

or

$$L (A v - \lambda v) = 0$$

However, since L is a (semi-orthogonal) basis matrix with full column rank, the above equality holds if and only if

$$A v - \lambda v = 0 \text{ or } A v = \lambda v. \quad \square$$

The following lemma is related to Proposition 2.1 and Lemma S1.3 of Cook, Li, and Chiaromonte (2010). We state this result here only for expositional purposes.

Lemma D-8:

Let M be a $N \times N$ real, symmetric matrix, and let $\mathcal{R}$ be a κ -dimensional subspace with

$\kappa < N$. Then, $\mathcal{R}$ reduces M if and only if M can be written in the form

$$M = L_1 \Lambda_1 L_1' + L_2 \Lambda_2 L_2' \quad (137)$$

where the columns of the $N \times \kappa$ matrix L_1 are the orthonormal eigenvectors of M such that

$$\text{span}(L_1) = \mathcal{R},$$

where the columns of the $N \times (N - \kappa)$ matrix L_2 are the orthonormal eigenvectors of M such that

$$\text{span}(L_2) = \mathcal{R}^\perp$$

and where $\begin{pmatrix} L_1 & L_2 \end{pmatrix}$ is a $N \times N$ orthogonal matrix. Here, Λ_1 is a diagonal matrix whose diagonal elements are those eigenvalues associated with the eigenvectors in L_1 and Λ_2 is a diagonal matrix whose diagonal elements are those eigenvalues associated with the eigenvectors in L_2 .

Proof of Lemma D-8:

To proceed, first suppose that M can be written in the representation specified in expression (137). Now, for any $v \in \mathcal{R}$, there exists $\varpi \in \mathbb{R}^\kappa$ such that $v = L_1 \varpi$, from which it follows that

$$\begin{aligned} Mv &= (L_1 \Lambda_1 L_1' + L_2 \Lambda_2 L_2') L_1 \varpi \\ &= L_1 \Lambda_1 \varpi \quad (\text{since } L_1' L_1 = I_\kappa \text{ and } L_2' L_1 = 0) \\ &= L_1 \varpi^* \quad (\text{by setting } \varpi^* = \Lambda_1 \varpi) \\ &\in \mathcal{R} \quad (\text{since } \text{span}(L_1) = \mathcal{R}) \end{aligned}$$

Moreover, for any $v^* \in \mathcal{R}^\perp$, there exists $\omega \in \mathbb{R}^{N-\kappa}$ such that $v^* = L_2 \omega$, from which it follows that

$$\begin{aligned} Mv^* &= (L_1 \Lambda_1 L_1' + L_2 \Lambda_2 L_2') L_2 \omega \\ &= L_2 \Lambda_2 \omega \quad (\text{since } L_2' L_2 = I_{N-\kappa} \text{ and } L_1' L_2 = 0) \\ &= L_2 \omega^* \quad (\text{by setting } \omega^* = \Lambda_2 \omega) \\ &\in \mathcal{R}^\perp \quad (\text{since } \text{span}(L_2) = \mathcal{R}^\perp). \end{aligned}$$

It follows that $\mathcal{R}$ is a reducing subspace of M .

Next, suppose that $\mathcal{R}$ is a reducing subspace of M . Now, let $H_1 \in \mathbb{R}^{N \times \kappa}$ be a semi-orthogonal matrix whose columns span $\mathcal{R}$ and let $H_2 \in \mathbb{R}^{N \times (N-\kappa)}$ be a semi-orthogonal matrix whose columns span $\mathcal{R}^\perp$, such that

$$H_{N \times N} = \begin{pmatrix} H_1 & H_2 \\ N \times \kappa & N \times (N-\kappa) \end{pmatrix}$$

is an orthogonal matrix. By the assumption that $\mathcal{R}$ is a reducing subspace of M , i.e.,

$$M\mathcal{R} \subseteq \mathcal{R} \text{ and } M\mathcal{R}^\perp \subseteq \mathcal{R}^\perp,$$

it follows that both $\mathcal{R}$ and $\mathcal{R}^\perp$ are invariant subspaces of M . It, thus, further follows from Lemma D-7, or Theorem 3.9 of Stewart and Sun (1990), that there exist matrices

$$\overline{B}_1 \in \mathbb{R}^{\kappa \times \kappa} \text{ and } \overline{B}_2 \in \mathbb{R}^{(N-\kappa) \times (N-\kappa)}$$

such that

$$MH_1 = H_1 \overline{B}_1 \text{ and } MH_2 = H_2 \overline{B}_2.$$

Hence,

$$MH = M \begin{pmatrix} H_1 & H_2 \end{pmatrix} = \begin{pmatrix} H_1 & H_2 \end{pmatrix} \begin{pmatrix} \overline{B}_1 & 0 \\ 0 & \overline{B}_2 \end{pmatrix}$$

or

$$M = H \begin{pmatrix} \overline{B}_1 & 0 \\ 0 & \overline{B}_2 \end{pmatrix} H',$$

given that $H = \begin{pmatrix} H_1 & H_2 \end{pmatrix}$ is an orthogonal matrix. Because M is real, symmetric, so must be $\overline{B}_1$ and $\overline{B}_2$. Hence, $\overline{B}_1$ and $\overline{B}_2$ have spectral decompositions given by

$$\overline{B}_1 = C_1 \Lambda_1 C_1' \text{ and } \overline{B}_2 = C_2 \Lambda_2 C_2'$$

for some diagonal matrices Λ_1 and Λ_2 and orthogonal matrices C_1 and C_2 . It follows, by the definition of a spectral decomposition, that the diagonal elements of Λ_1 are the eigenvalues of $\overline{B}_1$ whereas the columns of C_1 are the orthonormal eigenvectors of $\overline{B}_1$. Similarly, the diagonal elements of Λ_2 are the eigenvalues of $\overline{B}_2$ whereas the columns of C_2 are the orthonormal eigenvectors of $\overline{B}_2$. Now, let

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}.$$

Then,

$$\begin{aligned} C\Lambda C' &= \begin{pmatrix} C_1 & 0 \\ \kappa \times \kappa & \kappa \times (N-\kappa) \\ 0 & C_2 \\ (N-\kappa) \times \kappa & (N-\kappa) \times (N-\kappa) \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 \\ \kappa \times \kappa & \kappa \times (N-\kappa) \\ 0 & \Lambda_2 \\ (N-\kappa) \times \kappa & (N-\kappa) \times (N-\kappa) \end{pmatrix} \begin{pmatrix} C_1' & 0 \\ \kappa \times \kappa & \kappa \times (N-\kappa) \\ 0 & C_2' \\ (N-\kappa) \times \kappa & (N-\kappa) \times (N-\kappa) \end{pmatrix} \\ &= \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} \Lambda_1 C_1' & 0 \\ 0 & \Lambda_2 C_2' \end{pmatrix} \\ &= \begin{pmatrix} C_1 \Lambda_1 C_1' & 0 \\ 0 & C_2 \Lambda_2 C_2' \end{pmatrix}. \end{aligned}$$

from which it follows that

$$\begin{aligned}
M &= H \begin{pmatrix} \overline{B}_1 & 0 \\ 0 & \overline{B}_2 \end{pmatrix} H' \\
&= H \begin{pmatrix} C_1 \Lambda_1 C_1' & 0 \\ 0 & C_2 \Lambda_2 C_2' \end{pmatrix} H' \\
&= HC \Lambda C' H' \\
&= L \Lambda L',
\end{aligned}$$

where

$$\begin{aligned}
L &= HC \\
&= \begin{pmatrix} H_1 & H_2 \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \\
&= \begin{pmatrix} H_1 C_1 & H_2 C_2 \end{pmatrix} \\
&= \begin{pmatrix} L_1 & L_2 \end{pmatrix}
\end{aligned}$$

Next, note that both L_1 and L_2 are semi-orthogonal matrices since

$$\begin{aligned}
L_1' L_1 &= C_1' H_1' H_1 C_1 \\
&= C_1' C_1 \quad (\text{since } H_1 \text{ is a semi-orthogonal matrix, so that } H_1' H_1 = I_\kappa) \\
&= I_\kappa \quad (\text{since } C_1 \text{ is a orthogonal matrix}),
\end{aligned} \tag{138}$$

and

$$\begin{aligned}
L_2' L_2 &= C_2' H_2' H_2 C_2 \\
&= C_2' C_2 \quad (\text{since } H_2 \text{ is a semi-orthogonal matrix, so that } H_2' H_2 = I_{N-\kappa}) \\
&= I_{N-\kappa} \quad (\text{since } C_2 \text{ is a orthogonal matrix})
\end{aligned} \tag{139}$$

Furthermore, by Lemma D-7 (or Theorem 3.9 of Stewart and Sun, 1990), each diagonal element of Λ_1 is an eigenvalue of the matrix M , and the i^{th} column of L_1 (i.e., $\ell_{1,i} = H_1 c_{1,i} = H_1 C_1 e_{i,\kappa}$) gives the eigenvector associated with λ_{1i} , the i^{th} diagonal element of Λ_1 for $i \in \{1, \dots, \kappa\}$. Similarly, by Lemma D-7 (or Theorem 3.9 of Stewart and Sun, 1990), each diagonal element of Λ_2 is also an eigenvalue of the matrix M , and the j^{th} column of L_2 (i.e., $\ell_{2,j} = H c_{2,j} = H C_2 e_{j,N-\kappa}$) gives the eigenvector associated with $\lambda_{2,j}$, the j^{th} diagonal element of Λ_2 for $j \in \{1, \dots, N - \kappa\}$.

Finally, we show that

$$span(L_1) = span(H_1) \text{ and } span(L_2) = span(H_2).$$

To proceed, note first that since $H_1 \in \mathbb{R}^{N \times \kappa}$ is a semi-orthogonal matrix whose columns span $\mathcal{R}$ and since C_1 is a $\kappa \times \kappa$ orthogonal (and, thus, nonsingular) matrix, it follows that the

columns of L_1 span the same subspace as the columns of H_1 , i.e.,

$$\text{span}(L_1) = \text{span}(H_1 C_1) = \text{span}(H_1) = \mathcal{R}$$

Similarly, note that since $H_2 \in \mathbb{R}^{N \times (N-\kappa)}$ is a semi-orthogonal matrix whose columns span $\mathcal{R}^\perp$ and since C_2 is a $(N-\kappa) \times (N-\kappa)$ orthogonal (and, thus, nonsingular) matrix, it follows that the columns of L_2 span the same subspace as the columns of H_2 , i.e.,

$$\text{span}(L_2) = \text{span}(H_2 C_2) = \text{span}(H_2) = \mathcal{R}^\perp,$$

from which it also follows that $L_1' L_2 = 0$ so that, given expressions (138) and (139), $\begin{pmatrix} L_1 & L_2 \end{pmatrix}$ is a $N \times N$ orthogonal matrix. $\square$

Lemma D-9: Let

$$\hat{\lambda}_1 = \left\| \hat{\Sigma}_X^S \right\|_2 = \sqrt{\lambda_{\max} \left(\hat{\Sigma}_X^{S'} \hat{\Sigma}_X^S \right)}$$

and let

$$\eta_1 = B_1 \sqrt{\frac{\ln N}{T}}$$

for some sufficiently large constant $B_1 > 0$ be the threshold function used in the specification of $\hat{\Sigma}_X^S$, as previously defined in Lemma C-6. Suppose that Assumptions 4-1, 4-2, 4-3, 6-1, and 6-2, hold; and suppose that as $N, T \rightarrow \infty$

$$\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} = o(1).$$

Then, the following results hold.

(a) For $0 \leq q < 1$,

$$\left| \frac{\hat{\lambda}_1 - \lambda_1}{\lambda_1} \right| = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) = o_p(1),$$

where $\lambda_1 = \lambda_{\max}(\bar{\Sigma}_X)$.

(b) For any fixed real number r and for $0 \leq q < 1$,

$$\left(\frac{\hat{\lambda}_1}{\lambda_1} \right)^r = 1 + O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) = 1 + o_p(1).$$

(c) For any fixed real number $v > 0$,

$$\left\| \frac{1}{\hat{\lambda}_1^v} \hat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-2} \right\|_2 = O_p \left(\frac{1}{\lambda_1^v} \right) = o_p(1) \text{ as } N, T \rightarrow \infty,$$

where

$$\hat{D}_{\lambda,\kappa} = \begin{pmatrix} \hat{\lambda}_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \hat{\lambda}_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \hat{\lambda}_1^{\kappa-\frac{1}{2}} \end{pmatrix} \text{ and } D_{\lambda,\kappa} = \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{\kappa-\frac{1}{2}} \end{pmatrix}$$

(d)

$$\left\| \hat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right), \text{ for } 0 \leq q < 1,$$

where

$$\hat{D}_{\lambda,\kappa} = \begin{pmatrix} \hat{\lambda}_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \hat{\lambda}_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \hat{\lambda}_1^{\kappa-\frac{1}{2}} \end{pmatrix} \text{ and } D_{\lambda,\kappa} = \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{\kappa-\frac{1}{2}} \end{pmatrix}.$$

Proof of Lemma D-9:

For part (a), note first that, by the definition of the spectral norm $\|\cdot\|_2$, we have

$$\begin{aligned} \|\bar{\Sigma}_X\|_2 &= \sqrt{\lambda_{\max} \left\{ \bar{\Sigma}_X' \bar{\Sigma}_X \right\}} \\ &= \sqrt{[\lambda_{\max}(\bar{\Sigma}_X)]^2} \text{ (since } \bar{\Sigma}_X \text{ is symmetric and psd)} \\ &= \lambda_{\max}(\bar{\Sigma}_X) \\ &= \lambda_1 \end{aligned}$$

In addition, note that the conditions of this lemma is the same as that of Lemma C-6. Hence, by making use of the result given in Lemma C-6, i.e.,

$$\left\| \frac{\hat{\Sigma}_X^S - \bar{\Sigma}_X}{\lambda_1} \right\|_2 = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) = o_p(1), \quad (140)$$

we see that

$$\begin{aligned}
\left| \frac{\hat{\lambda}_1 - \lambda_1}{\lambda_1} \right| &= \left| \frac{\|\hat{\Sigma}_X^S\|_2 - \lambda_1}{\lambda_1} \right| \\
&= \left| \frac{\|\hat{\Sigma}_X^S - \bar{\Sigma}_X + \bar{\Sigma}_X\|_2 - \lambda_1}{\lambda_1} \right| \\
&\leq \left| \frac{\|\hat{\Sigma}_X^S - \bar{\Sigma}_X\|_2 + \|\bar{\Sigma}_X\|_2 - \lambda_1}{\lambda_1} \right| \\
&= \left| \frac{\|\hat{\Sigma}_X^S - \bar{\Sigma}_X\|_2 + \lambda_1 - \lambda_1}{\lambda_1} \right| \\
&\leq \frac{\|\hat{\Sigma}_X^S - \bar{\Sigma}_X\|_2}{\lambda_1} \\
&= O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right). \tag{141}
\end{aligned}$$

For part (b), note that, by the binomial theorem,

$$\begin{aligned}
\left(\frac{\hat{\lambda}_1}{\lambda_1} \right)^r &= \left(1 + \frac{\hat{\lambda}_1 - \lambda_1}{\lambda_1} \right)^r \\
&= 1 + r \left(\frac{\hat{\lambda}_1 - \lambda_1}{\lambda_1} \right) + \frac{r(r-1)}{2!} \left(\frac{\hat{\lambda}_1 - \lambda_1}{\lambda_1} \right)^2 + \dots \\
&= 1 + O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \\
&\quad \text{(by the result given in part (a) of this lemma)} \\
&= 1 + o_p(1) \text{ for any fixed real number } r.
\end{aligned}$$

To show part (c), note first that

$$\begin{aligned}
& \frac{1}{\widehat{\lambda}_1^v} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-2} \\
= & \frac{1}{\widehat{\lambda}_1^v} \begin{pmatrix} \widehat{\lambda}_1 & 0 & \cdots & 0 \\ 0 & \widehat{\lambda}_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \widehat{\lambda}_1^{2\kappa-1} \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(2\kappa-1)} \end{pmatrix} \\
= & \frac{1}{\lambda_1^v} \frac{1}{\left(\widehat{\lambda}_1/\lambda_1\right)^v} \begin{pmatrix} \left(\widehat{\lambda}_1/\lambda_1\right) & 0 & \cdots & 0 \\ 0 & \left(\widehat{\lambda}_1/\lambda_1\right)^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \left(\widehat{\lambda}_1/\lambda_1\right)^{(2\kappa-1)} \end{pmatrix} \\
= & \frac{1}{\lambda_1^v} \left(\frac{\widehat{\lambda}_1}{\lambda_1}\right)^{-v} \begin{pmatrix} \left(\widehat{\lambda}_1/\lambda_1\right) & 0 & \cdots & 0 \\ 0 & \left(\widehat{\lambda}_1/\lambda_1\right)^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \left(\widehat{\lambda}_1/\lambda_1\right)^{(2\kappa-1)} \end{pmatrix}
\end{aligned}$$

It follows that

$$\begin{aligned}
& \left\| \frac{1}{\widehat{\lambda}_1^v} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-2} \right\|_2 \\
&= \frac{1}{\widehat{\lambda}_1^v} \left\| \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-2} \right\|_2 \\
&= \frac{1}{\lambda_1^v} \left(\frac{\widehat{\lambda}_1}{\lambda_1} \right)^{-v} \left\| \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-2} \right\|_2 \\
&\leq \frac{1}{\lambda_1^v} \left(\frac{\widehat{\lambda}_1}{\lambda_1} \right)^{-v} \left\| \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-2} \right\|_F \\
&= \frac{1}{\lambda_1^v} \left(\frac{\widehat{\lambda}_1}{\lambda_1} \right)^{-v} \sqrt{\sum_{j=1}^{\kappa} \left[\left(\frac{\widehat{\lambda}_1}{\lambda_1} \right)^{j-\frac{1}{2}} \right]^4} \\
&\leq \frac{1}{\lambda_1^v} \left(\frac{\widehat{\lambda}_1}{\lambda_1} \right)^{-v} \sum_{j=1}^{\kappa} \left[\left(\frac{\widehat{\lambda}_1}{\lambda_1} \right)^{j-\frac{1}{2}} \right]^2 \\
&\quad \left(\text{using the inequality } \left| \sum_{i=1}^m a_i \right|^r \leq \sum_{i=1}^m |a_i|^r \text{ for } 0 < r \leq 1 \right) \\
&\leq \frac{1}{\lambda_1^v} \left(\frac{\widehat{\lambda}_1}{\lambda_1} \right)^{-v} \sum_{j=1}^{\kappa} \left(\frac{\widehat{\lambda}_1}{\lambda_1} \right)^{2j-1} \\
&= \frac{1}{\lambda_1^v} \left[1 + O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \right] \\
&\quad \times \sum_{j=1}^{\kappa} \left[1 + O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \right] \\
&\quad (\text{since by the result given in part (b) above} \\
&\quad \left(\frac{\widehat{\lambda}_1}{\lambda_1} \right)^r = 1 + O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \text{ for any fixed real number } r) \\
&= O_p \left(\frac{1}{\lambda_1^v} \right) = o_p(1).
\end{aligned}$$

Finally, to show part (d), note that

$$\begin{aligned} & \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \\ = & \begin{pmatrix} \left(\lambda_1/\widehat{\lambda}_1\right)^{\frac{1}{2}} - 1 & 0 & \cdots & 0 \\ 0 & \left(\lambda_1/\widehat{\lambda}_1\right)^{\frac{3}{2}} - 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \left(\lambda_1/\widehat{\lambda}_1\right)^{\kappa-\frac{1}{2}} - 1 \end{pmatrix} \end{aligned}$$

Hence,

$$\begin{aligned} \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_2 & \leq \left\| \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right\|_F \\ & = \sqrt{\text{tr} \left\{ \left(\widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \right)^2 \right\}} \quad \left(\text{since } \widehat{D}_{\lambda,\kappa}^{-1} D_{\lambda,\kappa} - I_\kappa \text{ is a symmetric matrix} \right) \\ & = \sqrt{\sum_{j=0}^{\kappa-1} \left[\left(\lambda_1/\widehat{\lambda}_1 \right)^{\frac{1}{2}+j} - 1 \right]^2} \\ & \leq \sum_{j=0}^{\kappa-1} \left| \left(\lambda_1/\widehat{\lambda}_1 \right)^{\frac{1}{2}+j} - 1 \right| \\ & \quad \left(\text{using the inequality } \left| \sum_{i=1}^m a_i \right|^r \leq \sum_{i=1}^m |a_i|^r \text{ for } 0 < r \leq 1 \right) \\ & = \left| \left(\frac{\widehat{\lambda}_1}{\lambda_1} \right)^{-\frac{1}{2}} - 1 \right| + \left| \left(\frac{\widehat{\lambda}_1}{\lambda_1} \right)^{-\frac{3}{2}} - 1 \right| + \cdots + \left| \left(\frac{\widehat{\lambda}_1}{\lambda_1} \right)^{-(\kappa-\frac{1}{2})} - 1 \right| \\ & = O_p \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right), \end{aligned}$$

where the last line follows from the result given in part (b) above. $\square$

Lemma D-10: Suppose that Assumption 6-3 holds. Suppose also that $\lambda_1 \rightarrow \infty$ as $N, T \rightarrow \infty$; then, the following results hold.

(a)

$$\frac{c_0^3(N)}{\lambda_1^{\frac{7}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{3}{2}(1-q)} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

(b)

$$\frac{[a(N)]^{\frac{3}{2}}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{3}{2}(1-\frac{q}{2})} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

(c)

$$\begin{aligned} & \frac{c_0(N) a(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \\ &= o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right). \end{aligned}$$

(d)

$$\frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{1-q} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)$$

and, for any fixed real number $\bar{\rho} > 0$,

$$\frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{1-q} \frac{1}{\lambda_1^{\bar{\rho}}} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

(e)

$$\begin{aligned} & \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \\ &= o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \end{aligned}$$

and, for any fixed real number $\bar{\rho} > 0$,

$$\begin{aligned} & \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \frac{1}{\lambda_1^{\bar{\rho}}} \\ &= o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \end{aligned}$$

(f)

$$\frac{a(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)$$

and, for any fixed real number $\bar{\rho} > 0$,

$$\frac{a(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \frac{1}{\lambda_1^{\bar{\rho}}} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)$$

(g) For any fixed real number $\bar{\rho} > 0$,

$$\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \frac{1}{\lambda_1^{\bar{\rho}+\frac{1}{2}}} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)$$

(h)

$$\frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} = O \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)$$

and for any fixed real number $\bar{\rho} > 0$,

$$\frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \frac{1}{\lambda_1^{\bar{\rho}}} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)$$

(i) For any fixed real number $\bar{\rho} > 0$,

$$\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \frac{1}{\lambda_1^{\bar{\rho}+\frac{1}{2}}} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)$$

(j)

$$\begin{aligned} & \frac{c_0^2(N) \sqrt{a(N)}}{\lambda_1^3} \left[\frac{\ln N}{T} \right]^{1-q} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \\ &= o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \end{aligned}$$

Proof of Lemma D-10: To show part (a), note first that in the case where

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \geq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

we have

$$\begin{aligned} & \frac{c_0^3(N)}{\lambda_1^{\frac{7}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{3}{2}(1-q)} \\ &= \left(\left[\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right]^2 \right) \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \\ &\ll \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left(\text{since } \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} = o(1) \right) \end{aligned}$$

On the other hand, in the case where

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

note that

$$\begin{aligned} \frac{c_0^3(N)}{\lambda_1^{\frac{7}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{3}{2}(1-q)} &= \left(\left[\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right]^2 \right) \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \\ &\ll \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \end{aligned}$$

Together, these two results imply that

$$\frac{c_0^3(N)}{\lambda_1^{\frac{7}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{3}{2}(1-q)} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

To show part (b), note first that in the case where

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \geq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

we have

$$\begin{aligned}
& \frac{[a(N)]^{\frac{3}{2}}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{3}{2}(1-\frac{q}{2})} \\
&= \left[\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right]^2 \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \\
&\ll \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \left(\text{since } \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} = o(1) \right) \\
&\leq \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}
\end{aligned}$$

On the other hand, in the case where

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

note that

$$\begin{aligned}
\frac{[a(N)]^{\frac{3}{2}}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{3}{2}(1-\frac{q}{2})} &= \left[\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right]^2 \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \\
&\ll \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}
\end{aligned}$$

Together, these two results imply that

$$\frac{[a(N)]^{\frac{3}{2}}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{3}{2}(1-\frac{q}{2})} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

To show part (c), note first that if

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \geq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

then

$$\begin{aligned}
& \frac{c_0(N) a(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \\
&= \left[\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right]^2 \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \\
&\ll \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left(\text{since } \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} = o(1) \right)
\end{aligned}$$

On the other hand, if

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

note that

$$\begin{aligned}
& \frac{c_0(N) a(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \\
&= \left[\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right]^2 \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \\
&\ll \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}
\end{aligned}$$

Together, these two results imply that

$$\frac{c_0(N) a(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

To show the first result of part (d), note first that if

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \geq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

then

$$\begin{aligned}
& \frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{1-q} \\
&= \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \ll \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \\
&\quad \left(\text{since } \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} = o(1) \right)
\end{aligned}$$

On the other hand, if

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

note that

$$\begin{aligned}
\frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{1-q} &= \left(\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right) \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \\
&\ll \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}
\end{aligned}$$

Together, these two results imply the first result of part (d), i.e.,

$$\frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{1-q} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

To show the second result of part (d), note first that

$$\frac{1}{\lambda_1^{\bar{\rho}}} = o(1) \text{ for } \bar{\rho} > 0 \text{ given that } \lambda_1 \rightarrow \infty.$$

It, thus, follows from the first result of part (d) that, for all fixed real number $\bar{\rho} > 0$,

$$\begin{aligned}
\frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{1-q} \frac{1}{\lambda_1^{\bar{\rho}}} &= o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) o(1) \\
&= o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).
\end{aligned}$$

To show the first result of part (e), note first that in the case where

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \geq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

we have

$$\begin{aligned} & \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \\ &= \left[\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right] \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \\ &\ll \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left(\text{since } \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} = o(1) \right) \end{aligned}$$

On the other hand, in the case where

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

note that

$$\begin{aligned} \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} &= \left[\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right] \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \\ &\ll \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \end{aligned}$$

Together, these two results imply the first result of part (e), i.e.,

$$\begin{aligned} & \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \\ &= o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \end{aligned}$$

Now, to show the second result of part (e), note first that

$$\frac{1}{\lambda_1^\rho} = o(1) \text{ for } \bar{\rho} > 0 \text{ given that } \lambda_1 \rightarrow \infty.$$

It, thus, follows from the first result of part (e) that, for all fixed real number $\bar{\rho} > 0$,

$$\begin{aligned}
& \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \frac{1}{\lambda_1^{\bar{\rho}}} \\
&= o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) o(1) \\
&= o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).
\end{aligned}$$

To show part (f), note first that if

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

then

$$\begin{aligned}
& \frac{a(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \\
&= \left[\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right] \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \\
&\ll \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \left(\text{since } \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} = o(1) \right)
\end{aligned}$$

On the other hand, if

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \geq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

note that

$$\begin{aligned}
\frac{a(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} &= \left[\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right] \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \\
&\ll \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \leq \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}
\end{aligned}$$

Together, these two results imply the first result of part (f), i.e.,

$$\frac{a(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)$$

Now, to show the second result of part (f), note again that

$$\frac{1}{\lambda_1^{\bar{\rho}}} = o(1) \text{ for } \bar{\rho} > 0 \text{ given that } \lambda_1 \rightarrow \infty.$$

It, thus, follows from the first result of part (f) that, for any fixed real number $\bar{\rho} > 0$,

$$\begin{aligned} \frac{a(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \frac{1}{\lambda_1^{\bar{\rho}}} &= o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) o(1) \\ &= o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \end{aligned}$$

To show the result given in part (g), note first that if

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \geq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

then, for any fixed real number $\bar{\rho} > 0$,

$$\begin{aligned} &\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \frac{1}{\lambda_1^{\bar{\rho}+\frac{1}{2}}} \\ &= \frac{1}{\lambda_1^{\bar{\rho}}} \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \\ &\ll \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left(\text{since } \frac{1}{\lambda_1^{\bar{\rho}}} = o(1) \text{ for any fixed real number } \bar{\rho} > 0 \right) \end{aligned}$$

On the other hand, if

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

note that

$$\begin{aligned} \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \frac{1}{\lambda_1^{\bar{\rho}+\frac{1}{2}}} &= \frac{1}{\lambda_1^{\bar{\rho}}} \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \\ &\ll \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \end{aligned}$$

Together, these two results imply that, for any fixed real number $\bar{\rho} > 0$,

$$\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \frac{1}{\lambda_1^{\bar{\rho}+\frac{1}{2}}} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

To show part (h), note first that, in this case, it is trivial that

$$\frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} = O \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

Moreover, by assumption, $\lambda_1 \rightarrow \infty$ so that

$$\frac{1}{\lambda_1^{\bar{\rho}}} = o(1) \text{ for all fixed real number } \bar{\rho} > 0.$$

from which it follows that

$$\begin{aligned} &\frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \frac{1}{\lambda_1^{\bar{\rho}}} \\ &= O \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) o(1) \\ &= o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right). \end{aligned}$$

Turning our attention now to part (i), note that in this case, we can rewrite

$$\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \frac{1}{\lambda_1^{\bar{\rho}+\frac{1}{2}}} = \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \frac{1}{\lambda_1^{\bar{\rho}}}$$

so that it follows immediately from the second result given in part (h) above that for all

fixed real number $\bar{\rho} > 0$

$$\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \frac{1}{\lambda_1^{\bar{\rho}+\frac{1}{2}}} = o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).$$

Finally, to show part (j), note first that in the case where

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \geq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

we have

$$\begin{aligned} & \frac{c_0^2(N) \sqrt{a(N)}}{\lambda_1^3} \left[\frac{\ln N}{T} \right]^{1-q} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \\ &= \left[\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right] \left[\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right] \left[\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right] \\ &\ll \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \\ &\quad \left(\text{since } \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} = o(1) \text{ and } \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} = o(1) \right) \end{aligned}$$

On the other hand, in the case where

$$\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})},$$

note that

$$\begin{aligned} & \frac{c_0^2(N) \sqrt{a(N)}}{\lambda_1^3} \left[\frac{\ln N}{T} \right]^{1-q} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \\ &= \left[\sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right] \left[\frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right] \left[\frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \right] \\ &\ll \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \leq \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \end{aligned}$$

Together, these two results imply that

$$\begin{aligned} & \frac{c_0^2(N) \sqrt{a(N)}}{\lambda_1^3} \left[\frac{\ln N}{T} \right]^{1-q} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \\ &= o \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right). \quad \square \end{aligned}$$

Lemma D-11: Let c be a positive constant. Then, for any given $\gamma_2 > 0$, there exists a constant $C^\# > 0$ such that

$$\sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \leq C^\# < \infty$$

for all positive integer T .

Proof of Lemma D-11:

Note that

$$\exp \left(-\frac{cm^{\gamma_2}}{4} \right) \geq 0 \text{ for all positive integer } m.$$

Hence, we make use of the integral test for convergence of an infinite series. To proceed, consider the integral

$$\int_1^\infty \exp \left(-\frac{cx^{\gamma_2}}{4} \right) dx \text{ for } c > 0 \text{ and } \gamma_2 > 0.$$

Let

$$u = \frac{cx^{\gamma_2}}{4} \iff x = \left(\frac{4}{c} \right)^{\frac{1}{\gamma_2}} u^{\frac{1}{\gamma_2}}$$

so that

$$dx = \left(\frac{4}{c} \right)^{\frac{1}{\gamma_2}} \frac{1}{\gamma_2} u^{\frac{1}{\gamma_2}-1} du$$

It follows that

$$\begin{aligned} \int_1^\infty \exp \left(-\frac{cx^{\gamma_2}}{4} \right) dx &= \left(\frac{4}{c} \right)^{\frac{1}{\gamma_2}} \frac{1}{\gamma_2} \int_{c/4}^\infty u^{\frac{1}{\gamma_2}-1} \exp(-u) du \\ &\leq \left(\frac{4}{c} \right)^{\frac{1}{\gamma_2}} \frac{1}{\gamma_2} \int_0^\infty u^{\frac{1}{\gamma_2}-1} \exp(-u) du \\ &= \left(\frac{4}{c} \right)^{\frac{1}{\gamma_2}} \frac{1}{\gamma_2} \Gamma \left(\frac{1}{\gamma_2} \right) \\ &< \infty \end{aligned}$$

The last two lines above follow from the fact that

$$\Gamma(z) = \int_0^\infty u^{z-1} \exp(-u) du$$

is the integral representation of the gamma function, and it is well known that this integral is absolutely convergent for all $z > 0$, a condition that is satisfied here given that $\gamma_2 > 0$. It further follows from the integral test that $\sum_{m=1}^T \exp(-cm^{\gamma_2}/4)$ is a convergent series, so that there exists a constant $C^\# > 0$ such that

$$\sum_{m=1}^T \exp\left(-\frac{cm^{\gamma_2}}{4}\right) \leq C^\# < \infty \text{ for all positive integer } T. \quad \square$$

Appendix E: Supporting Lemmas for the Proof of Theorems 4 and 7

The lemmas given in this appendix pertain to the diffusion index model

$$y_{t+h} = \alpha + (F_t - \mu_F)' \varphi + \varepsilon_{t+h}, \quad (142)$$

$$X_t = \mu_X + \Gamma(F_t - \mu_F) + u_t, \text{ for } t = 1, \dots, T-h, \quad (143)$$

where $\mu_F = E[F_t]$ and $\mu_X = E[X_t]$. The following assumptions are imposed on the model given in expressions (142) and (143) above.

Assumption (i): $\Gamma'\Gamma/\underline{\lambda}_{1,1} = I_K$ and $\bar{\Sigma}_F = \underline{\Lambda}/\underline{\lambda}_{1,1}$, where

$$\bar{\Sigma}_F = \frac{1}{T_h} \sum_{t=1}^{T_h} E[(F_t - \mu_F)(F_t - \mu_F)']$$

denotes the factor covariance matrix and where $\underline{\Lambda}$ is a diagonal matrix whose diagonal elements are the nonzero eigenvalues of $\Gamma'\bar{\Sigma}_F\Gamma$.

Assumption (ii): (a) Suppose that $E[u_t] = 0$ and $E[(F_t - \mu_F)u_t] = 0$, for all t . (b) Suppose that there exists some constant $C^\dagger > 0$ such that $\sup_t \lambda_{\max}(\Sigma_{u,t}) \leq C^\dagger < \infty$ and $E\left\{(F_{t,\cdot g} - \mu_{F,\cdot g})^4\right\} \leq C^\dagger < \infty$, for all $g \in \{1, \dots, K\}$ and for all t , where $\Sigma_{u,t} = E[u_t u_t']$ and where $F_{t,\cdot g}$ and $\mu_{F,\cdot g}$ denote the g^{th} components of F_t and μ_F , respectively.

Assumption (iii): $E[\varepsilon_t] = 0$ for all t . In addition, let $h \geq 1$ be a fixed finite integer, and assume that $E[(F_t - \mu_F)\varepsilon_{t+h}] = 0$ for all t and $E[u_{it}\varepsilon_{t+h}] = 0$ for all i and t .

Assumption (iv): Let $\Gamma_{(i,g)}$ denote the $(i, g)^{th}$ element of Γ , and suppose that there exists some constant $c^\ddagger > 1$ such that $\max_{1 \leq i \leq N, 1 \leq g \leq K} |\Gamma_{(i,g)}| \leq c^\ddagger < \infty$ for all N .

Assumption (v): Let γ , γ_1 , and γ_2 be positive constants satisfying the relationship $\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}$ or $\gamma = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2}$, such that $0 < \gamma < 1$. Suppose that the following conditions hold: (a)

There exists a constant $b_1 > 0$, such that for any $s > 0$, for any $\delta \in \mathbb{R}^N$, with $\|\delta\|_2 = 1$, and for $g \in \{1, \dots, K\}$

$$\begin{aligned} \sup_{t \in \mathbb{Z}} P(|u'_t \delta| > s) &\leq \exp \left\{ 1 - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\}, \\ \sup_{t \in \mathbb{Z}} P(|F_{t,g} - \mu_{F,g}| > s) &\leq \exp \left\{ 1 - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\}, \\ \sup_{t \in \mathbb{Z}} P(|\varepsilon_t| > s) &\leq \exp \left\{ 1 - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\}, \end{aligned}$$

where $\bar{C}$ is a positive constant such that $\bar{C} \geq \exp\{1\}$, where $\mu_F = E[F_t]$, and where $F_{t,g}$ and $\mu_{F,g}$ are the g^{th} element of F_t and μ_F , respectively. (b) Let $\underline{V}_t = (\varepsilon_t, F'_t, u'_t)'$. Define $\underline{\mathcal{F}}_{-\infty}^t = \sigma(\dots, \underline{V}_{t-2}, \underline{V}_{t-1}, \underline{V}_t)$ and $\underline{\mathcal{F}}_{t+m}^\infty = \sigma(\underline{V}_{t+m}, \underline{V}_{t+m+1}, \underline{V}_{t+m+2}, \dots)$. Suppose that there exist positive constants γ_2 , a , and c , such that:

$$\alpha(m) = \sup_t \alpha(\underline{\mathcal{F}}_{-\infty}^t, \underline{\mathcal{F}}_{t+m}^\infty) \leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1,$$

where

$$\alpha(\underline{\mathcal{F}}_{-\infty}^t, \underline{\mathcal{F}}_{t+m}^\infty) = \sup_{G \in \underline{\mathcal{F}}_{-\infty}^t, H \in \underline{\mathcal{F}}_{t+m}^\infty} |P(G \cap H) - P(G)P(H)|.$$

Remark: Note that Assumptions (i)-(iii) given above are the same as assumptions (i)-(iii) given in the statement of Theorem 4.

For Lemma E-1, we consider the diffusion index model given by expressions (142) and (143) under two different scenarios which have been explained in Section 5 of the paper:

Scenario 1 ($1 \leq \kappa < K$ Case): Assume that the following conditions hold:

Assumption S1-1: There exists a constant $\bar{c} > 1$ such that $0 < \frac{1}{\bar{c}} \leq \min_{1 \leq g \leq \kappa} |\varphi_{1,g}| \leq \max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \leq \bar{c} < \infty$, whereas $\varphi_2 = (\varphi_{2,1}, \dots, \varphi_{2,K-\kappa})' = 0$.

Assumption S1-2: (a) Suppose that $\underline{\lambda}_{1,g} \rightarrow \infty$ and $\underline{\lambda}_{2,j} \rightarrow \infty$, as $N, T \rightarrow \infty$ (for $g = 1, \dots, \kappa$ and $j = 1, \dots, K - \kappa$) such that $\underline{\lambda}_{1,1} \asymp \dots \asymp \underline{\lambda}_{1,\kappa} \asymp \underline{\lambda}_{2,1} \asymp \dots \asymp \underline{\lambda}_{2,K-\kappa}$. (b) Suppose also that there exists a constant, $\underline{c}$, with $0 < \underline{c} < 1$, such that $(\underline{\lambda}_{1,g} - \underline{\lambda}_{1,k}) / \underline{\lambda}_{1,1} \geq \underline{c}$, for all $g < k$, with $g \in \{1, \dots, \kappa - 1\}$ and $k \in \{2, \dots, \kappa\}$, for all N and T sufficiently large.

Scenario 2 ($\kappa = K$ Case): Assume that the following conditions hold:

Assumption S2-1: There exists a constant $\bar{c} > 1$ such that $0 < \frac{1}{\bar{c}} \leq \min_{1 \leq g \leq K} |\varphi_{1,g}| \leq \max_{1 \leq g \leq K} |\varphi_{1,g}| \leq \bar{c} < \infty$.

Assumption S2-2 (a) Suppose that $\underline{\lambda}_{1,g} \rightarrow \infty$ as $N, T \rightarrow \infty$ (for $g = 1, \dots, K$) such that $\underline{\lambda}_{1,1} \asymp \dots \asymp \underline{\lambda}_{1,K}$. (b) Suppose also that there exists a constant, $\underline{c}$, with $0 < \underline{c} < 1$, such that $(\underline{\lambda}_{1,g} - \underline{\lambda}_{1,k}) / \underline{\lambda}_{1,1} \geq \underline{c}$, for all $g < k$, with $g \in \{1, \dots, K - 1\}$ and $k \in \{2, \dots, K\}$, for all N and T sufficiently large.

Remark: It should be clear that under Scenario 1, among the K available factors, only $\kappa < K$ of them are relevant for forecasting the target variable y_{t+h} since in this case there are $K - \kappa$ zero coefficients (i.e., $\varphi_{\kappa+1}, \dots, \varphi_K$ are zero). On the other hand, under Scenario

2 all K available factors are relevant for the purpose of forecasting since all coefficients are non-zero in this case.

Lemma E-1: Consider the diffusion index model given by expressions (142)-(143). Suppose that Assumptions (i)-(iii) hold, and let $\bar{\beta} = G (G' \bar{\Sigma}_X G)^{-1} G' \bar{\sigma}_{Xy}$ be the population PLS estimator, where

$$G = \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{pmatrix}, \bar{\Sigma}_X = \frac{1}{T_h} \sum_{t=1}^{T_h} E [(X_t - \mu_X) (X_t - \mu_X)'] , \text{ and}$$

$$\bar{\sigma}_{Xy} = \frac{1}{T_h} \sum_{t=1}^{T_h} E \{(X_t - \mu_X) (y_{t+h} - \alpha)\}.$$

with $\kappa \in \{1, \dots, K\}$, with the target variable y_{t+h} being generated by expression (142), and with the vector of predictor variables X_t being generated by expression (143). Under these conditions, consider the following two scenarios.

(a) **Scenario 1 ($1 \leq \kappa < K$ Case):** Suppose that Assumptions S1-1 and S1-2 hold. Let

$$\beta = \frac{\Gamma_{\lambda,1} \varphi_1}{\sqrt{\Delta_{1,1}}}, \text{ where } \Gamma_{\lambda,1} = \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \text{ and } \varphi_1 = (\varphi_{1,1}, \dots, \varphi_{1,\kappa})'$$

Then,

$$\|\bar{\beta} - \beta\|_2 = O\left(\frac{1}{\lambda_1^{\frac{3}{2}}}\right), \text{ where } \lambda_1 = \lambda_{\max}(\bar{\Sigma}_X).$$

(b) **Scenario 2 ($\kappa = K$ Case):** Suppose that Assumptions S2-1 and S2-2 hold. Let

$$\beta = \frac{\Gamma_{\lambda} \varphi}{\sqrt{\Delta_{1,1}}}, \text{ where } \Gamma_{\lambda} = \frac{\Gamma}{\sqrt{\Delta_{1,1}}} \text{ and } \varphi = (\varphi_{1,1}, \dots, \varphi_{1,K})'$$

Then,

$$\|\bar{\beta} - \beta\|_2 = O\left(\frac{1}{\lambda_1^{\frac{3}{2}}}\right), \text{ where } \lambda_1 = \lambda_{\max}(\bar{\Sigma}_X).$$

Proof of Lemma E-1:

To show part (a), note first that, for the diffusion index model defined by expressions

(142) and (143), we have by direct calculation:

$$\begin{aligned}
\bar{\Sigma}_X &= \frac{1}{T_h} \sum_{t=1}^{T_h} E[(X_t - \mu_X)(X_t - \mu_X)'] \\
&= \Gamma \left(\frac{1}{T_h} \sum_{t=1}^{T_h} E[(F_t - \mu_F)(F_t - \mu_F)'] \right) \Gamma' + \frac{1}{T_h} \sum_{t=1}^{T_h} E[u_t u_t'] \quad (\text{given Assumption (ii)}) \\
&= \Gamma \bar{\Sigma}_F \Gamma' + \bar{\Sigma}_u \\
&= \Gamma \left(\frac{\Lambda}{\Delta_{1,1}} \right) \Gamma' + \bar{\Sigma}_u \quad (\text{given Assumption (i)}) \\
&= \frac{\Gamma}{\sqrt{\Delta_{1,1}}} \Lambda \frac{\Gamma'}{\sqrt{\Delta_{1,1}}} + \bar{\Sigma}_u \\
&= \frac{1}{\sqrt{\Delta_{1,1}}} \begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \begin{pmatrix} \Gamma'_1 \\ \Gamma'_2 \end{pmatrix} \frac{1}{\sqrt{\Delta_{1,1}}} + \bar{\Sigma}_u \\
&= \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \Lambda_1 \frac{\Gamma'_1}{\sqrt{\Delta_{1,1}}} + \frac{\Gamma_2}{\sqrt{\Delta_{1,1}}} \Lambda_2 \frac{\Gamma'_2}{\sqrt{\Delta_{1,1}}} + \bar{\Sigma}_u \\
&= \Gamma_{\lambda,1} \Lambda_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \Lambda_2 \Gamma'_{\lambda,2} + \bar{\Sigma}_u
\end{aligned}$$

where $\Gamma_{\lambda,1} = \Gamma_1 / \sqrt{\Delta_{1,1}}$ and $\Gamma_{\lambda,2} = \Gamma_2 / \sqrt{\Delta_{1,1}}$. Moreover,

$$\begin{aligned}
\bar{\sigma}_{Xy} &= \frac{1}{T_h} \sum_{t=1}^{T_h} E\{(X_t - \mu_X)(y_{t+h} - \alpha)\} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} E\left\{(\Gamma(F_t - \mu_F) + u_t) \left((F_{1,t} - \mu_{F,1})' \varphi_1 + \varepsilon_{t+h}\right)\right\} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} \Gamma E\left[(F_t - \mu_F)(F_{1,t} - \mu_{F,1})'\right] \varphi_1 + \frac{1}{T_h} \sum_{t=1}^{T_h} E\left[u_t (F_{1,t} - \mu_{F,1})'\right] \varphi_1 \\
&\quad + \frac{1}{T_h} \sum_{t=1}^{T_h} E\{\Gamma(F_t - \mu_F) \varepsilon_{t+h}\} + \frac{1}{T_h} \sum_{t=1}^{T_h} E[u_t \varepsilon_{t+h}] \\
&= \Gamma \frac{1}{T_h} \sum_{t=1}^{T_h} E\left[(F_t - \mu_F)(F_{1,t} - \mu_{F,1})'\right] \varphi_1 \quad (\text{given Assumptions (ii) and (iii)})
\end{aligned}$$

Next note that

$$\begin{aligned}
& \Gamma \frac{1}{T_h} \sum_{t=1}^{T_h} E \left[(F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \right] \varphi_1 \\
&= \Gamma \frac{1}{T_h} \sum_{t=1}^{T_h} E \left[(F_t - \mu_F) (F_t - \mu_F)' \right] \begin{pmatrix} \varphi_1 \\ 0 \end{pmatrix} \\
&= \Gamma \left(\frac{\underline{\Lambda}}{\underline{\lambda}_{1,1}} \right) \varphi \quad \left(\text{since, in this case, } \varphi = \begin{pmatrix} \varphi_1 \\ 0 \end{pmatrix} \text{ and since, by Assumption (i),} \right. \\
&\quad \left. \bar{\Sigma}_F = \frac{1}{T_h} \sum_{t=1}^{T_h} E \left[(F_t - \mu_F) (F_t - \mu_F)' \right] = \frac{\underline{\Lambda}}{\underline{\lambda}_{1,1}} \right) \\
&= \begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix} \frac{1}{\underline{\lambda}_{1,1}} \begin{pmatrix} \underline{\Lambda}_1 & 0 \\ 0 & \underline{\Lambda}_2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\underline{\lambda}_{1,1}} \begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix} \begin{pmatrix} \underline{\Lambda}_1 \varphi_1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\underline{\lambda}_{1,1}} \Gamma_1 \underline{\Lambda}_1 \varphi_1
\end{aligned}$$

so that, in this case,

$$\bar{\sigma}_{Xy} = \Gamma_1 \frac{\underline{\Lambda}_1}{\underline{\lambda}_{1,1}} \varphi_1 = \sqrt{\underline{\lambda}_{1,1}} \frac{\Gamma_1}{\sqrt{\underline{\lambda}_{1,1}}} \frac{\underline{\Lambda}_1}{\underline{\lambda}_{1,1}} \varphi_1 = \Gamma_{\lambda,1} \frac{\underline{\Lambda}_1}{\sqrt{\underline{\lambda}_{1,1}}} \varphi_1, \text{ given that } \Gamma_{\lambda,1} = \frac{\Gamma_1}{\sqrt{\underline{\lambda}_{1,1}}}$$

Next, we show that, in this case, the envelope parameter vector $\beta = \Gamma_{\lambda,1} (\varphi_1 / \sqrt{\underline{\lambda}_{1,1}})$ has an alternative representation given by

$$\Gamma_{\lambda,1} \left(\frac{1}{\sqrt{\underline{\lambda}_{1,1}}} D_{\varphi,\lambda}^* \right) D_{\varphi,\lambda}^{-1} \underline{w}_{\varphi},$$

where

$$\begin{aligned}
D_{\varphi,\lambda}^* &= \begin{pmatrix} \underline{\lambda}_{1,1} \varphi_{1,1} / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2} \varphi_{1,2} / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa} \varphi_{1,\kappa} / \lambda_1 \end{pmatrix}, \underline{w}_{\varphi} = \begin{pmatrix} \underline{\lambda}_{1,1}^2 \varphi_{1,1}^2 / \lambda_1^2 \\ \underline{\lambda}_{1,2}^2 \varphi_{1,2}^2 / \lambda_1^2 \\ \vdots \\ \underline{\lambda}_{1,\kappa}^2 \varphi_{1,\kappa}^2 / \lambda_1^2 \end{pmatrix}, \text{ and} \\
D_{\varphi,\lambda} &= \begin{pmatrix} \underline{\lambda}_{1,1}^3 \varphi_{1,1}^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \varphi_{1,2}^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa}^3 \varphi_{1,\kappa}^2 / \lambda_1^3 \end{pmatrix}.
\end{aligned}$$

To proceed, note that, by straightforward calculations,

$$\begin{aligned}
& \left(\frac{1}{\underline{\Delta}_{1,1}^{\frac{1}{2}}} D_{\varphi,\lambda}^* \right) D_{\varphi,\lambda}^{-1} w_{\varphi} \\
&= \frac{1}{\underline{\Delta}_{1,1}^{\frac{1}{2}}} \begin{pmatrix} \underline{\Delta}_{1,1} \varphi_{1,1} / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2} \varphi_{1,2} / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,\kappa} \varphi_{1,\kappa} / \lambda_1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} \underline{\Delta}_{1,1}^3 \varphi_{1,1}^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2}^3 \varphi_{1,2}^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,\kappa}^3 \varphi_{1,\kappa}^2 / \lambda_1^3 \end{pmatrix}^{-1} \begin{pmatrix} \underline{\Delta}_{1,1}^2 \varphi_{1,1}^2 / \lambda_1^2 \\ \underline{\Delta}_{1,2}^2 \varphi_{1,2}^2 / \lambda_1^2 \\ \vdots \\ \underline{\Delta}_{1,\kappa}^2 \varphi_{1,\kappa}^2 / \lambda_1^2 \end{pmatrix} \\
&= \frac{1}{\underline{\Delta}_{1,1}^{\frac{1}{2}}} \begin{pmatrix} \underline{\Delta}_{1,1} \varphi_{1,1} / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2} \varphi_{1,2} / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,\kappa} \varphi_{1,\kappa} / \lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1 / \underline{\Delta}_{1,1} \\ \lambda_1 / \underline{\Delta}_{1,2} \\ \vdots \\ \lambda_1 / \underline{\Delta}_{1,\kappa} \end{pmatrix} \\
&= \begin{pmatrix} \varphi_{1,1} / \underline{\Delta}_{1,1}^{\frac{1}{2}} \\ \varphi_{1,2} / \underline{\Delta}_{1,1}^{\frac{1}{2}} \\ \vdots \\ \varphi_{1,\kappa} / \underline{\Delta}_{1,1}^{\frac{1}{2}} \end{pmatrix} \\
&= \varphi_1 / \underline{\Delta}_{1,1}^{\frac{1}{2}}.
\end{aligned}$$

from which it follows immediately that

$$\beta = \Gamma_{\lambda,1} \frac{\varphi_1}{\sqrt{\underline{\Delta}_{1,1}}} = \Gamma_{\lambda,1} \left(\frac{D_{\varphi,\lambda}^*}{\underline{\Delta}_{1,1}^{\frac{1}{2}}} \right) D_{\varphi,\lambda}^{-1} w_{\varphi}. \quad (144)$$

Now, let

$$\underline{V} = \begin{pmatrix} 1 & \underline{\Delta}_{1,1} / \lambda_1 & \underline{\Delta}_{1,1}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,1}^{\kappa-1} / \lambda_1^{\kappa-1} \\ 1 & \underline{\Delta}_{1,2} / \lambda_1 & \underline{\Delta}_{1,2}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^{\kappa-1} / \lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underline{\Delta}_{1,\kappa} / \lambda_1 & \underline{\Delta}_{1,\kappa}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,\kappa}^{\kappa-1} / \lambda_1^{\kappa-1} \end{pmatrix},$$

and write

$$\begin{aligned}
& \sqrt{\lambda_{1,1}} (\bar{\beta} - \beta) \\
&= \sqrt{\lambda_{1,1}} \left[G (G' \bar{\Sigma}_X G)^{-1} G' \bar{\sigma}_{Xy} - \Gamma_{\lambda,1} \left(\frac{\varphi_1}{\sqrt{\lambda_{1,1}}} \right) \right] \\
&= \sqrt{\lambda_{1,1}} \frac{GD_{\lambda,\kappa}^{-1}}{\sqrt{\lambda_1}} \left[\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \Gamma_{\lambda,1} \varphi_1 \\
&= \left\{ \left(\sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} + \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right) \right. \\
&\quad \times \left(\left[\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} + (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right) \\
&\quad \times \left. \left(\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\varphi + \underline{V}' \underline{w}_\varphi \right) \right\} - \Gamma_{\lambda,1} \varphi_1 \quad (\text{where the nonsingularity of } \underline{V}' D_{\varphi,\lambda} \underline{V} \text{ has} \\
&\text{been shown in Lemma E-2 part (b)}) \\
&= \left(\sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right) \left(\left[\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right) \\
&\quad \times \left[\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\varphi \right] \\
&\quad + \left(\sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right) \left(\left[\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right) \underline{V}' \underline{w}_\varphi \\
&\quad + \left(\sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right) (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \left[\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\varphi \right] \\
&\quad + \left(\sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right) (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \underline{V}' \underline{w}_\varphi \\
&\quad + \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \left[\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\varphi \right] \\
&\quad + \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \left(\left[\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right) \left[\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\varphi \right] \\
&\quad + \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \left(\left[\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right) \underline{V}' \underline{w}_\varphi \\
&\quad + \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \underline{V}' \underline{w}_\varphi - \Gamma_{\lambda,1} \varphi_1.
\end{aligned}$$

Next, note that multiplying all three sides of expression (144) by $\sqrt{\underline{\Delta}_{1,1}}$, we see that

$$\begin{aligned}\Gamma_{\lambda,1}\varphi_1 &= \Gamma_{\lambda,1}D_{\varphi,\lambda}^*D_{\varphi,\lambda}^{-1}\underline{w}_\varphi \\ &= \Gamma_{\lambda,1}D_{\varphi,\lambda}^*\underline{V}\underline{V}^{-1}D_{\varphi,\lambda}^{-1}(\underline{V}')^{-1}\underline{V}'\underline{w}_\varphi \\ &= \Gamma_{\lambda,1}D_{\varphi,\lambda}^*\underline{V}(\underline{V}'D_{\varphi,\lambda}\underline{V})^{-1}\underline{V}'\underline{w}_\varphi\end{aligned}$$

from which it follows immediately that the last term on the right-hand side of expression (145) is zero, i.e.,

$$\Gamma_{\lambda,1}D_{\varphi,\lambda}^*\underline{V}(\underline{V}'D_{\varphi,\lambda}\underline{V})^{-1}\underline{V}'\underline{w}_\varphi - \Gamma_{\lambda,1}\varphi_1 = 0$$

Hence, in light of expression (145), we can write

$$\begin{aligned}& \left\| \sqrt{\underline{\Delta}_{1,1}} (\bar{\beta} - \beta) \right\|_2 \\ \leq & \left\{ \left\| \sqrt{\frac{\underline{\Delta}_{1,1}}{\lambda_1}} GD_{\lambda,\kappa}^{-1} - \Gamma_{\lambda,1}D_{\varphi,\lambda}^*\underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_X GD_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}'D_{\varphi,\lambda}\underline{V})^{-1} \right\|_2 \right. \\ & \times \left. \left\| \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1}G'\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}'\underline{w}_\varphi \right\|_2 \right\} \\ & + \left\| \sqrt{\frac{\underline{\Delta}_{1,1}}{\lambda_1}} GD_{\lambda,\kappa}^{-1} - \Gamma_{\lambda,1}D_{\varphi,\lambda}^*\underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_X GD_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}'D_{\varphi,\lambda}\underline{V})^{-1} \right\|_2 \|\underline{V}'\underline{w}_\varphi\|_2 \\ & + \left\| \sqrt{\frac{\underline{\Delta}_{1,1}}{\lambda_1}} GD_{\lambda,\kappa}^{-1} - \Gamma_{\lambda,1}D_{\varphi,\lambda}^*\underline{V} \right\|_2 \left\| (\underline{V}'D_{\varphi,\lambda}\underline{V})^{-1} \right\|_2 \left\| \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1}G'\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}'\underline{w}_\varphi \right\|_2 \\ & + \left\| \sqrt{\frac{\underline{\Delta}_{1,1}}{\lambda_1}} GD_{\lambda,\kappa}^{-1} - \Gamma_{\lambda,1}D_{\varphi,\lambda}^*\underline{V} \right\|_2 \left\| (\underline{V}'D_{\varphi,\lambda}\underline{V})^{-1} \right\|_2 \|\underline{V}'\underline{w}_\varphi\|_2 \\ & + \|\Gamma_{\lambda,1}D_{\varphi,\lambda}^*\underline{V}\|_2 \left\| (\underline{V}'D_{\varphi,\lambda}\underline{V})^{-1} \right\|_2 \left\| \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1}G'\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}'\underline{w}_\varphi \right\|_2 \\ & + \|\Gamma_{\lambda,1}D_{\varphi,\lambda}^*\underline{V}\|_2 \left\| \left[\left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_X GD_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}'D_{\varphi,\lambda}\underline{V})^{-1} \right\|_2 \left\| \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1}G'\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}'\underline{w}_\varphi \right\|_2 \\ & + \|\Gamma_{\lambda,1}D_{\varphi,\lambda}^*\underline{V}\|_2 \left\| \left[\left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1}G'\bar{\Sigma}_X GD_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}'D_{\varphi,\lambda}\underline{V})^{-1} \right\|_2 \|\underline{V}'\underline{w}_\varphi\|_2\end{aligned}\tag{146}$$

We shall analyze each term on the majorant side of expression (146) in turn. Consider initially the first term on the majorant side of expression (146). In this case, we can apply

parts (b), (c), and (d) of Lemma E-4 to obtain

$$\begin{aligned}
& \left\| \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right\|_2 \\
& \times \left\| \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\varphi \right\|_2 \\
& = O\left(\frac{1}{\lambda_1}\right) O\left(\frac{1}{\lambda_1}\right) O\left(\frac{1}{\lambda_1}\right) \\
& = O\left(\frac{1}{\lambda_1^3}\right). \tag{147}
\end{aligned}$$

To analyze the second term on the majorant side of expression (146), we first note that

$$\underline{V}' \underline{w}_\varphi = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{\lambda_{1,1}}{\lambda_1} & \frac{\lambda_{1,2}}{\lambda_1} & \cdots & \frac{\lambda_{1,\kappa}}{\lambda_1} \\ \frac{\lambda_{1,1}^2}{\lambda_1^2} & \frac{\lambda_{1,2}^2}{\lambda_1^2} & \cdots & \frac{\lambda_{1,\kappa}^2}{\lambda_1^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\lambda_{1,1}^{\kappa-1}}{\lambda_1^{\kappa-1}} & \frac{\lambda_{1,2}^{\kappa-1}}{\lambda_1^{\kappa-1}} & \cdots & \frac{\lambda_{1,\kappa}^{\kappa-1}}{\lambda_1^{\kappa-1}} \end{pmatrix} \begin{pmatrix} \frac{\lambda_{1,1}^2 \varphi_{1,1}^2}{\lambda_1^2} \\ \frac{\lambda_{1,2}^2 \varphi_{1,2}^2}{\lambda_1^2} \\ \vdots \\ \frac{\lambda_{1,\kappa}^2 \varphi_{1,\kappa}^2}{\lambda_1^2} \end{pmatrix} = \begin{pmatrix} \sum_{g=1}^{\kappa} \frac{\lambda_{1,g}^2 \varphi_{1,g}^2}{\lambda_1^2} \\ \sum_{g=1}^{\kappa} \frac{\lambda_{1,g}^3 \varphi_{1,g}^2}{\lambda_1^3} \\ \vdots \\ \sum_{g=1}^{\kappa} \frac{\lambda_{1,g}^{\kappa+1} \varphi_{1,g}^2}{\lambda_1^{\kappa+1}} \end{pmatrix}$$

so that

$$\begin{aligned}
& \|V' \underline{w}_\varphi\|_2 \\
&= \sqrt{\sum_{h=1}^{\kappa} \left(\sum_{g=1}^{\kappa} \frac{\lambda_{1,g}^{h+1} \varphi_{1,g}^2}{\lambda_1^{h+1}} \right)^2} \\
&\leq \sqrt{\sum_{h=1}^{\kappa} \kappa \sum_{g=1}^{\kappa} \frac{\lambda_{1,g}^{2(h+1)} \varphi_{1,g}^4}{\lambda_1^{2(h+1)}}} \left(\text{using the inequality } \left| \sum_{i=1}^m a_i \right|^2 \leq m \sum_{i=1}^m |a_i|^2 \right) \\
&\leq \kappa \left(\max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \right)^2 \sqrt{\sum_{h=1}^{\kappa} \left(\frac{\lambda_{1,1}^{2(h+1)}}{\lambda_1^{2(h+1)}} \right)} \quad (\text{given that } \lambda_{1,1} > \dots > \lambda_{1,\kappa}) \\
&\leq \kappa \left(\max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \right)^2 \sum_{h=1}^{\kappa} \left(\frac{\lambda_{1,1}^{(h+1)}}{\lambda_1^{(h+1)}} \right) \quad \left(\text{by the inequality } \left| \sum_{i=1}^m a_i \right|^r \leq \sum_{i=1}^m |a_i|^r \text{ for } 0 < r \leq 1 \right) \\
&\leq \kappa^2 \left(\max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \right)^2 (c^*)^{\kappa+1} \quad (\text{applying Lemma D-3(a) which implies that there exists} \\
&\quad \text{constant } c^* > 1 \text{ such that } 0 < \frac{1}{c^*} \leq \frac{\lambda_{1,1}}{\lambda_1} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large}) \\
&\leq \kappa^2 \bar{c}^2 (c^*)^{\kappa+1} < \infty \quad (\text{by Assumption S1-1}) \tag{148}
\end{aligned}$$

Making use of this result along with parts (c) and (d) of Lemma E-4, we see that

$$\begin{aligned}
& \left\| \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - \Gamma_{\lambda,1} D_{\varphi,\lambda}^* V \right\|_2 \left\| \left[\left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (V' D_{\varphi,\lambda} V)^{-1} \right\|_2 \|V' \underline{w}_\varphi\|_2 \\
&= O\left(\frac{1}{\lambda_1}\right) O\left(\frac{1}{\lambda_1}\right) O(1) \\
&= O\left(\frac{1}{\lambda_1^2}\right). \tag{149}
\end{aligned}$$

Next, to analyze the third term on the majorant side of expression (146), note first that

$$\begin{aligned}
& \left\| (V' D_{\varphi, \lambda} V)^{-1} \right\|_2 \\
& \leq \sqrt{\lambda_{\max} \left\{ \left[(V' D_{\varphi, \lambda} V)^{-1} \right]' (V' D_{\varphi, \lambda} V)^{-1} \right\}} \\
& = \sqrt{\lambda_{\max} \left\{ ([V' D_{\varphi, \lambda} V]')^{-1} (V' D_{\varphi, \lambda} V)^{-1} \right\}} \\
& = \sqrt{\lambda_{\max} \left\{ (V' D_{\varphi, \lambda} V)^{-2} \right\}} \quad (\text{since } V' D_{\varphi, \lambda} V \text{ is symmetric}) \\
& = \sqrt{\frac{1}{(\lambda_{\min} \{V' D_{\varphi, \lambda} V\})^2}} \\
& = \frac{1}{\lambda_{\min} \{V' D_{\varphi, \lambda} V\}} \\
& \leq \frac{1}{C_{\#}} < \infty \quad (\text{for some constant } C_{\#} > 0 \text{ by expression (162)} \\
& \quad \text{given in the proof of part (b) of Lemma E-2})
\end{aligned} \tag{150}$$

Making use of this result along with parts (b) and (c) of Lemma E-4, we obtain

$$\begin{aligned}
& \left\| \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda, \kappa}^{-1} - \Gamma_{\lambda, 1} D_{\varphi, \lambda}^* V \right\|_2 \left\| (V' D_{\varphi, \lambda} V)^{-1} \right\|_2 \left\| \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{xy}}{\sqrt{\lambda_1}} - V' w_{\varphi} \right\|_2 \\
& = O\left(\frac{1}{\lambda_1}\right) O(1) O\left(\frac{1}{\lambda_1}\right) \\
& = O\left(\frac{1}{\lambda_1^2}\right).
\end{aligned} \tag{151}$$

For the fourth term on the majorant side of expression (146), we have, by the results given in expressions (148) and (150) as well as part (c) of Lemma E-4,

$$\begin{aligned}
& \left\| \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda, \kappa}^{-1} - \Gamma_{\lambda, 1} D_{\varphi, \lambda}^* V \right\|_2 \left\| (V' D_{\varphi, \lambda} V)^{-1} \right\|_2 \|V' w_{\varphi}\|_2 \\
& = O\left(\frac{1}{\lambda_1}\right) O(1) O(1) \\
& = O\left(\frac{1}{\lambda_1}\right)
\end{aligned} \tag{152}$$

To analyze the fifth term on the majorant side of expression (146), note first that

$$\begin{aligned}
& D_{\varphi, \lambda}^* \underline{V} \\
&= \begin{pmatrix} \underline{\Delta}_{1,1} \varphi_{1,1} / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2} \varphi_{1,2} / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,\kappa} \varphi_{1,\kappa} / \lambda_1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & \underline{\Delta}_{1,1} / \lambda_1 & \underline{\Delta}_{1,1}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,1}^{\kappa-1} / \lambda_1^{\kappa-1} \\ 1 & \underline{\Delta}_{1,2} / \lambda_1 & \underline{\Delta}_{1,2}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^{\kappa-1} / \lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\Delta}_{1,\kappa} / \lambda_1 & \underline{\Delta}_{1,\kappa}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,\kappa}^{\kappa-1} / \lambda_1^{\kappa-1} \end{pmatrix} \\
&= \begin{pmatrix} \underline{\Delta}_{1,1} \varphi_{1,1} / \lambda_1 & \underline{\Delta}_{1,1}^2 \varphi_{1,1} / \lambda_1^2 & \underline{\Delta}_{1,1}^3 \varphi_{1,1} / \lambda_1^3 & \cdots & \underline{\Delta}_{1,1}^{\kappa} \varphi_{1,1} / \lambda_1^{\kappa} \\ \underline{\Delta}_{1,2} \varphi_{1,2} / \lambda_1 & \underline{\Delta}_{1,2}^2 \varphi_{1,2} / \lambda_1^2 & \underline{\Delta}_{1,2}^3 \varphi_{1,2} / \lambda_1^3 & \cdots & \underline{\Delta}_{1,2}^{\kappa} \varphi_{1,2} / \lambda_1^{\kappa} \\ \vdots & \vdots & \vdots & & \vdots \\ \underline{\Delta}_{1,\kappa} \varphi_{1,\kappa} / \lambda_1 & \underline{\Delta}_{1,\kappa}^2 \varphi_{1,\kappa} / \lambda_1^2 & \underline{\Delta}_{1,\kappa}^3 \varphi_{1,\kappa} / \lambda_1^3 & \cdots & \underline{\Delta}_{1,\kappa}^{\kappa} \varphi_{1,\kappa} / \lambda_1^{\kappa} \end{pmatrix}
\end{aligned}$$

from which it follows, by straightforward calculation, that

$$\text{tr} \left\{ \underline{V}' (D_{\varphi, \lambda}^*)' D_{\varphi, \lambda}^* \underline{V} \right\} = \sum_{h=1}^{\kappa} \sum_{g=1}^{\kappa} \frac{\underline{\Delta}_{1,g}^{2h} \varphi_{1,g}^2}{\lambda_1^{2h}}$$

Making use of this calculation, we see that

$$\begin{aligned}
\|\Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V}\|_2 &= \|\Gamma_{\lambda,1}\|_2 \|D_{\varphi,\lambda}^* \underline{V}\|_2 \\
&\leq \|\Gamma_{\lambda,1}\|_2 \|D_{\varphi,\lambda}^* \underline{V}\|_F \\
&\leq \sqrt{\lambda_{\max}(\Gamma'_{\lambda,1} \Gamma_{\lambda,1})} \sqrt{\text{tr} \left\{ \underline{V}' (D_{\varphi,\lambda}^*)' D_{\varphi,\lambda}^* \underline{V} \right\}} \\
&= \sqrt{\lambda_{\max}(I_\kappa)} \sqrt{\sum_{h=1}^{\kappa} \sum_{g=1}^{\kappa} \frac{\lambda_{1,g}^{2h} \varphi_{1,g}^2}{\lambda_1^{2h}}} \\
&\quad \left(\text{since } \Gamma'_{\lambda,1} \Gamma_{\lambda,1} = \frac{\Gamma'_1}{\sqrt{\lambda_{1,1}}} \frac{\Gamma_1}{\sqrt{\lambda_{1,1}}} = \frac{\Gamma'_1 \Gamma_1}{\lambda_{1,1}} = I_\kappa \right) \\
&\leq \sqrt{\sum_{h=1}^{\kappa} \sum_{g=1}^{\kappa} \frac{\lambda_{1,1}^{2h} \varphi_{1,g}^2}{\lambda_1^{2h}}} \\
&\leq \sqrt{\kappa} \max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \sqrt{\sum_{h=1}^{\kappa} \frac{\lambda_{1,1}^{2h}}{\lambda_1^{2h}}} \\
&\leq \sqrt{\kappa} \max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \sum_{h=1}^{\kappa} \frac{\lambda_{1,1}^h}{\lambda_1^h} \left(\text{applying the inequality } \left| \sum_{i=1}^m a_i \right|^r \leq \sum_{i=1}^m |a_i|^r \text{ for } 0 < r \leq 1 \right) \\
&\leq \kappa^{\frac{3}{2}} \max_{1 \leq g \leq \kappa} |\varphi_{1,g}| (c^*)^\kappa \quad (\text{applying Lemma D-3(a) which implies that there exists constant } c^* \\
&\quad \text{such that } 0 < \frac{1}{c^*} \leq \frac{\lambda_{1,1}}{\lambda_1} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large}) \\
&\leq \kappa^{\frac{3}{2}} \bar{c} (c^*)^\kappa < \infty \quad \left(\text{for some constant } \bar{c} \geq \max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \text{ by Assumption S1-1} \right)
\end{aligned}$$

Using this result along with expression (150) and part (b) of Lemma E-4, we see that

$$\begin{aligned}
&\|\Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V}\|_2 \left\| (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\varphi \right\|_2 \\
&= O(1) O(1) O\left(\frac{1}{\lambda_1}\right) \\
&= O\left(\frac{1}{\lambda_1}\right). \tag{154}
\end{aligned}$$

Turning our attention to the sixth term, note that, by expressions (153) and by the results

given in parts (b) and (d) of Lemma E-4, we have

$$\begin{aligned}
& \left\| \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right\|_2 \\
& \times \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\varphi \right\|_2 \\
& = O(1) O\left(\frac{1}{\lambda_1}\right) O\left(\frac{1}{\lambda_1}\right) \\
& = O\left(\frac{1}{\lambda_1^2}\right)
\end{aligned} \tag{155}$$

Now, for the last term on the majorant side of expression (146), we can apply part (d) of Lemma E-4 and the results given in expressions (148) and (??) to obtain

$$\begin{aligned}
& \left\| \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \underline{V}' \underline{w}_\varphi \right\|_2 \\
& = O(1) O\left(\frac{1}{\lambda_1}\right) O(1) \\
& = O\left(\frac{1}{\lambda_1}\right).
\end{aligned} \tag{156}$$

Combining the results given in expressions (147), (149), (151)-(152), and (154)-(156); we obtain

$$\begin{aligned}
& \left\| \sqrt{\underline{\lambda}_{1,1}} (\bar{\beta} - \beta) \right\|_2 \\
& = O\left(\frac{1}{\lambda_1^3}\right) + O\left(\frac{1}{\lambda_1^2}\right) + O\left(\frac{1}{\lambda_1^2}\right) + O\left(\frac{1}{\lambda_1}\right) + O\left(\frac{1}{\lambda_1}\right) + O\left(\frac{1}{\lambda_1^2}\right) + O\left(\frac{1}{\lambda_1}\right) \\
& = O\left(\frac{1}{\lambda_1}\right),
\end{aligned}$$

which, in turn, implies that

$$\left\| \bar{\beta} - \beta \right\|_2 = O\left(\frac{1}{\lambda_1 \sqrt{\underline{\lambda}_{1,1}}}\right) = O\left(\frac{1}{\lambda_1^{\frac{3}{2}}}\right),$$

since, by part (a) of Lemma D-3, there exists an absolute constant $c^* > 1$ such that

$$0 < \frac{1}{c^*} < \frac{\lambda_1}{\underline{\lambda}_{1,1}} \leq c^* < \infty \text{ for all } N \text{ and } T \text{ sufficiently large.}$$

To show part (b), note first that because in this case all K factors are relevant for the purpose of forecasting, there is no reason to split the nonzero eigenvalues of the common component covariance matrix $\Gamma \bar{\Sigma}_F \Gamma'$ and their associated eigenvectors into two groups. However, in this case, we will still write the nonzero eigenvalues with a double subscript (i.e., as $\underline{\lambda}_{1,1}, \dots, \underline{\lambda}_{1,K}$), and we shall interpret having the first subscript being 1 for all K of these eigenvalues as meaning that these are now all eigenvalues that are associated with eigenvectors which form a basis set for the envelope so that all the associated eigenvectors in this case contain useful information about the target variable of interest y_{t+h} . Hence, following similar derivation as given in part (a), we can write $\bar{\Sigma}_X$ in the form

$$\bar{\Sigma}_X = \frac{1}{T_h} \sum_{t=1}^{T_h} E [(X_t - \mu_X) (X_t - \mu_X)'] = \Gamma_\lambda \underline{\Lambda} \Gamma_\lambda' + \bar{\Sigma}_u$$

where $\underline{\Lambda} = \text{diag}(\underline{\lambda}_{1,1}, \dots, \underline{\lambda}_{1,K})$ and where $\Gamma_\lambda = \Gamma / \sqrt{\underline{\lambda}_{1,1}}$ is an $N \times K$ semi-orthogonal matrix whose columns are those orthonormal eigenvectors of $\Gamma \bar{\Sigma}_F \Gamma'$ associated with the nonzero eigenvalues $\underline{\lambda}_{1,1}, \dots, \underline{\lambda}_{1,K}$. Moreover, in this $\kappa = K$ case, we can also write $\bar{\sigma}_{Xy}$ in the form

$$\begin{aligned} \bar{\sigma}_{Xy} &= \frac{1}{T_h} \sum_{t=1}^{T_h} E \{(X_t - \mu_X) (y_{t+h} - \alpha)\} \\ &= \Gamma \frac{1}{T_h} \sum_{t=1}^{T_h} E [(F_t - \mu_F) (F_t - \mu_F)'] \varphi \\ &= \frac{1}{\underline{\lambda}_{1,1}} \Gamma \underline{\Lambda} \varphi \\ &= \Gamma_\lambda \underline{\Lambda} \frac{\varphi}{\sqrt{\underline{\lambda}_{1,1}}} \left(\text{setting } \Gamma_\lambda = \frac{\Gamma}{\sqrt{\underline{\lambda}_{1,1}}} \right) \end{aligned}$$

where $\varphi = (\varphi_{1,1}, \dots, \varphi_{1,K})'$. Here, note that none of the coefficients $\varphi_{1,j}$ (with $j = 1, \dots, K$) is zero and there is no vector of zero coefficients φ_2 in this case, but we still retain the double subscript notation on $\varphi_{1,1}, \dots, \varphi_{1,K}$ because again we interpret the first subscript being 1 as meaning that this is a coefficient associated with a relevant factor so it is not zero. Moreover, all columns of $\Gamma_\lambda = \Gamma / \sqrt{\underline{\lambda}_{1,1}}$ are now needed to form the basis set that spans the envelope since, in this case, they are all orthonormal eigenvectors that contain useful information about the target variable y_{t+h} . The envelope coefficient vector β in this case is, thus, given by the formula

$$\beta = \frac{\Gamma}{\sqrt{\underline{\lambda}_{1,1}}} \left(\frac{\varphi}{\sqrt{\underline{\lambda}_{1,1}}} \right) = \Gamma_\lambda \left(\frac{\varphi}{\sqrt{\underline{\lambda}_{1,1}}} \right).$$

Next, following similar calculations as that given in part (a), we can show that β also has a

representation given by

$$\Gamma_\lambda \left(\frac{1}{\sqrt{\Delta_{1,1}}} D_{\varphi,\lambda}^* \right) D_{\varphi,\lambda}^{-1} \underline{w}_\varphi,$$

where, in this case, $\kappa = K$ and we pre-multiply by Γ_λ instead of $\Gamma_{\lambda,1}$ since $\Gamma_\lambda = \Gamma_{\lambda,1}$ and there is no $\Gamma_{\lambda,2}$ in the $\kappa = K$ case. Moreover, in this case,

$$D_{\varphi,\lambda}^* = \begin{pmatrix} \Delta_{1,1}\varphi_{1,1}/\lambda_1 & 0 & \cdots & 0 \\ 0 & \Delta_{1,2}\varphi_{1,2}/\lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Delta_{1,K}\varphi_{1,K}/\lambda_1 \end{pmatrix}, \underline{w}_\varphi = \begin{pmatrix} \Delta_{1,1}^2\varphi_{1,1}^2/\lambda_1^2 \\ \Delta_{1,2}^2\varphi_{1,2}^2/\lambda_1^2 \\ \vdots \\ \Delta_{1,K}^2\varphi_{1,K}^2/\lambda_1^2 \end{pmatrix},$$

$$D_{\varphi,\lambda} = \begin{pmatrix} \Delta_{1,1}^3\varphi_{1,1}^2/\lambda_1^3 & 0 & \cdots & 0 \\ 0 & \Delta_{1,2}^3\varphi_{1,2}^2/\lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Delta_{1,K}^3\varphi_{1,K}^2/\lambda_1^3 \end{pmatrix}.$$

By making use of these minor modifications in notations, we can then follow essentially the

same argument as that given in the proof of part (a) to obtain

$$\begin{aligned}
& \left\| \sqrt{\underline{\Delta}_{1,1}} (\bar{\beta} - \beta) \right\|_2 \\
\leq & \left\{ \left\| \sqrt{\frac{\underline{\Delta}_{1,1}}{\lambda_1}} G D_{\lambda,K}^{-1} - \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,K}^{-1} G' \bar{\Sigma}_X G D_{\lambda,K}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right\|_2 \right. \\
& \times \left. \left\| \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,K}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\varphi \right\|_2 \right\} \\
& + \left\| \sqrt{\frac{\underline{\Delta}_{1,1}}{\lambda_1}} G D_{\lambda,K}^{-1} - \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,K}^{-1} G' \bar{\Sigma}_X G D_{\lambda,K}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \underline{V}' \underline{w}_\varphi \right\|_2 \\
& + \left\| \sqrt{\frac{\underline{\Delta}_{1,1}}{\lambda_1}} G D_{\lambda,K}^{-1} - \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right\|_2 \left\| (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,K}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\varphi \right\|_2 \\
& + \left\| \sqrt{\frac{\underline{\Delta}_{1,1}}{\lambda_1}} G D_{\lambda,K}^{-1} - \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right\|_2 \left\| (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \underline{V}' \underline{w}_\varphi \right\|_2 \\
& + \left\| \Gamma_\lambda D_{\varphi,\lambda}^* \underline{V} \right\|_2 \left\| (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,K}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\varphi \right\|_2 \\
& + \left\| \Gamma_\lambda D_{\varphi,\lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,K}^{-1} G' \bar{\Sigma}_X G D_{\lambda,K}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,K}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\varphi \right\|_2 \\
& + \left\| \Gamma_\lambda D_{\varphi,\lambda}^* \underline{V} \right\|_2 \left\| \left[\left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,K}^{-1} G' \bar{\Sigma}_X G D_{\lambda,K}^{-1}}{\lambda_1} \right]^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right\|_2 \left\| \underline{V}' \underline{w}_\varphi \right\|_2 \tag{157}
\end{aligned}$$

Finally, to derive the order of magnitude of the terms on the right-hand side of expression (157), we again make use of the same argument as that given in the proof of part (a), except that we apply part (b) of Lemma E-3 instead of part (b) of Lemma E-2 and apply parts (b), (c), and (d) of Lemma E-5 instead of part (b), (c), and (d) of Lemma E-4 while setting $\kappa = K$ and $\Gamma_{\lambda,1} = \Gamma_\lambda$. By making these minor modifications in the proof of part (a) above, we can show that

$$\begin{aligned}
\left\| \sqrt{\underline{\Delta}_{1,1}} (\bar{\beta} - \beta) \right\|_2 &= \sqrt{\underline{\Delta}_{1,1}} \left[G (G' \bar{\Sigma}_X G)^{-1} G' \bar{\sigma}_{Xy} - \Gamma_\lambda \left(\frac{\varphi}{\sqrt{\underline{\Delta}_{1,1}}} \right) \right] \\
&= O \left(\frac{1}{\lambda_1} \right)
\end{aligned}$$

from which it follows that

$$\|\bar{\beta} - \beta\|_2 = O\left(\frac{1}{\lambda_1 \sqrt{\Delta_{1,1}}}\right) = O\left(\frac{1}{\lambda_1^{\frac{3}{2}}}\right). \quad \square$$

Lemma E-2: Suppose that Assumptions (i)-(iii) hold. In addition, suppose that Assumptions S1-1 and S1-2 hold under Scenario 1. Under these conditions, the following statements are true.

(a)

$$\begin{aligned} & \left\| \left(\frac{\Delta_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W [\Gamma_{\lambda,1} \underline{\Delta}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Delta}_2 \Gamma'_{\lambda,2}] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right\|_F \\ &= O\left(\frac{1}{\lambda_1}\right), \end{aligned}$$

where

$$\begin{aligned} G_W &= \begin{pmatrix} \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \sqrt{\Delta_{1,1}} & \bar{\Sigma}_W \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \sqrt{\Delta_{1,1}} & \cdots & \bar{\Sigma}_W^{\kappa-1} \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \sqrt{\Delta_{1,1}} \end{pmatrix}, \\ D_{\lambda,\kappa} &= \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{\kappa-\frac{1}{2}} \end{pmatrix}, \quad G = \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{pmatrix}, \\ \bar{\Sigma}_X &= \bar{\Sigma}_W + \bar{\Sigma}_u = \Gamma_{\lambda,1} \underline{\Delta}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Delta}_2 \Gamma'_{\lambda,2} + \bar{\Sigma}_u, \quad \Gamma_{\lambda,1} = \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}}, \text{ and } \Gamma_{\lambda,2} = \frac{\Gamma_2}{\sqrt{\Delta_{1,1}}}. \end{aligned}$$

(b) Let $D_{\lambda,\kappa}$ and G_W be as defined in part (a) above under Scenario 1 and let

$$\begin{aligned} D_{\varphi,\lambda} &= \begin{pmatrix} \Delta_{1,1}^3 \varphi_{1,1}^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \Delta_{1,2}^3 \varphi_{1,2}^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Delta_{1,\kappa}^3 \varphi_{1,\kappa}^2 / \lambda_1^3 \end{pmatrix} \text{ and} \\ \underline{V} &= \begin{pmatrix} 1 & \Delta_{1,1} / \lambda_1 & \Delta_{1,1}^2 / \lambda_1^2 & \cdots & \Delta_{1,1}^{\kappa-1} / \lambda_1^{\kappa-1} \\ 1 & \Delta_{1,2} / \lambda_1 & \Delta_{1,2}^2 / \lambda_1^2 & \cdots & \Delta_{1,2}^{\kappa-1} / \lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \Delta_{1,\kappa} / \lambda_1 & \Delta_{1,\kappa}^2 / \lambda_1^2 & \cdots & \Delta_{1,\kappa}^{\kappa-1} / \lambda_1^{\kappa-1} \end{pmatrix} \end{aligned}$$

Then,

$$\left\| \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \underline{V}' D_{\varphi,\lambda} \underline{V} \right\|_2 = O\left(\frac{1}{\lambda_1}\right) = o(1) .$$

and there exists a constant $C_{\#} > 0$ such that

$$\lambda_{\min}(\underline{V}' D_{\varphi,\lambda} \underline{V}) \geq C_{\#} > 0 \text{ and } \lambda_{\min}\left(\left[\frac{\lambda_{1,1}}{\lambda_1}\right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1}\right) \geq C_{\#} > 0$$

or all N and T sufficiently large. Moreover, there exists a constant $C_* > 0$ such that

$$\lambda_{\min}\left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1}\right) \geq C_* > 0$$

for all N and T sufficiently large.

Proof of Lemma E-2: To show the result for part (a), note first that, by straightforward calculation

$$\begin{aligned} & \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \\ &= \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \left\{ \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_{Xy} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \end{pmatrix} \frac{\bar{\Sigma}_X}{\lambda_1} \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{pmatrix} \right. \\ & \quad \times \left. \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \right\} \\ &= \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \begin{pmatrix} \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} / \lambda_1^2 & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} / \lambda_1^3 & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa} \bar{\sigma}_{Xy} / \lambda_1^{\kappa+1} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} / \lambda_1^3 & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^3 \bar{\sigma}_{Xy} / \lambda_1^4 & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa+1} \bar{\sigma}_{Xy} / \lambda_1^{\kappa+2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa} \bar{\sigma}_{Xy} / \lambda_1^{\kappa+1} & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa+1} \bar{\sigma}_{Xy} / \lambda_1^{\kappa+2} & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{2\kappa-1} \bar{\sigma}_{Xy} / \lambda_1^{2\kappa} \end{pmatrix} \\ &= \begin{pmatrix} \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \lambda_1^3 & \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X^2 \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \lambda_1^4 & \cdots & \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X^{\kappa} \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \lambda_1^{\kappa+2} \\ \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X^2 \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \lambda_1^4 & \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X^3 \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \lambda_1^5 & \cdots & \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X^{\kappa+1} \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \lambda_1^{\kappa+3} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X^{\kappa} \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \lambda_1^{\kappa+2} & \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X^{\kappa+1} \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \lambda_1^{\kappa+3} & \cdots & \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X^{2\kappa-1} \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \lambda_1^{2\kappa+1} \end{pmatrix} \end{aligned}$$

given that $\bar{\sigma}_{Xy} = \Gamma_{\lambda,1}\underline{\Delta}_1\varphi_1/\sqrt{\underline{\Delta}_{1,1}}$. Moreover, in this case, let

$$\bar{\Sigma}_W = \Gamma_{\lambda,1}\underline{\Delta}_1\Gamma'_{\lambda,1} + \Gamma_{\lambda,2}\underline{\Delta}_2\Gamma'_{\lambda,2}$$

and

$$G_W = \begin{pmatrix} \Gamma_{\lambda,1}\underline{\Delta}_1\varphi_1/\sqrt{\underline{\Delta}_{1,1}} & \bar{\Sigma}_W\Gamma_{\lambda,1}\underline{\Delta}_1\varphi_1/\sqrt{\underline{\Delta}_{1,1}} & \cdots & \bar{\Sigma}_W^{\kappa-1}\Gamma_{\lambda,1}\underline{\Delta}_1\varphi_1/\sqrt{\underline{\Delta}_{1,1}} \end{pmatrix}$$

and note that

$$\begin{aligned} & \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1}\right) \frac{D_{\lambda,\kappa}^{-1}G'_W\bar{\Sigma}_WG_WD_{\lambda,\kappa}^{-1}}{\lambda_1} \\ &= \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1}\right) \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \underline{\Delta}_{1,1}^{-\frac{1}{2}}\varphi'_1\underline{\Delta}_1\Gamma'_{\lambda,1} \\ \underline{\Delta}_{1,1}^{-\frac{1}{2}}\varphi'_1\underline{\Delta}_1\Gamma'_{\lambda,1}\bar{\Sigma}_W \\ \vdots \\ \underline{\Delta}_{1,1}^{-\frac{1}{2}}\varphi'_1\underline{\Delta}_1\Gamma'_{\lambda,1}\bar{\Sigma}_W^{\kappa-1} \end{pmatrix} \frac{\bar{\Sigma}_W}{\lambda_1} \\ & \times \begin{pmatrix} \Gamma_{\lambda,1}\underline{\Delta}_1\varphi_1/\sqrt{\underline{\Delta}_{1,1}} & \bar{\Sigma}_W\Gamma_{\lambda,1}\underline{\Delta}_1\varphi_1/\sqrt{\underline{\Delta}_{1,1}} & \cdots & \bar{\Sigma}_W^{\kappa-1}\Gamma_{\lambda,1}\underline{\Delta}_1\varphi_1/\sqrt{\underline{\Delta}_{1,1}} \end{pmatrix} \\ & \times \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \\ &= \begin{pmatrix} \varphi'_1\underline{\Delta}_1\Gamma'_{\lambda,1}\bar{\Sigma}_W\Gamma_{\lambda,1}\underline{\Delta}_1\varphi_1/\lambda_1^3 & \cdots & \varphi'_1\underline{\Delta}_1\Gamma'_{\lambda,1}\bar{\Sigma}_W^\kappa\Gamma_{\lambda,1}\underline{\Delta}_1\varphi_1/\lambda_1^{\kappa+2} \\ \vdots & \ddots & \vdots \\ \varphi'_1\underline{\Delta}_1\Gamma'_{\lambda,1}\bar{\Sigma}_W^\kappa\Gamma_{\lambda,1}\underline{\Delta}_1\varphi_1/\lambda_1^{\kappa+2} & \cdots & \varphi'_1\underline{\Delta}_1\Gamma'_{\lambda,1}\bar{\Sigma}_W^{2\kappa-1}\Gamma_{\lambda,1}\underline{\Delta}_1\varphi_1/\lambda_1^{2\kappa+1} \end{pmatrix} \quad (158) \end{aligned}$$

It follows from these calculations that

$$\begin{aligned}
& \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W \bar{\Sigma}_W G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \\
&= \begin{pmatrix} \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^3 & \cdots & \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X^\kappa \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^{\kappa+2} \\ \vdots & \ddots & \vdots \\ \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X^\kappa \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^{\kappa+2} & \cdots & \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X^{2\kappa-1} \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^{2\kappa+1} \end{pmatrix} \\
&\quad - \begin{pmatrix} \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_W \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^3 & \cdots & \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_W^\kappa \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^{\kappa+2} \\ \vdots & \ddots & \vdots \\ \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_W^\kappa \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^{\kappa+2} & \cdots & \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_W^{2\kappa-1} \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^{2\kappa+1} \end{pmatrix} \\
&= \begin{pmatrix} \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} [\bar{\Sigma}_X - \bar{\Sigma}_W] \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^3 & \cdots & \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \Sigma [\bar{\Sigma}_X^\kappa - \bar{\Sigma}_W^\kappa] \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^{\kappa+2} \\ \vdots & \ddots & \vdots \\ \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} [\bar{\Sigma}_X^\kappa - \bar{\Sigma}_W^\kappa] \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^{\kappa+2} & \cdots & \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} [\bar{\Sigma}_X^{2\kappa-1} - \bar{\Sigma}_W^{2\kappa-1}] \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^{2\kappa+1} \end{pmatrix}
\end{aligned}$$

from which we further deduce that

$$\begin{aligned}
& \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W \bar{\Sigma}_W G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right\|_F^2 \\
&= \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right\|_F^2 \\
&= \sum_{j=1}^{\kappa} \left(\frac{\varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \left(\bar{\Sigma}_X^j - [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}]^j \right) \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\lambda_1^{j+2}} \right)^2 \\
&\quad + \sum_{j=1}^{\kappa} \left(\frac{\varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \left(\bar{\Sigma}_X^{j+1} - [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}]^{j+1} \right) \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\lambda_1^{j+3}} \right)^2 \\
&\quad + \cdots + \sum_{j=1}^{\kappa} \left(\frac{\varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \left(\bar{\Sigma}_X^{j+\kappa-1} - [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}]^{j+\kappa-1} \right) \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\lambda_1^{j+\kappa+1}} \right)^2 \\
&= \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \left(\frac{\varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \left(\bar{\Sigma}_X^{i+j-1} - [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}]^{i+j-1} \right) \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\lambda_1^{i+j+1}} \right)^2 \quad (159)
\end{aligned}$$

Next, note that, for any fixed integer $1 \leq k \leq 2\kappa - 1$, we have

$$\begin{aligned}
& \left| \frac{\varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \left(\bar{\Sigma}_X^k - [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}]^k \right) \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\lambda_1^{k+2}} \right| \\
&= \left| \frac{\varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \left(\bar{\Sigma}_X^k - \bar{\Sigma}_W^k \right) \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\lambda_1^{k+2}} \right| \\
&= \left| \frac{\varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \left([\bar{\Sigma}_W + \bar{\Sigma}_X - \bar{\Sigma}_W]^k - \bar{\Sigma}_W^k \right) \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\lambda_1^{k+2}} \right| \\
&\leq \frac{1}{\lambda_1^{k+2}} \sum_{\ell=1}^k \binom{k}{\ell} \|\bar{\Sigma}_X - \bar{\Sigma}_W\|_2^\ell \|\bar{\Sigma}_W\|_2^{k-\ell} \|\Gamma_{\lambda,1}\|_2^2 \|\underline{\Lambda}_1\|_2^2 \|\varphi_1\|_2^2 \\
&= \sum_{\ell=1}^k \binom{k}{\ell} \left\| \frac{\bar{\Sigma}_X - \bar{\Sigma}_W}{\lambda_1} \right\|_2^\ell \frac{\|\bar{\Sigma}_W\|_2^{k-\ell}}{\lambda_1^{k-\ell}} \frac{\|\underline{\Lambda}_1\|_2^2}{\lambda_1^2} \|\varphi_1\|_2^2 \quad (\text{given that } \|\Gamma_{\lambda,1}\|_2^2 = \Gamma'_{\lambda,1} \Gamma_{\lambda,1} = I_\kappa) \\
&= \sum_{\ell=1}^k \binom{k}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \frac{\|\bar{\Sigma}_W\|_2^{k-\ell}}{\lambda_{1,1}^{k-\ell}} \left(\frac{\lambda_{1,1}}{\lambda_1} \right)^{k-\ell+2} \|\varphi_1\|_2^2 \quad (\text{since } \bar{\Sigma}_u = \bar{\Sigma}_X - \bar{\Sigma}_W \text{ and since} \\
&\quad \|\underline{\Lambda}_1\|_2^2 = \lambda_{\max}(\underline{\Lambda}_1^2) = \lambda_{1,1}^2) \\
&\leq (c^*)^{2\kappa} \sum_{\ell=1}^k \binom{k}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \frac{\|\bar{\Sigma}_W\|_2^{k-\ell}}{\lambda_{1,1}^{k-\ell}} \|\varphi_1\|_2^2 \\
&\quad \left(\text{Lemma D-3(a) implies that there exists } c^* > 1 \text{ such that } 0 < \frac{1}{c^*} \leq \frac{\lambda_{1,1}}{\lambda_1} \leq c^* < \infty \right. \\
&\quad \left. \text{for all } N, T \text{ sufficiently large} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{\ell=1}^k \binom{k}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \|\varphi_1\|_2^2 \\
&\quad \left(\text{given that } \|\bar{\Sigma}_W\|_2 = \sqrt{\left(\lambda_{\max} \left\{ \bar{\Sigma}_W^2 \right\} \right)} = \lambda_{\max} \left\{ \bar{\Sigma}_W \right\} = \max \left\{ \underline{\lambda}_{1,1}, \underline{\lambda}_{2,1} \right\} \leq C \underline{\lambda}_{1,1} \right. \\
&\quad \left. \text{for some constant } C > 1 \text{ and for all } N, T \text{ sufficiently large since } \underline{\lambda}_{1,1} \asymp \underline{\lambda}_{2,1} \right) \\
&\leq C^{2\kappa} \kappa(\bar{c})^2 \sum_{\ell=1}^k \binom{k}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \\
&\quad \left(\text{by Assumption S1-1, } \kappa(\bar{c})^2 \geq \kappa \left(\max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \right)^2 \geq \|\varphi_1\|_2^2 \right) \\
&= (c^*)^{2\kappa} C^{2\kappa} \kappa(\bar{c})^2 \sum_{\ell=1}^k \binom{k}{\ell} \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right)^\ell \quad (\text{by symmetry and positive semidefiniteness of } \bar{\Sigma}_u, \\
&\quad \text{so that } \|\bar{\Sigma}_u\|_2 = \sqrt{\lambda_{\max}(\bar{\Sigma}_u' \bar{\Sigma}_u)} = \sqrt{[\lambda_{\max}(\bar{\Sigma}_u)]^2} = \lambda_{\max}(\bar{\Sigma}_u)) \tag{160}
\end{aligned}$$

Setting $k = i + j - 1$ in expression (160) and applying it to the analysis of expression (159),

we have

$$\begin{aligned}
& \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W \bar{\Sigma}_W G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right\|_F^2 \\
&= \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \left(\frac{\varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \left(\bar{\Sigma}_X^{i+j-1} - [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}]^{i+j-1} \right) \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\lambda_1^{i+j+1}} \right)^2 \\
&\leq \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \left[(c^*)^{2\kappa} C^{2\kappa} \kappa (\bar{c})^2 \sum_{\ell=1}^{i+j-1} \binom{i+j-1}{\ell} \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right)^\ell \right]^2 \\
&\leq (c^*)^{4\kappa} C^{4\kappa} \kappa^2 (\bar{c})^4 (2\kappa - 1) \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \sum_{\ell=1}^{i+j-1} \binom{i+j-1}{\ell}^2 \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right)^{2\ell} \\
&\quad \left(\text{using the inequality } \left| \sum_{i=1}^m a_i \right|^2 \leq m \sum_{i=1}^m |a_i|^2 \right) \\
&\leq (c^*)^{4\kappa} C^{4\kappa} \kappa^2 (\bar{c})^4 (2\kappa - 1) \sum_{i=1}^{\kappa} \sum_{j=1}^{\kappa} \sum_{\ell=1}^{i+j-1} \binom{i+j-1}{\ell}^2 \left(\frac{\bar{C}}{\lambda_1} \right)^{2\ell} \text{ for all } N, T \text{ sufficiently large} \\
&\quad \left(\text{given that, by Assumption (ii) and Lemma D-2, there exists some constant } \bar{C} > 0 \right. \\
&\quad \left. \text{such that } \lambda_{\max}(\bar{\Sigma}_u) \leq \bar{C} < \infty \text{ and given that } \underline{\lambda}_{1,1} \rightarrow \infty \text{ as } N, T \rightarrow \infty \right) \\
&\leq (c^*)^{4\kappa} C^{4\kappa} \kappa^4 (\bar{c})^4 [(2\kappa - 1)!]^2 \left(\frac{\bar{C}}{\lambda_1} \right)^2 \left(\text{since } \left(\frac{\bar{C}}{\lambda_1} \right)^{2\ell} \ll \left(\frac{\bar{C}}{\lambda_1} \right)^2 \right. \\
&\quad \left. \text{for } \ell > 1 \text{ given that } \lambda_1 \rightarrow \infty \text{ as } N, T \rightarrow \infty \right) \\
&\leq C^* \left(\frac{1}{\lambda_1} \right)^2 \\
&\quad \left(\text{for some constant } C^* > 0 \text{ such that } C^* \geq (c^*)^{4\kappa} C^{4\kappa} \kappa^4 (\bar{c})^4 (2\kappa - 1)^2 [(2\kappa - 1)!]^2 \bar{C}^2 \right) \\
&= O \left(\frac{1}{\lambda_1^2} \right),
\end{aligned}$$

from which it follows that

$$\left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right\|_F = O \left(\frac{1}{\lambda_1} \right)$$

To show part (b), note first that, for fixed positive integer k ,

$$\begin{aligned}
\frac{\varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_W^k \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1}{\lambda_1^{k+2}} &= \frac{\varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} [\Gamma_{\lambda,1} \underline{\Delta}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Delta}_2 \Gamma'_{\lambda,2}]^k \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1}{\lambda_1^{k+2}} \\
&= \frac{\varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} [\Gamma_{\lambda,1} \underline{\Delta}_1^k \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Delta}_2^k \Gamma'_{\lambda,2}] \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1}{\lambda_1^{k+2}} \\
&\quad (\text{given that } \Gamma'_{\lambda,1} \Gamma_{\lambda,1} = I_\kappa, \Gamma'_{\lambda,2} \Gamma_{\lambda,2} = I_{K-\kappa}, \text{ and } \Gamma'_{\lambda,2} \Gamma_{\lambda,1} = 0) \\
&= \frac{\varphi'_1 \underline{\Delta}_1^{k+2} \varphi_1}{\lambda_1^{k+2}} \\
&= \frac{1}{\lambda_1^{k+2}} \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^{k+2} \varphi_{1,g}^2
\end{aligned}$$

Making use of the above identity and of expression (158) in the proof of part (a) this lemma, we further obtain

$$\begin{aligned}
&\left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G'_W \bar{\Sigma}_W G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
&= \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G'_W [\Gamma_{\lambda,1} \underline{\Delta}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Delta}_2 \Gamma'_{\lambda,2}] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \\
&= \begin{pmatrix} \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_W \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \lambda_1^3 & \cdots & \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_W^\kappa \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \lambda_1^{\kappa+2} \\ \vdots & \ddots & \vdots \\ \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_W^\kappa \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \lambda_1^{\kappa+2} & \cdots & \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_W^{2\kappa-1} \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \lambda_1^{2\kappa+1} \end{pmatrix} \\
&= \begin{pmatrix} \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^3 \varphi_{1,g}^2 / \lambda_1^3 & \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^4 \varphi_{1,g}^2 / \lambda_1^4 & \cdots & \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^{\kappa+2} \varphi_{1,g}^2 / \lambda_1^{\kappa+2} \\ \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^4 \varphi_{1,g}^2 / \lambda_1^4 & \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^5 \varphi_{1,g}^2 / \lambda_1^5 & \cdots & \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^{\kappa+3} \varphi_{1,g}^2 / \lambda_1^{\kappa+3} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^{\kappa+2} \varphi_{1,g}^2 / \lambda_1^{\kappa+2} & \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^{\kappa+3} \varphi_{1,g}^2 / \lambda_1^{\kappa+3} & \cdots & \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^{2\kappa+1} \varphi_{1,g}^2 / \lambda_1^{2\kappa+1} \end{pmatrix}
\end{aligned}$$

Now,

$$\begin{aligned}
& \begin{pmatrix} \sum_{g=1}^{\kappa} \Delta_{1,g}^3 \varphi_{1,g}^2 / \lambda_1^3 & \sum_{g=1}^{\kappa} \Delta_{1,g}^4 \varphi_{1,g}^2 / \lambda_1^4 & \cdots & \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+2} \varphi_{1,g}^2 / \lambda_1^{\kappa+2} \\ \sum_{g=1}^{\kappa} \Delta_{1,g}^4 \varphi_{1,g}^2 / \lambda_1^4 & \sum_{g=1}^{\kappa} \Delta_{1,g}^5 \varphi_{1,g}^2 / \lambda_1^5 & \cdots & \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+3} \varphi_{1,g}^2 / \lambda_1^{\kappa+3} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+2} \varphi_{1,g}^2 / \lambda_1^{\kappa+2} & \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+3} \varphi_{1,g}^2 / \lambda_1^{\kappa+3} & \cdots & \sum_{g=1}^{\kappa} \Delta_{1,g}^{2\kappa+1} \varphi_{1,g}^2 / \lambda_1^{2\kappa+1} \end{pmatrix} \\
&= \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \right. \\
&\quad \times \begin{pmatrix} \sum_{g=1}^{\kappa} \Delta_{1,g}^3 \varphi_{1,g}^2 / \lambda_1^3 & \sum_{g=1}^{\kappa} \Delta_{1,g}^4 \varphi_{1,g}^2 / \lambda_1^3 & \cdots & \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+2} \varphi_{1,g}^2 / \lambda_1^3 \\ \sum_{g=1}^{\kappa} \Delta_{1,g}^4 \varphi_{1,g}^2 / \lambda_1^3 & \sum_{g=1}^{\kappa} \Delta_{1,g}^5 \varphi_{1,g}^2 / \lambda_1^3 & \cdots & \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+3} \varphi_{1,g}^2 / \lambda_1^3 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+2} \varphi_{1,g}^2 / \lambda_1^3 & \sum_{g=1}^{\kappa} \Delta_{1,g}^{\kappa+3} \varphi_{1,g}^2 / \lambda_1^3 & \cdots & \sum_{g=1}^{\kappa} \Delta_{1,g}^{2\kappa+1} \varphi_{1,g}^2 / \lambda_1^3 \end{pmatrix} \\
&\quad \times \left. \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \begin{pmatrix} \Delta_{1,1}^3 \varphi_{1,1}^2 / \lambda_1^3 & \Delta_{1,2}^3 \varphi_{1,2}^2 / \lambda_1^3 & \cdots & \Delta_{1,\kappa}^3 \varphi_{1,\kappa}^2 / \lambda_1^3 \\ \Delta_{1,1}^4 \varphi_{1,1}^2 / \lambda_1^3 & \Delta_{1,2}^4 \varphi_{1,2}^2 / \lambda_1^3 & \cdots & \Delta_{1,\kappa}^4 \varphi_{1,\kappa}^2 / \lambda_1^3 \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{1,1}^{\kappa+2} \varphi_{1,1}^2 / \lambda_1^3 & \Delta_{1,2}^{\kappa+2} \varphi_{1,2}^2 / \lambda_1^3 & \cdots & \Delta_{1,\kappa}^{\kappa+2} \varphi_{1,\kappa}^2 / \lambda_1^3 \end{pmatrix} \right. \\
&\quad \times \left. \begin{pmatrix} 1 & \Delta_{1,1} & \Delta_{1,1}^2 & \cdots & \Delta_{1,1}^{\kappa-1} \\ 1 & \Delta_{1,2} & \Delta_{1,2}^2 & \cdots & \Delta_{1,2}^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \Delta_{1,\kappa} & \Delta_{1,\kappa}^2 & \cdots & \Delta_{1,\kappa}^{\kappa-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \underline{\lambda}_{1,1} & \underline{\lambda}_{1,2} & \cdots & \underline{\lambda}_{1,\kappa} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\lambda}_{1,1}^{\kappa-1} & \underline{\lambda}_{1,2}^{\kappa-1} & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} \end{pmatrix} \right. \\
&\quad \times \begin{pmatrix} \underline{\lambda}_{1,1}^3 \varphi_{1,1}^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \varphi_{1,2}^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa}^3 \varphi_{1,\kappa}^2 / \lambda_1^3 \end{pmatrix} \\
&\quad \times \left. \begin{pmatrix} 1 & \underline{\lambda}_{1,1} & \underline{\lambda}_{1,1}^2 & \cdots & \underline{\lambda}_{1,1}^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2} & \underline{\lambda}_{1,2}^2 & \cdots & \underline{\lambda}_{1,2}^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underline{\lambda}_{1,\kappa} & \underline{\lambda}_{1,\kappa}^2 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \right\} \\
&= \underline{V}' D_{\varphi, \lambda} \underline{V}
\end{aligned}$$

where

$$\begin{aligned}
D_{\varphi, \lambda} &= \begin{pmatrix} \underline{\lambda}_{1,1}^3 \varphi_{1,1}^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \varphi_{1,2}^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa}^3 \varphi_{1,\kappa}^2 / \lambda_1^3 \end{pmatrix} \\
\text{and } \underline{V} &= \begin{pmatrix} 1 & \underline{\lambda}_{1,1} & \underline{\lambda}_{1,1}^2 & \cdots & \underline{\lambda}_{1,1}^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2} & \underline{\lambda}_{1,2}^2 & \cdots & \underline{\lambda}_{1,2}^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underline{\lambda}_{1,\kappa} & \underline{\lambda}_{1,\kappa}^2 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \\
&= \begin{pmatrix} 1 & \underline{\lambda}_{1,1} / \lambda_1 & \underline{\lambda}_{1,1}^2 / \lambda_1^2 & \cdots & \underline{\lambda}_{1,1}^{\kappa-1} / \lambda_1^{\kappa-1} \\ 1 & \underline{\lambda}_{1,2} / \lambda_1 & \underline{\lambda}_{1,2}^2 / \lambda_1^2 & \cdots & \underline{\lambda}_{1,2}^{\kappa-1} / \lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underline{\lambda}_{1,\kappa} / \lambda_1 & \underline{\lambda}_{1,\kappa}^2 / \lambda_1^2 & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} / \lambda_1^{\kappa-1} \end{pmatrix}.
\end{aligned}$$

It follows by the result given in part (a) of this lemma that

$$\begin{aligned}
& \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \underline{V}' D_{\varphi,\lambda} \underline{V} \right\|_2 \\
&= \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right\|_2 \\
&\leq \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \frac{D_{\lambda,\kappa}^{-1} G'_W [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}] G_W D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \right\|_F \\
&= O\left(\frac{1}{\lambda_1}\right) \quad . \tag{161}
\end{aligned}$$

Moreover, note that the eigenvalues $\underline{\lambda}_{1,1}, \dots, \underline{\lambda}_{1,\kappa}$ under Assumption S1-2 satisfy the same conditions as that given in part (a) of Assumption 4-2. Hence, we can apply the result of Lemma B-2 above as well as Assumptions S1-1 and S1-2 to obtain

$$\begin{aligned}
& \lambda_{\min}(\underline{V}' D_{\varphi,\lambda} \underline{V}) \\
&\geq \min_{1 \leq g \leq \kappa} (\underline{\lambda}_{1,g}^3 \varphi_{1,g}^2 / \lambda_1^3) \lambda_{\min}(\underline{V}' \underline{V}) \\
&\geq \left(\frac{\underline{\lambda}_{1,\kappa}}{\lambda_1} \right)^3 \left(\min_{1 \leq g \leq \kappa} |\varphi_{1,g}| \right)^2 \lambda_{\min}(\underline{V}' \underline{V}) \\
&\geq \left(\frac{\underline{\lambda}_{1,\kappa}}{\lambda_1} \right)^3 \left(\frac{1}{\bar{c}} \right)^2 \underline{C} \quad (\text{by Lemma B-2 and Assumption S1-1}) \\
&\geq \left(\frac{1}{c^*} \right)^3 \left(\frac{1}{\bar{c}} \right)^2 \underline{C} \quad (\text{applying Lemma D-3(a) which implies that there exists constant } c^* > 1 \\
&\quad \text{such that } 0 < \frac{1}{c^*} \leq \frac{\underline{\lambda}_{1,g}}{\lambda_1} \leq c^* < \infty \text{ for } g \in \{1, \dots, \kappa\} \text{ and for all } N, T \text{ sufficiently large}) \\
&\geq C_{\#} \left(\text{for some constant } C_{\#} > 0 \text{ such that } C_{\#} \leq \left(\frac{1}{c^*} \right)^3 \left(\frac{1}{\bar{c}} \right)^2 \underline{C} \right) \tag{162}
\end{aligned}$$

Furthermore, by Lemma D-4 and by expression (161) above, we see that

$$\begin{aligned}
& \left| \lambda_{\min} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) - \lambda_{\min}(\underline{V}' D_{\varphi,\lambda} \underline{V}) \right| \\
&\leq \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} - \underline{V}' D_{\varphi,\lambda} \underline{V} \right\|_2 \\
&= O\left(\frac{1}{\lambda_1}\right) = o(1) \tag{163}
\end{aligned}$$

Hence, by applying Lemma B-2, the result given in expression (163) above, as well as Assumptions S1-1 and S1-2; we obtain

$$\begin{aligned}
& \lambda_{\min} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \\
&= \lambda_{\min} (\underline{V}' D_{\varphi,\lambda} \underline{V}) + \lambda_{\min} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) - \lambda_{\min} (\underline{V}' D_{\varphi,\lambda} \underline{V}) \\
&\geq \lambda_{\min} (\underline{V}' D_{\varphi,\lambda} \underline{V}) - \left| \lambda_{\min} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) - \lambda_{\min} (\underline{V}' D_{\varphi,\lambda} \underline{V}) \right| \\
&\geq \min_{1 \leq g \leq \kappa} (\underline{\lambda}_{1,g}^3 \varphi_{1,g}^2 / \lambda_1^3) \lambda_{\min} (\underline{V}' \underline{V}) - \left| \lambda_{\min} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) - \lambda_{\min} (\underline{V}' D_{\varphi,\lambda} \underline{V}) \right| \\
&\geq \left(\frac{\underline{\lambda}_{1,\kappa}}{\lambda_1} \right)^3 \left(\min_{1 \leq g \leq \kappa} |\varphi_{1,g}| \right)^2 \lambda_{\min} (\underline{V}' \underline{V}) + o(1) \\
&\geq C_{\#} \quad (\text{for some constant } C_{\#} > 0 \text{ and for all } N, T \text{ sufficiently large})
\end{aligned}$$

Furthermore, note that, for all N, T sufficiently large,

$$\begin{aligned}
& \lambda_{\min} \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \\
&= \frac{\lambda_1}{\underline{\lambda}_{1,1}} \lambda_{\min} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right) \\
&\geq \frac{C_{\#}}{c^*} \quad (\text{since by Lemma D-3 part (a), there exists a constant } c^* > 1 \\
&\quad \text{such that } 0 < \frac{1}{c^*} \leq \frac{\lambda_1}{\underline{\lambda}_{1,1}} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large}) \\
&\geq C_* \quad \text{for some constant } C_* > 0 \text{ such that } C_* \leq \frac{C_{\#}}{c^*}. \quad \square
\end{aligned}$$

Lemma E-3: Suppose that Assumptions (i)-(iii) hold. In addition, suppose that Assumptions S2-1 and S2-2 hold under Scenario 2. Under these conditions, the following statements are true.

(a)

$$\left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left(\frac{D_{\lambda,K}^{-1} G' \bar{\Sigma}_X G D_{\lambda,K}^{-1}}{\lambda_1} - \frac{D_{\lambda,K}^{-1} G'_{W,K} [\Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda}] G_{W,K} D_{\lambda,K}^{-1}}{\lambda_1} \right) \right\|_F = O \left(\frac{1}{\lambda_1} \right),$$

where, in this case,

$$\begin{aligned}
G_{W,K} &= \begin{pmatrix} \Gamma_\lambda \underline{\Delta} \varphi / \sqrt{\underline{\Delta}_{1,1}} & \bar{\Sigma}_W \Gamma_\lambda \underline{\Delta} \varphi / \sqrt{\underline{\Delta}_{1,1}} & \cdots & \bar{\Sigma}_W^{K-1} \Gamma_\lambda \underline{\Delta} \varphi / \sqrt{\underline{\Delta}_{1,1}} \end{pmatrix}, \\
D_{\lambda,K} &= \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{K-\frac{1}{2}} \end{pmatrix}, \quad G = \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{K-1} \bar{\sigma}_{Xy} \end{pmatrix}, \\
\underline{\Delta} &= \text{diag}(\underline{\Delta}_{1,1}, \dots, \underline{\Delta}_{1,K}), \quad \varphi = (\varphi_{1,1}, \dots, \varphi_{1,K})', \\
\bar{\Sigma}_X &= \bar{\Sigma}_W + \bar{\Sigma}_u = \Gamma_\lambda \underline{\Delta}_1 \Gamma'_\lambda + \bar{\Sigma}_u \text{ and } \Gamma_{\lambda, N \times K} = \frac{\Gamma}{\sqrt{\underline{\Delta}_{1,1}}}.
\end{aligned}$$

(b) Let $D_{\underline{\Delta},K}$ and $G_{W,K}$ be as defined in part (a) above under Scenario 2 and let

$$\begin{aligned}
D_{\varphi,\lambda} &= \begin{pmatrix} \underline{\Delta}_{1,1}^3 \varphi_{1,1}^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2}^3 \varphi_{1,2}^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,K}^3 \varphi_{1,K}^2 / \lambda_1^3 \end{pmatrix} \text{ and} \\
\underline{V} &= \begin{pmatrix} 1 & \underline{\Delta}_{1,1} / \lambda_1 & \underline{\Delta}_{1,1}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,1}^{K-1} / \lambda_1^{K-1} \\ 1 & \underline{\Delta}_{1,2} / \lambda_1 & \underline{\Delta}_{1,2}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^{K-1} / \lambda_1^{K-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underline{\Delta}_{1,K} / \lambda_1 & \underline{\Delta}_{1,K}^2 / \lambda_1^2 & \cdots & \underline{\Delta}_{1,K}^{K-1} / \lambda_1^{K-1} \end{pmatrix}
\end{aligned}$$

Then,

$$\left\| \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,K}^{-1} G' \bar{\Sigma}_X G D_{\lambda,K}^{-1}}{\lambda_1} - \underline{V}' D_{\varphi,\lambda} \underline{V} \right\|_2 = O\left(\frac{1}{\lambda_1}\right) = o(1).$$

and there exists positive constants $C_\#$ and C_* such that

$$\lambda_{\min}(\underline{V}' D_{\varphi,\lambda} \underline{V}) \geq C_\# > 0 \text{ and } \lambda_{\min}\left(\frac{D_{\lambda,K}^{-1} G' \bar{\Sigma}_X G D_{\lambda,K}^{-1}}{\lambda_1}\right) \geq C_* > 0$$

for all T and N sufficiently large.

Proof of Lemma E-3: The results given in parts (a) and (b) of this lemma can be shown using the same argument as that given, respectively, in the proof of parts (a) and (b) of Lemma E-2, except that we apply Assumptions S2-1 and S2-2 instead of Assumptions S1-1

and S1-2 and change the notations used in the proof of Lemma E-2 by setting

$$\Gamma_{\lambda,1} = \Gamma_{\lambda} = \frac{\Gamma}{\sqrt{\Delta_{1,1}}} \text{ and } \underline{\Delta}_1 = \text{diag}(\lambda_{1,1}, \dots, \lambda_{1,K}) = \underline{\Delta}$$

(noting that there are no $\Gamma_{\lambda,2}$ and $\underline{\Delta}_2$ in this case)

since, under Scenario 2 considered here, all available factors are relevant for the purpose of forecasting and there is no reason to split the nonzero eigenvalues of the common component covariance matrix $\Gamma \bar{\Sigma}_F \Gamma'$ and their associated eigenvectors into two groups as all the columns of the $N \times K$ semi-orthogonal matrix $\Gamma / \sqrt{\Delta_{1,1}}$ are needed to span the envelope. In this case, we also take $\varphi_1 = (\varphi_{1,1}, \dots, \varphi_{1,K})' = \varphi$ since none of the coefficients is equal to zero. $\square$

Lemma E-4: Suppose that Assumptions (i)-(iii) hold. In addition, suppose that Assumptions S1-1 and S1-2 hold under Scenario 1. Under these conditions, the following statements are true.

(a)

$$\frac{D_{\lambda,\kappa}^{-1} G'_W [\Gamma_{\lambda,1} \underline{\Delta}_1 \Gamma'_{\lambda,1}] \Gamma_{\lambda,1} (\varphi_1 / \sqrt{\Delta_{1,1}})}{\lambda_1^{\frac{1}{2}}} = \underline{V}' \underline{w}_{\varphi},$$

where

$$D_{\lambda,\kappa} = \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 & \dots & 0 \\ 0 & \lambda_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_1^{(\kappa-\frac{1}{2})} \end{pmatrix}, \underline{w}_{\varphi} = \begin{pmatrix} \Delta_{1,1}^2 \varphi_{1,1}^2 / \lambda_1^2 \\ \Delta_{1,2}^2 \varphi_{1,2}^2 / \lambda_1^2 \\ \vdots \\ \Delta_{1,\kappa}^2 \varphi_{1,\kappa}^2 / \lambda_1^2 \end{pmatrix},$$

$$\underline{V} = \begin{pmatrix} 1 & \Delta_{1,1} / \lambda_1 & \Delta_{1,1}^2 / \lambda_1^2 & \dots & \Delta_{1,1}^{\kappa-1} / \lambda_1^{\kappa-1} \\ 1 & \Delta_{1,2} / \lambda_1 & \Delta_{1,2}^2 / \lambda_1^2 & \dots & \Delta_{1,2}^{\kappa-1} / \lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \Delta_{1,\kappa} / \lambda_1 & \Delta_{1,\kappa}^2 / \lambda_1^2 & \dots & \Delta_{1,\kappa}^{\kappa-1} / \lambda_1^{\kappa-1} \end{pmatrix}$$

where

$$G_W = \begin{pmatrix} \frac{\Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1}{\sqrt{\Delta_{1,1}}} & [\Gamma_{\lambda,1} \underline{\Delta}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Delta}_2 \Gamma'_{\lambda,2}] \frac{\Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1}{\sqrt{\Delta_{1,1}}} & \dots & [\Gamma_{\lambda,1} \underline{\Delta}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Delta}_2 \Gamma'_{\lambda,2}]^{\kappa-1} \frac{\Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1}{\sqrt{\Delta_{1,1}}} \end{pmatrix}$$

and where

$$\Gamma_{\lambda,1} = \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \text{ and } \Gamma_{\lambda,2} = \frac{\Gamma_2}{\sqrt{\Delta_{1,1}}}$$

(b)

$$\left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} - \underline{V}' \underline{w}_\varphi \right\|_2 = O\left(\frac{1}{\lambda_1}\right).$$

where

$$G = \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{pmatrix} \text{ for } 1 \leq \kappa < K$$

(c)

$$\left\| \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right\|_2 = O\left(\frac{1}{\lambda_1}\right),$$

where

$$D_{\varphi,\lambda}^* = \begin{pmatrix} \underline{\lambda}_{1,1} \varphi_{1,1} / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2} \varphi_{1,2} / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa} \varphi_{1,\kappa} / \lambda_1 \end{pmatrix}.$$

(d)

$$\left\| \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right\|_2 = O\left(\frac{1}{\lambda_1}\right),$$

where $\underline{V}$ is the Vandermonde matrix defined in part (a) above and where

$$D_{\phi,\lambda} = \begin{pmatrix} \underline{\lambda}_{1,1}^3 \varphi_{1,1}^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \varphi_{1,2}^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,\kappa}^3 \varphi_{1,\kappa}^2 / \lambda_1^3 \end{pmatrix}.$$

Proof of Lemma E-4:

To show part (a), first write

$$\begin{aligned} & G_W \\ &= \begin{pmatrix} \frac{\Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\sqrt{\underline{\Lambda}_{1,1}}} & [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}] \frac{\Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\sqrt{\underline{\Lambda}_{1,1}}} & \cdots & [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}]^{\kappa-1} \frac{\Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\sqrt{\underline{\Lambda}_{1,1}}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\sqrt{\underline{\Lambda}_{1,1}}} & [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}] \frac{\Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\sqrt{\underline{\Lambda}_{1,1}}} & \cdots & [\Gamma_{\lambda,1} \underline{\Lambda}_1^{\kappa-1} \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2^{\kappa-1} \Gamma'_{\lambda,2}] \frac{\Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\sqrt{\underline{\Lambda}_{1,1}}} \end{pmatrix} \\ & \quad (\text{given that } \Gamma'_{\lambda,1} \Gamma_{\lambda,1} = I_\kappa, \Gamma'_{\lambda,2} \Gamma_{\lambda,2} = I_{K-\kappa}, \text{ and } \Gamma'_{\lambda,2} \Gamma_{\lambda,1} = 0) \\ &= \begin{pmatrix} \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \sqrt{\underline{\Lambda}_{1,1}} & \Gamma_{\lambda,1} \underline{\Lambda}_1^2 \varphi_1 / \sqrt{\underline{\Lambda}_{1,1}} & \cdots & \Gamma_{\lambda,1} \underline{\Lambda}_1^\kappa \varphi_1 / \sqrt{\underline{\Lambda}_{1,1}} \end{pmatrix} \end{aligned}$$

Making use of this expression for G_W , we obtain by straightforward calculation

$$\begin{aligned}
& \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G'_W [\Gamma_{\lambda,1} \underline{\Delta}_1 \Gamma'_{\lambda,1}] \Gamma_{\lambda,1} (\varphi_1 / \sqrt{\lambda_{1,1}})}{\lambda_1^{\frac{1}{2}}} \\
&= \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G'_W \Gamma_{\lambda,1} \underline{\Delta}_1 (\varphi_1 / \sqrt{\lambda_{1,1}})}{\lambda_1^{\frac{1}{2}}} \\
&= \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \lambda_{1,1}^{-\frac{1}{2}} \varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \\ \lambda_{1,1}^{-\frac{1}{2}} \varphi'_1 \underline{\Delta}_1^2 \Gamma'_{\lambda,1} \\ \vdots \\ \lambda_{1,1}^{-\frac{1}{2}} \varphi'_1 \underline{\Delta}_1^\kappa \Gamma'_{\lambda,1} \end{pmatrix} \frac{\Gamma_{\lambda,1} \underline{\Delta}_1 (\varphi_1 / \sqrt{\lambda_{1,1}})}{\lambda_1^{\frac{1}{2}}} \\
&= \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \lambda_1^{-1} \varphi'_1 \underline{\Delta}_1^2 \\ \lambda_1^{-1} \varphi'_1 \underline{\Delta}_1^3 \\ \vdots \\ \lambda_1^{-1} \varphi'_1 \underline{\Delta}_1^{\kappa+1} \end{pmatrix} \frac{\varphi_1}{\lambda_1^{\frac{1}{2}}} \\
&= \begin{pmatrix} \varphi'_1 \underline{\Delta}_1^2 \varphi_1 / \lambda_1^2 \\ \varphi'_1 \underline{\Delta}_1^3 \varphi_1 / \lambda_1^3 \\ \vdots \\ \varphi'_1 \underline{\Delta}_1^{\kappa+1} \varphi_1 / \lambda_1^{\kappa+1} \end{pmatrix} \tag{164}
\end{aligned}$$

Next, note that, for positive integer k , we have

$$\frac{\varphi'_1 \underline{\Delta}_1^k \varphi_1}{\lambda_1^k} = \frac{1}{\lambda_1^k} \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^k \varphi_{1,g}^2$$

from which we further obtain

$$\begin{pmatrix} \varphi'_1 \underline{\Delta}_1^2 \varphi_1 / \lambda_1^2 \\ \varphi'_1 \underline{\Delta}_1^3 \varphi_1 / \lambda_1^3 \\ \vdots \\ \varphi'_1 \underline{\Delta}_1^{\kappa+1} \varphi_1 / \lambda_1^{\kappa+1} \end{pmatrix} = \begin{pmatrix} \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^2 \varphi_{1,g}^2 / \lambda_1^2 \\ \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^3 \varphi_{1,g}^2 / \lambda_1^3 \\ \vdots \\ \sum_{g=1}^{\kappa} \underline{\Delta}_{1,g}^{\kappa+1} \varphi_{1,g}^2 / \lambda_1^{\kappa+1} \end{pmatrix} \tag{165}$$

Making use of expression (165), we see that, by straightforward calculation,

$$\begin{aligned}
& \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G'_W [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1}] \Gamma_{\lambda,1} (\varphi_1 / \sqrt{\underline{\Lambda}_{1,1}})}{\lambda_1^{\frac{1}{2}}} \\
&= \begin{pmatrix} \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^2 \varphi_{1,g}^2 / \lambda_1^2 \\ \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^3 \varphi_{1,g}^2 / \lambda_1^3 \\ \vdots \\ \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^{\kappa+1} \varphi_{1,g}^2 / \lambda_1^{\kappa+1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \begin{pmatrix} \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^2 \varphi_{1,g}^2 / \lambda_1^2 \\ \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^3 \varphi_{1,g}^2 / \lambda_1^3 \\ \vdots \\ \sum_{g=1}^{\kappa} \underline{\lambda}_{1,g}^{\kappa+1} \varphi_{1,g}^2 / \lambda_1^{\kappa+1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_1^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-1)} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \underline{\lambda}_{1,1} & \underline{\lambda}_{1,2} & \cdots & \underline{\lambda}_{1,\kappa} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\lambda}_{1,1}^{\kappa-1} & \underline{\lambda}_{1,2}^{\kappa-1} & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} \end{pmatrix} \begin{pmatrix} \underline{\lambda}_{1,1}^2 \varphi_{1,1}^2 / \lambda_1^2 \\ \underline{\lambda}_{1,2}^2 \varphi_{1,2}^2 / \lambda_1^2 \\ \vdots \\ \underline{\lambda}_{1,\kappa}^2 \varphi_{1,\kappa}^2 / \lambda_1^2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \underline{\lambda}_{1,1} / \lambda_1 & \underline{\lambda}_{1,2} / \lambda_1 & \cdots & \underline{\lambda}_{1,\kappa} / \lambda_1 \\ \underline{\lambda}_{1,1}^2 / \lambda_1^2 & \underline{\lambda}_{1,2}^2 / \lambda_1^2 & \cdots & \underline{\lambda}_{1,\kappa}^2 / \lambda_1^2 \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\lambda}_{1,1}^{\kappa-1} / \lambda_1^{\kappa-1} & \underline{\lambda}_{1,2}^{\kappa-1} / \lambda_1^{\kappa-1} & \cdots & \underline{\lambda}_{1,\kappa}^{\kappa-1} / \lambda_1^{\kappa-1} \end{pmatrix} \begin{pmatrix} \underline{\lambda}_{1,1}^2 \varphi_{1,1}^2 / \lambda_1^2 \\ \underline{\lambda}_{1,2}^2 \varphi_{1,2}^2 / \lambda_1^2 \\ \vdots \\ \underline{\lambda}_{1,\kappa}^2 \varphi_{1,\kappa}^2 / \lambda_1^2 \end{pmatrix} \\
&= \underline{V}' \underline{w}_{\varphi},
\end{aligned}$$

where $\underline{w}_{\varphi} = (\underline{\lambda}_{1,1}^2 \varphi_1^2 / \lambda_1^2, \underline{\lambda}_{1,2}^2 \varphi_2^2 / \lambda_1^2, \dots, \underline{\lambda}_{1,\kappa}^2 \varphi_{\kappa}^2 / \lambda_1^2)'$.

To show part (b), note first that

$$\frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\lambda_1^{\frac{1}{2}}} = \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_{Xy} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \end{pmatrix} \frac{\bar{\sigma}_{Xy}}{\lambda_1^{\frac{1}{2}}} = \begin{pmatrix} \bar{\sigma}'_{Xy} \bar{\sigma}_{Xy} / \lambda_1 \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} / \lambda_1^2 \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} / \lambda_1^{\kappa} \end{pmatrix} \quad (166)$$

Moreover, note that, for integer $k \geq 0$,

$$\begin{aligned}
\left(\frac{\lambda_{1,1}}{\lambda_1}\right) \frac{\overline{\sigma}'_{Xy} \overline{\Sigma}_X^k \overline{\sigma}_{Xy}}{\lambda_1^{k+1}} &= \left(\frac{\lambda_{1,1}}{\lambda_1}\right) \frac{1}{\lambda_1^{k+1}} \varphi'_1 \left(\frac{\underline{\lambda}_1}{\underline{\lambda}_{1,1}}\right) \Gamma'_1 \overline{\Sigma}_X^k \Gamma_1 \left(\frac{\underline{\lambda}_1}{\underline{\lambda}_{1,1}}\right) \varphi_1 \\
&= \left(\frac{\lambda_{1,1}}{\lambda_1}\right) \left(\frac{1}{\underline{\lambda}_{1,1}}\right) \frac{1}{\lambda_1^{k+1}} \varphi'_1 \underline{\lambda}_1 \frac{\Gamma'_1}{\sqrt{\underline{\lambda}_{1,1}}} \overline{\Sigma}_X^k \frac{\Gamma_1}{\sqrt{\underline{\lambda}_{1,1}}} \underline{\lambda}_1 \varphi_1 \\
&= \frac{1}{\lambda_1^{k+2}} \varphi'_1 \underline{\lambda}_1 \Gamma'_{\lambda,1} \overline{\Sigma}_X^k \Gamma_{\lambda,1} \underline{\lambda}_1 \varphi_1, \text{ where } \Gamma_{\lambda,1} = \frac{\Gamma_1}{\sqrt{\underline{\lambda}_{1,1}}}. \quad (167)
\end{aligned}$$

By making use of expression (164) in part (a) above as well as expressions (166) and (167),

we have

$$\begin{aligned}
& \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\lambda_1^{\frac{1}{2}}} - \underline{V}' \underline{w}_\varphi \\
&= \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\lambda_1^{\frac{1}{2}}} - \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G'_W [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1}] \Gamma_{\lambda,1} (\varphi_1 / \sqrt{\underline{\lambda}_{1,1}})}{\lambda_1^{\frac{1}{2}}} \\
& \left(\text{since } \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G'_W [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1}] \Gamma_{\lambda,1} (\varphi_1 / \sqrt{\underline{\lambda}_{1,1}})}{\lambda_1^{\frac{1}{2}}} = \underline{V}' \underline{w}_\phi \text{ by the result given in part (a)} \right) \\
&= \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \begin{pmatrix} \bar{\sigma}'_{Xy} \bar{\sigma}_{Xy} / \lambda_1 \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} / \lambda_1^2 \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} / \lambda_1^\kappa \end{pmatrix} - \begin{pmatrix} \varphi'_1 \underline{\Lambda}_1^2 \varphi_1 / \lambda_1^2 \\ \varphi'_1 \underline{\Lambda}_1^3 \varphi_1 / \lambda_1^3 \\ \vdots \\ \varphi'_1 \underline{\Lambda}_1^{\kappa+1} \varphi_1 / \lambda_1^{\kappa+1} \end{pmatrix} \\
& \text{(by the results given in expressions (164) and (166))} \\
&= \begin{pmatrix} \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^2 \\ \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^3 \\ \vdots \\ \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X^{\kappa-1} \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^{\kappa+1} \end{pmatrix} - \begin{pmatrix} \varphi'_1 \underline{\Lambda}_1^2 \varphi_1 / \lambda_1^2 \\ \varphi'_1 \underline{\Lambda}_1^3 \varphi_1 / \lambda_1^3 \\ \vdots \\ \varphi'_1 \underline{\Lambda}_1^{\kappa+1} \varphi_1 / \lambda_1^{\kappa+1} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} (\bar{\Sigma}_X - \Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1}) \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^3 \\ \vdots \\ \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} [\bar{\Sigma}_X^{\kappa-1} - (\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1})^{\kappa-1}] \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^{\kappa+1} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} (\bar{\Sigma}_X - [\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2}]) \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^3 \\ \vdots \\ \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} [\bar{\Sigma}_X^{\kappa-1} - (\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2 \Gamma'_{\lambda,2})^{\kappa-1}] \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^{\kappa+1} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} (\bar{\Sigma}_X - \Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda}) \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^3 \\ \vdots \\ \varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} [\bar{\Sigma}_X^{\kappa-1} - (\Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda})^{\kappa-1}] \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1 / \lambda_1^{\kappa+1} \end{pmatrix}
\end{aligned}$$

where

$$\Gamma_{\lambda,1} = \frac{\Gamma_1}{\sqrt{\underline{\lambda}_{1,1}}}, \Gamma_{\lambda,2} = \frac{\Gamma_2}{\sqrt{\underline{\lambda}_{1,1}}}, \Gamma_{\lambda} = \begin{pmatrix} \Gamma_{\lambda,1} & \Gamma_{\lambda,2} \end{pmatrix}, \text{ and } \underline{\Lambda} = \begin{pmatrix} \underline{\Lambda}_1 & 0 \\ 0 & \underline{\Lambda}_2 \end{pmatrix}$$

In addition, using argument similar to that given in the proof of Lemma E-2 part (a), we have, for any fixed integer $2 \leq j \leq \kappa$,

$$\begin{aligned}
& \left| \frac{\varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \left[\bar{\Sigma}_X^{j-1} - (\Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda)^{j-1} \right] \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\lambda_1^{j+1}} \right| \\
&= \left| \frac{\varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \left[(\Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda + \bar{\Sigma}_X - \Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda)^{j-1} - (\Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda)^{j-1} \right] \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\lambda_1^{j+1}} \right| \\
&\leq \frac{1}{\lambda_1^{j+1}} \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \|\bar{\Sigma}_X - \Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda\|_2^\ell \|\Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda\|_2^{j-1-\ell} \|\Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1\|_2^2 \\
&\leq \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \left\| \frac{\bar{\Sigma}_X - \Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda}{\lambda_1} \right\|_2^\ell \frac{\|\Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda\|_2^{j-1-\ell}}{\lambda_1^{j-1-\ell}} \left\| \frac{\underline{\Lambda}_1}{\lambda_1} \right\|_2^2 \|\varphi_1\|_2^2 \quad (\text{since } \|\Gamma_{\lambda,1}\|_2^2 = 1) \\
&= \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \frac{\|\Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda\|_2^{j-1-\ell}}{\underline{\lambda}_{1,1}^{j-1-\ell}} \left\| \frac{\underline{\Lambda}_1}{\underline{\lambda}_{1,1}} \right\|_2^2 \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right)^{j+1-\ell} \|\varphi_1\|_2^2 \quad (\text{since } \bar{\Sigma}_u = \bar{\Sigma}_X - \Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda) \\
&\leq (c^*)^\kappa \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \frac{\|\Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda\|_2^{j-1-\ell}}{\underline{\lambda}_{1,1}^{j-1-\ell}} \left\| \frac{\underline{\Lambda}_1}{\underline{\lambda}_{1,1}} \right\|_2^2 \|\varphi_1\|_2^2 \\
&\quad \left(\text{since Lemma D-3(a) implies that there exists a constant } c^* > 1 \text{ such that } \frac{\lambda_{1,1}}{\lambda_1} \leq c^* < \infty \right. \\
&\quad \left. \text{for all } N, T \text{ sufficiently large and since } j+1-\ell \leq \kappa \text{ for all } j \leq \kappa \text{ and } \ell \geq 1 \right) \\
&\leq (c^*)^\kappa C^\kappa \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \|\varphi_1\|_2^2 \\
&\quad (\text{since } \|\Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda\|_2 = \lambda_{\max} \{\underline{\Lambda}\} = \max \{\underline{\lambda}_{1,1}, \underline{\lambda}_{2,1}\} \leq C \underline{\lambda}_{1,1} \text{ for some constant } C > 1 \text{ and for all } \\
&\quad N, T \text{ sufficiently large given that } \underline{\lambda}_{1,1} \asymp \underline{\lambda}_{2,1} \text{ and since } \|\underline{\Lambda}_1\|_2^2 = \lambda_{\max} \{\underline{\Lambda}_1^2\} = (\lambda_{\max} \{\underline{\Lambda}_1\})^2 = \underline{\lambda}_{1,1}^2) \\
&\leq (c^*)^\kappa C^\kappa \bar{c}^2 \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^\ell \quad (\text{since, by Assumption S1-1, there exists a constant } \bar{c} \\
&\quad \text{such that } \bar{c} \geq \max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \text{ so that } \kappa \bar{c}^2 \geq \kappa \left(\max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \right)^2 \geq \|\varphi_1\|_2^2)
\end{aligned}$$

$$\begin{aligned}
&= (c^*)^\kappa C^\kappa \kappa \bar{c}^2 \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right)^\ell \left(\text{since } \|\bar{\Sigma}_u\|_2 = \sqrt{\lambda_{\max}(\bar{\Sigma}'_u \bar{\Sigma}_u)} = \sqrt{[\lambda_{\max}(\bar{\Sigma}_u)]^2} \right) \\
&\leq (c^*)^\kappa C^\kappa \kappa \bar{c}^2 \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \left(\frac{\bar{C}}{\lambda_1} \right)^\ell \left(\text{given that } \lambda_{\max}(\bar{\Sigma}_u) \leq \bar{C} < \infty \text{ for some constant } \bar{C} > 0 \right. \\
&\quad \left. \text{by Assumption (ii) and Lemma D-2} \right) \\
&\leq (c^*)^\kappa C^\kappa \kappa \bar{c}^2 (\kappa - 1) ([\kappa - 1]!) \left(\frac{\bar{C}}{\lambda_1} \right)^\ell \left(\text{noting that } \left(\frac{\bar{C}}{\lambda_1} \right)^\ell \ll \frac{\bar{C}}{\lambda_1} \text{ for all integer } \ell \geq 2 \right. \\
&\quad \left. \text{and for all } N \text{ and } T \text{ sufficiently large since } \lambda_1 \rightarrow \infty \right) \\
&= O\left(\frac{1}{\lambda_1}\right).
\end{aligned}$$

Making use of this result, we see that

$$\begin{aligned}
&\left\| \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\lambda_1^{\frac{1}{2}}} - \underline{V}' \underline{w}_\varphi \right\|_2^2 \\
&= \sum_{j=2}^{\kappa} \left(\frac{\varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \left[\bar{\Sigma}_X^{j-1} - (\Gamma_{\lambda,1} \underline{\Delta} \Gamma'_{\lambda,1})^{j-1} \right] \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1}{\lambda_1^{j+1}} \right)^2 \\
&= \sum_{j=2}^{\kappa} \left(\frac{\varphi'_1 \underline{\Delta}_1 \Gamma'_{\lambda,1} \left[\bar{\Sigma}_X^{j-1} - (\Gamma_{\lambda,1} \underline{\Delta} \Gamma'_{\lambda,1})^{j-1} \right] \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1}{\lambda_1^{j+1}} \right)^2 \\
&\leq \sum_{j=2}^{\kappa} \left[(c^*)^\kappa C^\kappa \kappa \bar{c}^2 \sum_{\ell=1}^{j-1} \binom{j-1}{\ell} \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right)^\ell \right]^2 \\
&\leq \sum_{j=2}^{\kappa} \left[(c^*)^\kappa C^\kappa \kappa \bar{c}^2 (\kappa - 1) ([\kappa - 1]!) \left(\frac{\bar{C}}{\lambda_1} \right) \right]^2 \text{ for all } N \text{ and } T \text{ sufficiently large} \\
&= (c^*)^{2\kappa} C^{2\kappa} \kappa^3 \bar{c}^4 [\kappa - 1]^2 [(\kappa - 1)!]^2 \left(\frac{\bar{C}}{\lambda_1} \right)^2 \\
&= O\left(\frac{1}{\lambda_1^2}\right)
\end{aligned}$$

It follows that

$$\left\| \left(\frac{\underline{\Delta}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,\kappa}^{-1} G' \bar{\sigma}_{Xy}}{\lambda_1^{\frac{1}{2}}} - \underline{V}' \underline{w}_\varphi \right\|_2 = O\left(\frac{1}{\lambda_1}\right).$$

Turning our attention to part (c), again let $\Gamma_{\lambda,1} = \Gamma_1 / \sqrt{\underline{\Delta}_{1,1}}$, and note that, by straight-

forward calculation

$$\begin{aligned}
& \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} \\
&= \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} \left(\begin{array}{cccc} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \end{array} \right) \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \\
&= \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} \left(\begin{array}{cccc} \bar{\sigma}_{Xy}/\lambda_1^{\frac{1}{2}} & \bar{\Sigma}_X \bar{\sigma}_{Xy}/\lambda_1^{\frac{3}{2}} & \cdots & \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy}/\lambda_1^{\kappa-\frac{1}{2}} \end{array} \right) \\
&= \left(\begin{array}{cccc} \Gamma_{\lambda,1}(\underline{\Delta}_1/\lambda_1) \varphi_1 & \bar{\Sigma}_X \Gamma_{\lambda,1}(\underline{\Delta}_1/\lambda_1) \varphi_1/\lambda_1 & \cdots & \bar{\Sigma}_X^{\kappa-1} \Gamma_{\lambda,1}(\underline{\Delta}_1/\lambda_1) \varphi_1/\lambda_1^{\kappa-1} \end{array} \right) \\
&\quad \left(\text{since } \bar{\sigma}_{Xy} = \Gamma_{\lambda,1} \left(\frac{\underline{\Delta}_1}{\sqrt{\underline{\Delta}_{1,1}}} \right) \varphi_1, \text{ where } \Gamma_{\lambda,1} = \frac{\Gamma_1}{\sqrt{\underline{\Delta}_{1,1}}} \right) \tag{168}
\end{aligned}$$

and

$$\begin{aligned}
& \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G_W D_{\lambda,\kappa}^{-1} \\
&= \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} \left(\begin{array}{cccc} \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \sqrt{\underline{\Delta}_{1,1}} & \Gamma_{\lambda,1} \underline{\Delta}_1^2 \varphi_1 / \sqrt{\underline{\Delta}_{1,1}} & \cdots & \Gamma_{\lambda,1} \underline{\Delta}_1^\kappa \varphi_1 / \sqrt{\underline{\Delta}_{1,1}} \end{array} \right) \\
&\quad \times \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \\
&= \left(\begin{array}{cccc} \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1 / \sqrt{\lambda_1} & \Gamma_{\lambda,1} \underline{\Delta}_1^2 \varphi_1 / \sqrt{\lambda_1} & \cdots & \Gamma_{\lambda,1} \underline{\Delta}_1^\kappa \varphi_1 / \sqrt{\lambda_1} \end{array} \right) \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \\
&= \left(\begin{array}{cccc} \Gamma_{\lambda,1}(\underline{\Delta}_1/\lambda_1) \varphi_1 & (\Gamma_{\lambda,1} \underline{\Delta}_1 \Gamma'_{\lambda,1}) \Gamma_{\lambda,1}(\underline{\Delta}_1/\lambda_1) \varphi_1/\lambda_1 & \cdots & \cdots \\ \cdots & \cdots & (\Gamma_{\lambda,1} \underline{\Delta}_1 \Gamma'_{\lambda,1})^{\kappa-1} \Gamma_{\lambda,1}(\underline{\Delta}_1/\lambda_1) \varphi_1/\lambda_1^{\kappa-1} & \end{array} \right) \tag{169}
\end{aligned}$$

It follows immediately from expressions (168) and (169) that

$$\begin{aligned}
& \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G_W D_{\lambda,\kappa}^{-1} \\
&= \begin{pmatrix} \Gamma_{\lambda,1}(\underline{\Lambda}_1/\lambda_1) \varphi_1 & \bar{\Sigma}_X \Gamma_{\lambda,1}(\underline{\Lambda}_1/\lambda_1) \varphi_1/\lambda_1 & \cdots & \bar{\Sigma}_X^{\kappa-1} \Gamma_{\lambda,1}(\underline{\Lambda}_1/\lambda_1) \varphi_1/\lambda_1^{\kappa-1} \\ -(\Gamma_{\lambda,1}(\underline{\Lambda}_1/\lambda_1) \varphi_1 & (\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1}) \Gamma_{\lambda,1}(\underline{\Lambda}_1/\lambda_1) \varphi_1/\lambda_1 & \cdots & \cdots \\ & \cdots & (\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1})^{\kappa-1} \Gamma_{\lambda,1}(\underline{\Lambda}_1/\lambda_1) \varphi_1/\lambda_1^{\kappa-1} \end{pmatrix} \\
&= \begin{pmatrix} 0 & [\bar{\Sigma}_X - \Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1}] \Gamma_{\lambda,1}(\underline{\Lambda}_1/\lambda_1) \varphi_1/\lambda_1 & \cdots & \cdots \\ \cdots & \cdots & [\bar{\Sigma}_X^{\kappa-1} - (\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1})^{\kappa-1}] \Gamma_{\lambda,1}(\underline{\Lambda}_1/\lambda_1) \varphi_1/\lambda_1^{\kappa-1} \end{pmatrix}
\end{aligned}$$

Note that for $\kappa = 1$, by the above calculations

$$\sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,1}^{-1} - \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G_W D_{\lambda,1}^{-1} = 0$$

On the other hand, for $\kappa \geq 2$ and for any fixed integer j such that $1 \leq j \leq \kappa - 1$, we have

$$\begin{aligned}
& \left\| \frac{[\bar{\Sigma}_X^j - (\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1})^j] \Gamma_{\lambda,1} (\underline{\Lambda}_1 / \lambda_1) \varphi_1}{\lambda_1^j} \right\|_2 \\
&= \left\| \frac{([\Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda} + \bar{\Sigma}_X - \Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda}]^j - [\Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda}]^j) \Gamma_{\lambda,1} (\underline{\Lambda}_1 / \lambda_1) \varphi_1}{\lambda_1^j} \right\|_2 \\
&\quad \left(\text{since } (\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1})^j \Gamma_{\lambda,1} = [\Gamma_{\lambda,1} \underline{\Lambda}_1^j \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2^j \Gamma'_{\lambda,2}] \Gamma_{\lambda,1} = [\Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda}]^j \Gamma_{\lambda,1} \text{ given that} \right. \\
&\quad \left. \Gamma_{\lambda,1} \underline{\Lambda}_1^j \Gamma'_{\lambda,1} + \Gamma_{\lambda,2} \underline{\Lambda}_2^j \Gamma'_{\lambda,2} = \begin{pmatrix} \Gamma_{\lambda,1} & \Gamma_{\lambda,2} \end{pmatrix} \begin{pmatrix} \underline{\Lambda}_1^j & 0 \\ 0 & \underline{\Lambda}_2^j \end{pmatrix} \begin{pmatrix} \Gamma'_{\lambda,1} \\ \Gamma'_{\lambda,2} \end{pmatrix} = \Gamma_{\lambda} \underline{\Lambda}^j \Gamma'_{\lambda} = [\Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda}]^j \right) \\
&= \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_X - \Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda}}{\lambda_1} \right\|_2^{\ell} \frac{\|\Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda}\|_2^{j-\ell}}{\lambda_1^{j-\ell}} \|\Gamma_{\lambda,1} (\underline{\Lambda}_1 / \lambda_1) \varphi_1\|_2 \\
&\leq \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^{\ell} \frac{\|\Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda}\|_2^{j-\ell}}{\lambda_1^{j-\ell}} \left\| \frac{\underline{\Lambda}_1}{\lambda_1} \right\|_2 \|\varphi_1\|_2 \quad \left(\text{since } \bar{\Sigma}_u = \bar{\Sigma}_X - \Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda} \text{ and since} \right. \\
&\quad \left. \|\Gamma_{\lambda,1} (\underline{\Lambda}_1 / \lambda_1) \varphi_1\|_2 \leq \|\Gamma_{\lambda,1}\|_2 \left\| \frac{\underline{\Lambda}_1}{\lambda_1} \right\|_2 \|\varphi_1\|_2 = \left\| \frac{\underline{\Lambda}_1}{\lambda_1} \right\|_2 \|\varphi_1\|_2 \text{ given that } \|\Gamma_{\lambda,1}\|_2 = 1 \right) \\
&= \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^{\ell} \frac{\|\Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda}\|_2^{j-\ell}}{\underline{\lambda}_{1,1}^{j-\ell}} \left\| \frac{\underline{\Lambda}_1}{\underline{\lambda}_{1,1}} \right\|_2 \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right)^{j-\ell+1} \|\varphi_1\|_2 \\
&\leq (c^*)^{\kappa} \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^{\ell} \frac{\|\Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda}\|_2^{j-\ell}}{\underline{\lambda}_{1,1}^{j-\ell}} \left\| \frac{\underline{\Lambda}_1}{\underline{\lambda}_{1,1}} \right\|_2 \|\varphi_1\|_2 \\
&\quad \left(\text{since Lemma D-3(a) implies that there exists a constant } c^* > 1 \text{ such that } \frac{\underline{\lambda}_{1,1}}{\lambda_1} \leq c^* < \infty \right. \\
&\quad \left. \text{for all } N, T \text{ sufficiently large and since, for } \kappa \geq 2, j - \ell + 1 < \kappa \text{ for all } j \leq \kappa - 1 \text{ and } \ell \geq 1 \right) \\
&\leq C^{\kappa} (c^*)^{\kappa} \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^{\ell} \|\varphi_1\|_2 \quad \left(\text{given that } \left\| \frac{\underline{\Lambda}_1}{\underline{\lambda}_{1,1}} \right\|_2 = 1 \text{ since } \|\underline{\Lambda}_1\|_2 = \lambda_{\max}(\underline{\Lambda}_1) = \underline{\lambda}_{1,1} \right) \\
&\quad \left(\text{given also that } \|\Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda}\|_2 = \sqrt{(\lambda_{\max} \{ \Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda} \Gamma_{\lambda} \underline{\Lambda} \Gamma'_{\lambda} \})} = \sqrt{\lambda_{\max} \{ \underline{\Lambda}^2 \}} = \max \{ \underline{\lambda}_{1,1}, \underline{\lambda}_{2,1} \} \right. \\
&\quad \left. \leq C \underline{\lambda}_{1,1} \text{ for some constant } C > 1 \text{ and for all } N, T \text{ sufficiently large since } \underline{\lambda}_{1,1} \asymp \underline{\lambda}_{2,1} \right) \\
&\leq C^{\kappa} (c^*)^{\kappa} \sqrt{\kappa \bar{c}} \sum_{\ell=1}^j \binom{j}{\ell} \left\| \frac{\bar{\Sigma}_u}{\lambda_1} \right\|_2^{\ell} \quad \left(\text{since, by Assumption S1-1, there exists a constant } \bar{c} \text{ such that} \right. \\
&\quad \left. \bar{c} \geq \max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \text{ so that } \sqrt{\kappa \bar{c}} \geq \sqrt{\kappa \left(\max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \right)^2} \geq \sqrt{\|\varphi_1\|_2^2} = \|\varphi_1\|_2 \right)
\end{aligned}$$

$$\begin{aligned}
&= C^\kappa (c^*)^\kappa \sqrt{\kappa \bar{c}} \sum_{\ell=1}^j \binom{j}{\ell} \left(\frac{\lambda_{\max}(\bar{\Sigma}_u)}{\lambda_1} \right)^\ell \quad (\text{by symmetry and positive semidefiniteness of } \bar{\Sigma}_u, \\
&\quad \text{so that } \|\bar{\Sigma}_u\|_2 = \sqrt{\lambda_{\max}(\bar{\Sigma}_u' \bar{\Sigma}_u)} = \sqrt{[\lambda_{\max}(\bar{\Sigma}_u)]^2} = \lambda_{\max}(\bar{\Sigma}_u)) \\
&\leq C^\kappa (c^*)^\kappa \sqrt{\kappa \bar{c}} \sum_{\ell=1}^j \binom{j}{\ell} \left(\frac{\bar{C}}{\lambda_1} \right)^\ell \quad \text{for all } N \text{ and } T \text{ sufficiently large} \\
&\quad (\text{since } \lambda_{\max}(\bar{\Sigma}_u) \leq \bar{C} < \infty \text{ for some constant } \bar{C} > 0 \text{ by Assumption (ii) and Lemma D-2}) \\
&\leq C^\kappa (c^*)^\kappa \sqrt{\kappa \bar{c}} \left(\frac{\bar{C}}{\lambda_1} \right) \sum_{\ell=1}^j \binom{j}{\ell} \left(\text{since } \left(\frac{\bar{C}}{\lambda_1} \right)^\ell \leq \frac{\bar{C}}{\lambda_1} \text{ for all integer } \ell \geq 2 \text{ and for all} \right. \\
&\quad \left. N, T \text{ sufficiently large given that } \lambda_1 \rightarrow \infty \text{ as } N, T \rightarrow \infty \right) \\
&\leq C^\kappa (c^*)^\kappa \sqrt{\kappa \bar{c}} (\kappa - 1) ((\kappa - 1)!) \left(\frac{\bar{C}}{\lambda_1} \right) \\
&= O\left(\frac{1}{\lambda_1}\right)
\end{aligned}$$

Making use of this result, we see that

$$\begin{aligned}
&\left\| \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda, \kappa}^{-1} - \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G_W D_{\lambda, \kappa}^{-1} \right\|_2^2 \\
&\leq \left\| \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda, \kappa}^{-1} - \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G_W D_{\lambda, \kappa}^{-1} \right\|_F^2 \\
&= \text{tr} \left\{ \left(\sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda, \kappa}^{-1} - \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G_W D_{\lambda, \kappa}^{-1} \right)' \left(\sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda, \kappa}^{-1} - \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G_W D_{\lambda, \kappa}^{-1} \right) \right\} \\
&= \sum_{j=1}^{\kappa-1} \left\| \frac{[\bar{\Sigma}_X^j - (\Gamma_{\lambda,1} \underline{\Lambda}_1 \Gamma'_{\lambda,1})^j] \Gamma_{\lambda,1} (\underline{\Lambda}_1 / \underline{\lambda}_{1,1}) \varphi_1}{\lambda_1^j} \right\|_2^2 \\
&= O\left(\frac{1}{\lambda_1^2}\right),
\end{aligned}$$

from which it follows that

$$\left\| \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda, \kappa}^{-1} - \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G_W D_{\lambda, \kappa}^{-1} \right\|_2 = O\left(\frac{1}{\lambda_1}\right). \quad (170)$$

Moreover, note that, from expression (169), we see that

$$\begin{aligned}
& \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G_W D_{\lambda,\kappa}^{-1} \\
&= \begin{pmatrix} \Gamma_{\lambda,1} (\underline{\Delta}_1/\lambda_1) \varphi_1 & (\Gamma_{\lambda,1} \underline{\Delta}_1 \Gamma'_{\lambda,1}) \Gamma_{\lambda,1} (\underline{\Delta}_1/\lambda_1) \varphi_1/\lambda_1 & \cdots \\ \cdots & \cdots & (\Gamma_{\lambda,1} \underline{\Delta}_1 \Gamma'_{\lambda,1})^{\kappa-1} \Gamma_{\lambda,1} (\underline{\Delta}_1/\lambda_1) \varphi_1/\lambda_1^{\kappa-1} \end{pmatrix} \\
&= \begin{pmatrix} \Gamma_{\lambda,1} \underline{\Delta}_1 \varphi_1/\lambda_1 & \Gamma_{\lambda,1} \underline{\Delta}_1^2 \varphi_1/\lambda_1^2 & \cdots & \Gamma_{\lambda,1} \underline{\Delta}_1^\kappa \varphi_1/\lambda_1^\kappa \end{pmatrix} \\
&= \Gamma_{\lambda,1} \begin{pmatrix} \underline{\Delta}_{1,1} \varphi_{1,1}/\lambda_1 & \underline{\Delta}_{1,1}^2 \varphi_{1,1}/\lambda_1^2 & \cdots & \underline{\Delta}_{1,1}^\kappa \varphi_{1,1}/\lambda_1^\kappa \\ \underline{\Delta}_{1,2} \varphi_{1,2}/\lambda_1 & \underline{\Delta}_{1,2}^2 \varphi_{1,2}/\lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^\kappa \varphi_{1,2}/\lambda_1^\kappa \\ \vdots & \vdots & & \vdots \\ \underline{\Delta}_{1,\kappa} \varphi_{1,\kappa}/\lambda_1 & \underline{\Delta}_{1,\kappa}^2 \varphi_{1,\kappa}/\lambda_1^2 & \cdots & \underline{\Delta}_{1,\kappa}^\kappa \varphi_{1,\kappa}/\lambda_1^\kappa \end{pmatrix} \\
&= \Gamma_{\lambda,1} \begin{pmatrix} \underline{\Delta}_{1,1} \varphi_{1,1}/\lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2} \varphi_{1,2}/\lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,\kappa} \varphi_{1,\kappa}/\lambda_1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & \underline{\Delta}_{1,1}/\lambda_1 & \underline{\Delta}_{1,1}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,1}^{\kappa-1}/\lambda_1^{\kappa-1} \\ 1 & \underline{\Delta}_{1,2}/\lambda_1 & \underline{\Delta}_{1,2}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^{\kappa-1}/\lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\Delta}_{1,\kappa}/\lambda_1 & \underline{\Delta}_{1,\kappa}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,\kappa}^{\kappa-1}/\lambda_1^{\kappa-1} \end{pmatrix} \\
&= \Gamma_{\lambda,1} D_{\varphi,\underline{\Delta}}^* \underline{V}
\end{aligned}$$

where

$$\begin{aligned}
D_{\varphi,\lambda}^* &= \begin{pmatrix} \underline{\Delta}_{1,1} \varphi_{1,1}/\lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\Delta}_{1,2} \varphi_{1,2}/\lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\Delta}_{1,\kappa} \varphi_{1,\kappa}/\lambda_1 \end{pmatrix}, \\
\underline{V} &= \begin{pmatrix} 1 & \underline{\Delta}_{1,1}/\lambda_1 & \underline{\Delta}_{1,1}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,1}^{\kappa-1}/\lambda_1^{\kappa-1} \\ 1 & \underline{\Delta}_{1,2}/\lambda_1 & \underline{\Delta}_{1,2}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,2}^{\kappa-1}/\lambda_1^{\kappa-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \underline{\Delta}_{1,\kappa}/\lambda_1 & \underline{\Delta}_{1,\kappa}^2/\lambda_1^2 & \cdots & \underline{\Delta}_{1,\kappa}^{\kappa-1}/\lambda_1^{\kappa-1} \end{pmatrix}.
\end{aligned}$$

Hence, the result given in expression (170) can be restated as

$$\left\| \sqrt{\frac{\lambda_{1,1}}{\lambda_1}} G D_{\lambda,\kappa}^{-1} - \Gamma_{\lambda,1} D_{\varphi,\lambda}^* \underline{V} \right\|_2 = O\left(\frac{1}{\lambda_1}\right).$$

Finally, to show part (d), note first that $\underline{V}' D_{\varphi, \lambda} \underline{V}$ is clearly a symmetric matrix. Moreover, expression (162) given in the proof of part (b) of Lemma E-2 further shows that there exists a positive constant $C_{\#}$ such that

$$\lambda_{\min} (\underline{V}' D_{\varphi, \lambda} \underline{V}) \geq C_{\#} > 0$$

for all N and T sufficiently large, so that $\underline{V}' D_{\varphi, \lambda} \underline{V}$ is also positive definite and, thus, non-singular for all N and T sufficiently large. Making use of these properties of the matrix $\underline{V}' D_{\varphi, \lambda} \underline{V}$, we see that for N and T sufficiently large

$$\begin{aligned} \left\| (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} \right\|_2 &= \sqrt{\lambda_{\max} \left\{ \left[(\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} \right]' (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} \right\}} \\ &= \sqrt{\lambda_{\max} \left\{ [(\underline{V}' D_{\varphi, \lambda} \underline{V})']^{-1} (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} \right\}} \\ &\quad \text{(since we can interchange the transposition and the inverse operations)} \\ &= \sqrt{\lambda_{\max} \left\{ (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} \right\}} \\ &\quad \text{(since } \underline{V}' D_{\varphi, \lambda} \underline{V} \text{ is a symmetric matrix)} \\ &= \sqrt{\left[\lambda_{\max} \left\{ (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} \right\} \right]^2} \\ &= \sqrt{\left(\frac{1}{\lambda_{\min} (\underline{V}' D_{\varphi, \lambda} \underline{V})} \right)^2} \\ &= \frac{1}{\lambda_{\min} (\underline{V}' D_{\varphi, \lambda} \underline{V})} \\ &\leq \frac{1}{C_{\#}} \quad \text{(given that } \lambda_{\min} (\underline{V}' D_{\varphi, \lambda} \underline{V}) \geq C_{\#}) \\ &< \infty \quad \text{(since } C_{\#} > 0) \end{aligned}$$

Next, define

$$E^{\dagger} = \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} - \underline{V}' D_{\varphi, \lambda} \underline{V}$$

and applying the result given in expression (161) in the proof of part (b) of Lemma E-2, we

have

$$\begin{aligned}
\left\| (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} E^\dagger \right\|_2 &\leq \left\| (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} \right\|_2 \|E^\dagger\|_2 \\
&\leq \frac{1}{C_\#} \left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\underline{\lambda}, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\underline{\lambda}, \kappa}^{-1}}{\lambda_1} - \underline{V}' D_{\varphi, \lambda} \underline{V} \right\|_2 \\
&= O(1) O\left(\frac{1}{\lambda_1}\right) \\
&= O\left(\frac{1}{\lambda_1}\right) = o(1)
\end{aligned}$$

Also, part (b) of Lemma E-2 gives a result showing that there exists a positive constant $C_\#$ such that

$$\lambda_{\min} \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\underline{\lambda}, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\underline{\lambda}, \kappa}^{-1}}{\lambda_1} \right) \geq C_\# > 0 \text{ for all } N, T \text{ sufficiently large,}$$

so that $(\underline{\lambda}_{1,1}/\lambda_1) D_{\underline{\lambda}, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\underline{\lambda}, \kappa}^{-1}/\lambda_1$ is positive definite and, thus, nonsingular for all N and T sufficiently large. Hence, applying the last inequality given in part (d) of Lemma B-1, we further deduce that

$$\begin{aligned}
\left\| \left(\left[\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right] \frac{D_{\underline{\lambda}, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\underline{\lambda}, \kappa}^{-1}}{\lambda_1} \right)^{-1} - (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} \right\|_2 &\leq \left\| (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} \right\|_2 \frac{\left\| (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} E^\dagger \right\|_2}{1 - \left\| (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} E^\dagger \right\|_2} \\
&\leq \frac{1}{C_\#} \frac{\left\| (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} E^\dagger \right\|_2}{1 - \left\| (\underline{V}' D_{\varphi, \lambda} \underline{V})^{-1} E^\dagger \right\|_2} \\
&= O(1) O\left(\frac{1}{\lambda_1}\right) \\
&= O\left(\frac{1}{\lambda_1}\right). \square
\end{aligned}$$

Lemma E-5: Suppose that Assumptions (i)-(iii) hold. In addition, suppose that Assumptions S2-1 and S2-2 hold under Scenario 2. Under these conditions, the following statements are true.

(a)

$$\frac{D_{\lambda, K}^{-1} G'_{W, K} [\Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda] \Gamma_\lambda (\varphi / \sqrt{\underline{\lambda}_{1,1}})}{\lambda_1^{\frac{1}{2}}} = \underline{V}' w_\varphi,$$

where

$$D_{\lambda,K} = \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{(K-\frac{1}{2})} \end{pmatrix}, \underline{w}_\varphi = \begin{pmatrix} \underline{\lambda}_{1,1}^2 \varphi_1^2 / \lambda_1^2 \\ \underline{\lambda}_{1,2}^2 \varphi_2^2 / \lambda_1^2 \\ \vdots \\ \underline{\lambda}_{1,K}^2 \varphi_K^2 / \lambda_1^2 \end{pmatrix},$$

$$\underline{V} = \begin{pmatrix} 1 & \underline{\lambda}_{1,1}/\lambda_1 & \underline{\lambda}_{1,1}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,1}^{K-1}/\lambda_1^{K-1} \\ 1 & \underline{\lambda}_{1,2}/\lambda_1 & \underline{\lambda}_{1,2}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,2}^{K-1}/\lambda_1^{K-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \underline{\lambda}_{1,K}/\lambda_1 & \underline{\lambda}_{1,K}^2/\lambda_1^2 & \cdots & \underline{\lambda}_{1,K}^{K-1}/\lambda_1^{K-1} \end{pmatrix}$$

where

$$G_{W,K} = \begin{pmatrix} \frac{\Gamma_\lambda \underline{\Lambda} \varphi}{\sqrt{\underline{\lambda}_{1,1}}} & [\Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda] \frac{\Gamma_\lambda \underline{\Lambda} \varphi}{\sqrt{\underline{\lambda}_{1,1}}} & \cdots & [\Gamma_\lambda \underline{\Lambda} \Gamma'_\lambda]^{K-1} \frac{\Gamma_\lambda \underline{\Lambda} \varphi}{\sqrt{\underline{\lambda}_{1,1}}} \end{pmatrix}$$

$$= \begin{pmatrix} \Gamma_\lambda \underline{\Lambda} \varphi / \sqrt{\underline{\lambda}_{1,1}} & \bar{\Sigma}_W \Gamma_\lambda \underline{\Lambda} \varphi / \sqrt{\underline{\lambda}_{1,1}} & \cdots & \bar{\Sigma}_W^{K-1} \Gamma_\lambda \underline{\Lambda} \varphi / \sqrt{\underline{\lambda}_{1,1}} \end{pmatrix}$$

and where

$$\Gamma_\lambda = \frac{\Gamma}{\sqrt{\underline{\lambda}_{1,1}}}, \underline{\Lambda} = \text{diag}(\underline{\lambda}_{1,1}, \dots, \underline{\lambda}_{1,K}), \text{ and } \varphi = (\varphi_{1,1}, \dots, \varphi_{1,K})'.$$

(b)

$$\left\| \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \frac{D_{\lambda,K}^{-1} G' \bar{\sigma}_{Xy}}{\lambda_1^{\frac{1}{2}}} - \underline{V}' \underline{w}_\varphi \right\|_2 = O\left(\frac{1}{\lambda_1}\right).$$

where

$$G = \begin{pmatrix} \bar{\sigma}_{Xy} & \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\Sigma}_X^{K-1} \bar{\sigma}_{Xy} \end{pmatrix}$$

(c)

$$\left\| \sqrt{\frac{\underline{\lambda}_{1,1}}{\lambda_1}} G D_{\lambda,K}^{-1} - \Gamma_\lambda D_{\varphi,\lambda}^* \underline{V} \right\|_2 = O\left(\frac{1}{\lambda_1}\right),$$

where

$$D_{\varphi,\lambda}^* = \begin{pmatrix} \underline{\lambda}_{1,1} \varphi_1 / \lambda_1 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2} \varphi_2 / \lambda_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,K} \varphi_K / \lambda_1 \end{pmatrix}.$$

(d)

$$\left\| \left(\left[\frac{\lambda_{1,1}}{\lambda_1} \right] \frac{D_{\underline{\lambda},K}^{-1} G' \bar{\Sigma}_X G D_{\underline{\lambda},K}^{-1}}{\lambda_1} \right)^{-1} - (\underline{V}' D_{\varphi,\lambda} \underline{V})^{-1} \right\|_2 = O\left(\frac{1}{\lambda_1}\right),$$

where $\underline{V}$ is the Vandermonde matrix defined in part (a) above and where

$$D_{\varphi,\lambda} = \begin{pmatrix} \underline{\lambda}_{1,1}^3 \varphi_{1,1}^2 / \lambda_1^3 & 0 & \cdots & 0 \\ 0 & \underline{\lambda}_{1,2}^3 \varphi_{1,2}^2 / \lambda_1^3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \underline{\lambda}_{1,K}^3 \varphi_{1,K}^2 / \lambda_1^3 \end{pmatrix}.$$

Proof of Lemma E-5: The results given in parts (a), (b), (c), and (d) of this lemma can be shown by using the same argument as that given, respectively, in the proof of parts (a), (b), (c), and (d) of Lemma E-4, except that we apply Assumptions S2-1 and S2-2 instead of Assumptions S1-1 and S1-2 and change the notations used in the proof of Lemma E-4 by setting

$$\Gamma_{\lambda,1} = \Gamma_\lambda = \frac{\Gamma}{\sqrt{\underline{\lambda}_{1,1}}} \text{ and } \underline{\Lambda}_1 = \text{diag}(\lambda_{1,1}, \dots, \lambda_{1,K}) = \underline{\Lambda}$$

(noting that there are no $\Gamma_{\lambda,2}$ and $\underline{\Lambda}_2$ in this case)

since, under Scenario 2 considered here, all K available factors are relevant for the purpose of forecasting and there is no reason to split the nonzero eigenvalues of the common component covariance matrix $\Gamma \bar{\Sigma}_F \Gamma'$ and their associated eigenvectors into two groups as all the columns of the $N \times K$ semi-orthogonal matrix $\Gamma / \sqrt{\underline{\lambda}_{1,1}}$ are needed to span the envelope. In this case, we also take $\varphi_1 = (\varphi_{1,1}, \dots, \varphi_{1,K})' = \varphi$ since none of the coefficients is equal to zero. $\square$

Lemma E-6: Suppose that Assumption S1-1 holds under Scenario 1. In addition, suppose that Assumptions (iv) and (v) hold, and let γ_1 and b_1 be the positive constants defined in Assumption (v). Moreover, let $\bar{b}_2$ be some positive constant such that $\bar{b}_2 \geq \max \left\{ b_1 K \bar{c} [1 + \ln K]^{\frac{1}{\gamma_1}}, b_1 K c^\dagger [1 + \ln K]^{\frac{1}{\gamma_1}} \right\} \geq b_1$, where $\bar{c} > 1$ and $c^\dagger > 1$ are the constants defined in Assumption S1-1 and Assumption (iv). Under these conditions, for $r = \gamma_1/2 > 0$, for some constant $\bar{b} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \bar{b}_2^2 > 0$, and for all $s > 0$; the following statements are true.

(a) There exists a positive constant $\bar{C} \geq \exp\{1\}$ such that, for all $t \in \mathbb{Z}$,

$$P \left(\left| (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\}.$$

(b) There exists a positive constant $\bar{C} \geq \exp\{1\}$ such that, for all integer i with $1 \leq i \leq N$

and for all $t \in \mathbb{Z}$,

$$P \left(\left| e'_{i,N} \Gamma(F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\}.$$

- (c) There exists a positive constant $\bar{C} \geq \exp \{1\}$ such that, for all integer i with $1 \leq i \leq N$ and for all $t \in \mathbb{Z}$,

$$P \left(\left| u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\}.$$

Proof of Lemma E-6: To show part (a), note first that, for any $s > 0$,

$$\begin{aligned} & P \left(\left| (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) \\ = & P \left(\left| \sum_{g=1}^{\kappa} (F_{1,t,g} - \mu_{F,1,g}) \varphi_{1,g} \right| > s \right) \\ & \text{(where } F_{1,t,g} \text{ and } \mu_{F,1,g} \text{ are, respectively, the } g^{th} \text{ element of } F_{1,t} \text{ and } \mu_{F,1}) \\ \leq & P \left(\sum_{g=1}^{\kappa} |(F_{1,t,g} - \mu_{F,1,g}) \varphi_{1,g}| > s \right) \\ \leq & P \left(\sum_{g=1}^{\kappa} |F_{1,t,g} - \mu_{F,1,g}| > \frac{s}{\max_{1 \leq g \leq \kappa} |\varphi_{1,g}|} \right) \\ \leq & P \left(\sum_{g=1}^{\kappa} |F_{1,t,g} - \mu_{F,1,g}| > \frac{s}{\bar{c}} \right) \quad \text{(given Assumption S1-1)} \\ \leq & P \left(\bigcup_{g=1}^{\kappa} \left\{ |F_{1,t,g} - \mu_{F,1,g}| > \frac{s}{\kappa \bar{c}} \right\} \right) \\ \leq & \sum_{g=1}^{\kappa} P \left(|F_{1,t,g} - \mu_{F,1,g}| > \frac{s}{\kappa \bar{c}} \right) \\ \leq & \kappa \exp \left\{ 1 - \left(\frac{s}{b_1 \kappa \bar{c}} \right)^{\gamma_1} \right\} \end{aligned}$$

Next, note that $\bar{b}_2 \geq \max \left\{ b_1 K \bar{c} [1 + \ln K]^{\frac{1}{\gamma_1}}, b_1 K c^\dagger [1 + \ln K]^{\frac{1}{\gamma_1}} \right\} \geq b_1$ given that $K \geq 1$,

$\bar{c} > 1$, and $c^\dagger > 1$. Hence, if $s > \bar{b}_2$, we further obtain

$$\begin{aligned}
& P \left(\left| (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) \\
& \leq \kappa \exp \left\{ 1 - \left(\frac{s}{b_1 \kappa \bar{c}} \right)^{\gamma_1} \right\} \\
& = \kappa \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \left(\frac{\bar{b}_2}{b_1 \kappa \bar{c}} \right)^{\gamma_1} \right\} \\
& \leq \kappa \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} [1 + \ln \kappa]^{\frac{1}{\gamma_1}} \right)^{\gamma_1} \right\} \\
& \quad \left(\text{since } \bar{b}_2 \geq b_1 K \bar{c} [1 + \ln K]^{\frac{1}{\gamma_1}} \geq b_1 \kappa \bar{c} [1 + \ln \kappa]^{\frac{1}{\gamma_1}} \text{ given that } \kappa < K \text{ in this case} \right) \\
& = \kappa \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} [1 + \ln \kappa] \right\} \\
& \leq \kappa \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} - \ln \kappa \right\} \quad \left(\text{since } \frac{s}{\bar{b}_2} > 1 \text{ and } \frac{\gamma_1}{2} > 0 \right) \\
& = \frac{\kappa}{\kappa} \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
& = \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\}.
\end{aligned}$$

On the other hand, when $s \leq \bar{b}_2$, we have

$$\begin{aligned}
P \left(\left| (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) & \leq 1 \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
& = \exp \{1\} \exp \left\{ - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
& \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \quad (\text{since } \bar{C} \geq \exp \{1\}).
\end{aligned}$$

To show part (b), note that, by the results given in part (a) of this lemma and part (a) of Lemma E-8, there exist constants $\gamma_1 > 0$ and $\bar{b}_2 > 0$, such that for any $s > 0$ and for any integer i with $1 \leq i \leq N$,

$$\begin{aligned}
P \left(\left| (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) & \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
P \left(\left| e'_{i,N} \Gamma(F_t - \mu_F) \right| > s \right) & \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\}
\end{aligned}$$

Hence, for any $s > 0$ and for any integer i with $1 \leq i \leq N$,

$$\begin{aligned}
& P \left(\left| e'_{i,N} \Gamma (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) \\
&= P \left(\left\{ \left| e'_{i,N} \Gamma (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right\} \cap \left\{ \left| (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \leq \sqrt{s} \right\} \right) \\
&\quad + P \left(\left\{ \left| e'_{i,N} \Gamma (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right\} \cap \left\{ \left| (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > \sqrt{s} \right\} \right) \\
&\leq P \left(\sqrt{s} \left| e'_{i,N} \Gamma (F_t - \mu_F) \right| > s \right) \\
&\quad + P \left(\left\{ \left| e'_{i,N} \Gamma (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right\} \cap \left\{ \left| (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > \sqrt{s} \right\} \right) \\
&\leq P \left(\left| e'_{i,N} \Gamma (F_t - \mu_F) \right| > \sqrt{s} \right) + P \left(\left| (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > \sqrt{s} \right) \\
&\leq \exp \left\{ 1 - \left(\frac{\sqrt{s}}{\bar{b}_2} \right)^{\gamma_1} \right\} + \exp \left\{ 1 - \left(\frac{\sqrt{s}}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{\bar{b}_2} \right)^{\gamma_1} \right\}
\end{aligned}$$

Moreover, since, by definition, $\bar{b} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \bar{b}_2^2$, it follows that

$$\frac{\bar{b}}{\bar{b}_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \geq 1$$

This, in turn, implies that when $s > \bar{b}$ that

$$\begin{aligned}
& P \left(\left| e'_{i,N} \Gamma (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) \\
& \leq 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
& = 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
& = 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \left(\frac{\bar{b}}{\bar{b}_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
& \leq 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} (1 + \ln 2)^{\frac{2}{\gamma_1}} \right)^{\frac{\gamma_1}{2}} \right\} \quad \left(\text{given that } \frac{\bar{b}}{\bar{b}_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \right) \\
& = 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} (1 + \ln 2) \right\} \\
& = 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \ln 2 \right\} \\
& \leq 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} - \ln 2 \right\} \quad \left(\text{since } \frac{s}{\bar{b}} > 1 \text{ and } \frac{\gamma_1}{2} > 0 \right) \\
& = \frac{2}{2} \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \right\} \\
& = \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{by setting } r = \frac{\gamma_1}{2} \right) \\
& \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right)
\end{aligned}$$

On the other hand, when $s \leq \bar{b}$

$$\begin{aligned}
P \left(\left| e'_{i,N} \Gamma (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) & \leq 1 \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \\
& = \exp \{1\} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \\
& \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right)
\end{aligned}$$

which shows part (b).

To show part (c), note that, for any $s > 0$ and for any integer i such that $1 \leq i \leq N$,

$$\begin{aligned}
& P \left(\left| u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) \\
= & P \left(\left\{ \left| u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right\} \cap \{|u_{it}| \leq \sqrt{s}\} \right) \\
& + P \left(\left\{ \left| u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right\} \cap \{|u_{it}| > \sqrt{s}\} \right) \\
& \text{(since } \{|u_{it}| \leq \sqrt{s}\} \text{ is the complement of the event } \{|u_{it}| > \sqrt{s}\}) \\
\leq & P \left(\sqrt{s} \left| (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) + P (|u_{it}| > \sqrt{s}) \\
= & P \left(\left| (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > \sqrt{s} \right) + P (|u_{it}| > \sqrt{s}) \\
= & P \left(\left| (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > \sqrt{s} \right) + P (|u'_t e_{i,N}| > \sqrt{s}) \\
& \text{(noting that, in this case, } \delta = e_{i,N} \text{ and } \|e_{i,N}\|_2 = 1) \\
\leq & \exp \left\{ 1 - \left(\frac{\sqrt{s}}{\bar{b}_2} \right)^{\gamma_1} \right\} + \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_1} \right)^{\gamma_1} \right\} \\
& \text{(by part (a) of this lemma and Assumption (v))} \\
\leq & 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{\bar{b}_2} \right)^{\gamma_1} \right\} \left(\text{since } \bar{b}_2 \geq \max \left\{ b_1 K \bar{c} [1 + \ln K]^{\frac{1}{\gamma_1}}, b_1 K c^\ddagger [1 + \ln K]^{\frac{1}{\gamma_1}} \right\} \geq b_1 \right. \\
& \left. \text{given that } K \geq 1, \bar{c} > 1, \text{ and } c^\ddagger > 1 \right)
\end{aligned}$$

When $s > \bar{b}$, the inequality

$$\frac{\bar{b}}{\bar{b}_2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \geq 1$$

further implies that

$$\begin{aligned}
P\left(\left|u_{it}(F_{1,t} - \mu_{F,1})' \varphi_1\right| > s\right) &\leq 2 \exp\left\{1 - \left(\frac{\sqrt{s}}{\bar{b}_2}\right)^{\gamma_1}\right\} \\
&= 2 \exp\left\{1 - \left(\frac{s}{\bar{b}_2^2}\right)^{\frac{\gamma_1}{2}}\right\} \\
&= 2 \exp\left\{1 - \left(\frac{s}{\bar{b}}\right)^{\frac{\gamma_1}{2}} \left(\frac{\bar{b}}{\bar{b}_2^2}\right)^{\frac{\gamma_1}{2}}\right\} \\
&\leq 2 \exp\left\{1 - \left(\frac{s}{\bar{b}}(1 + \ln 2)^{\frac{2}{\gamma_1}}\right)^{\frac{\gamma_1}{2}}\right\} \\
&\quad \left(\text{given that } \frac{\bar{b}}{\bar{b}_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}}\right) \\
&= 2 \exp\left\{1 - \left(\frac{s}{\bar{b}}\right)^{\frac{\gamma_1}{2}} (1 + \ln 2)\right\} \\
&= 2 \exp\left\{1 - \left(\frac{s}{\bar{b}}\right)^{\frac{\gamma_1}{2}} - \left(\frac{s}{\bar{b}}\right)^{\frac{\gamma_1}{2}} \ln 2\right\} \\
&\leq 2 \exp\left\{1 - \left(\frac{s}{\bar{b}}\right)^{\frac{\gamma_1}{2}} - \ln 2\right\} \quad \left(\text{since } \frac{s}{\bar{b}} > 1 \text{ and } \frac{\gamma_1}{2} > 0\right) \\
&= \frac{2}{2} \exp\left\{1 - \left(\frac{s}{\bar{b}}\right)^{\frac{\gamma_1}{2}}\right\} \\
&= \exp\left\{1 - \left(\frac{s}{\bar{b}}\right)^r\right\} \quad \left(\text{by setting } r = \frac{\gamma_1}{2}\right) \\
&\leq \bar{C} \exp\left\{-\left(\frac{s}{\bar{b}}\right)^r\right\} \quad \left(\text{since } \bar{C} \geq \exp\{1\}\right)
\end{aligned}$$

On the other hand, when $s \leq \bar{b}$, we have

$$\begin{aligned}
P\left(\left|u_{it}(F_{1,t} - \mu_{F,1})' \varphi_1\right| > s\right) &\leq 1 \leq \exp\left\{1 - \left(\frac{s}{\bar{b}}\right)^r\right\} \\
&= \exp\{1\} \exp\left\{-\left(\frac{s}{\bar{b}}\right)^r\right\} \\
&\leq \bar{C} \exp\left\{-\left(\frac{s}{\bar{b}}\right)^r\right\} \quad \left(\text{since } \bar{C} \geq \exp\{1\}\right),
\end{aligned}$$

which shows part (c). $\square$

Lemma E-7: Suppose that Assumption S2-1 holds under Scenario 2 where $\kappa = K$. In addition, suppose that Assumptions (iv) and (v) hold, and let γ_1 and b_1 be the positive constants defined in Assumption (v). Moreover, let $\bar{b}_2$ be some positive constant such that $\bar{b}_2 \geq \max \left\{ b_1 K \bar{c} [1 + \ln K]^{\frac{1}{\gamma_1}}, b_1 K c^\dagger [1 + \ln K]^{\frac{1}{\gamma_1}} \right\} \geq b_1$, where $\bar{c} > 1$ and $c^\dagger > 1$ are the constants defined in Assumption S2-1 and Assumption (iv). Under these conditions, for $r = \gamma_1/2 > 0$, for some constant $\bar{b} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \bar{b}_2^2 > 0$, and for all $s > 0$; the following statements are true.

- (a) There exists a positive constant $\bar{C} \geq \exp \{1\}$ such that, for all $t \in \mathbb{Z}$,

$$P \left(|(F_t - \mu_F)' \varphi| > s \right) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\},$$

where $\varphi = (\varphi_{1,1}, \dots, \varphi_{1,K})'$ so that all components of φ are nonzero in this case.

- (b) There exists a positive constant $\bar{C} \geq \exp \{1\}$ such that, for all integer i with $1 \leq i \leq N$ and for all $t \in \mathbb{Z}$,

$$P \left(|e'_{i,N} \Gamma(F_t - \mu_F)(F_t - \mu_F)' \varphi| > s \right) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\}.$$

- (c) There exists a positive constant $\bar{C} \geq \exp \{1\}$ such that, for all integer i with $1 \leq i \leq N$ and for all $t \in \mathbb{Z}$,

$$P \left(|u_{it}(F_t - \mu_F)' \varphi| > s \right) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\}.$$

Proof of Lemma E-7: The results given in parts (a), (b), and (c) of this lemma can be shown using the same argument as that given, respectively, in the proof of parts (a), (b), and (c) of Lemma E-6, except that we apply Assumption S2-1 instead of Assumption S1-1 and change the notations used in the proof of Lemma E-6 by setting

$$\kappa = K, F_{1,t} = F_t, \mu_{F,1} = \mu_F, \text{ and } \varphi_1 = \varphi;$$

since, under Scenario 2 considered here, all available factors are relevant for the purpose of forecasting and there is no reason to partition $F_t = (F'_{1,t}, F'_{2,t})'$, $\mu_F = (\mu'_{F,1}, \mu'_{F,2})'$, and $\varphi = (\varphi'_1, \varphi'_2)'$. $\square$

Lemma E-8: Suppose that Assumptions (iv) and (v) hold, and let γ_1 and b_1 be the positive constants defined in Assumption (v). Moreover, let $\bar{b}_2$ be some positive constant such that $\bar{b}_2 \geq \max \left\{ b_1 K \bar{c} [1 + \ln K]^{\frac{1}{\gamma_1}}, b_1 K c^\dagger [1 + \ln K]^{\frac{1}{\gamma_1}} \right\} \geq b_1$, where $c^\dagger > 1$ is a constant previously defined in Assumption (iv) and where $\bar{c} > 1$ is a constant such that $\max_{1 \leq g \leq K} |\varphi_{1,g}| \leq \bar{c}$.

Under these conditions, for $r = \gamma_1/2 > 0$, for some constant $\bar{b} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \bar{b}_2^2 > 0$, and for all $s > 0$; the following statements are true.

- (a) There exists a positive constant $\bar{C} \geq \exp\{1\}$ such that, for all integer i with $1 \leq i \leq N$ and for all $t \in \mathbb{Z}$,

$$P(|e'_{i,N} \Gamma(F_t - \mu_F)| > s) \leq \exp\left\{1 - \left(\frac{s}{\bar{b}_2}\right)^{\gamma_1}\right\} \leq \bar{C} \exp\left\{-\left(\frac{s}{\bar{b}_2}\right)^{\gamma_1}\right\}$$

- (b) There exists a positive constant $\bar{C} \geq \exp\{1\}$ such that, for all integers i and j with $1 \leq i, j \leq N$ and for all $t \in \mathbb{Z}$;

$$P(|e'_{i,N} \Gamma(F_t - \mu_F) (F_t - \mu_F)' \Gamma' e_{j,N}| > s) \leq \exp\left\{1 - \left(\frac{s}{\bar{b}}\right)^r\right\} \leq \bar{C} \exp\left\{-\left(\frac{s}{\bar{b}}\right)^r\right\},$$

- (c) There exists a positive constant $\bar{C} \geq \exp\{1\}$ such that, for all integers i and j with $1 \leq i, j \leq N$ and for all $t \in \mathbb{Z}$;

$$P(|e'_{i,N} \Gamma(F_t - \mu_F) u_{jt}| > s) \leq \exp\left\{1 - \left(\frac{s}{\bar{b}}\right)^r\right\} \leq \bar{C} \exp\left\{-\left(\frac{s}{\bar{b}}\right)^r\right\}$$

- (d) There exists a positive constant $\bar{C} \geq \exp\{1\}$ such that, for all integers i and j with $1 \leq i, j \leq N$ and for all $t \in \mathbb{Z}$;

$$P(|u_{it} u_{jt}| > s) \leq \exp\left\{1 - \left(\frac{s}{\bar{b}}\right)^r\right\} \leq \bar{C} \exp\left\{-\left(\frac{s}{\bar{b}}\right)^r\right\}$$

- (e) Let $h \geq 1$ be a fixed finite integer. There exists a positive constant $\bar{C} \geq \exp\{1\}$ such that, for all integer i with $1 \leq i \leq N$ and for all $t \in \mathbb{Z}$,

$$P(|e'_{i,N} \Gamma(F_t - \mu_F) \varepsilon_{t+h}| > s) \leq \exp\left\{1 - \left(\frac{s}{\bar{b}}\right)^r\right\} \leq \bar{C} \exp\left\{-\left(\frac{s}{\bar{b}}\right)^r\right\}$$

- (f) Let $h \geq 1$ be a fixed finite integer. There exists a positive constant $\bar{C} \geq \exp\{1\}$ such that, for all integer i with $1 \leq i \leq N$ and for all $t \in \mathbb{Z}$,

$$P(|u_{it} \varepsilon_{t+h}| > s) \leq \exp\left\{1 - \left(\frac{s}{\bar{b}}\right)^r\right\} \leq \bar{C} \exp\left\{-\left(\frac{s}{\bar{b}}\right)^r\right\}$$

Proof of Lemma E-8: To show part (a), let $\Gamma_{(i,g)}$ denote the $(i, g)^{th}$ element of Γ and note

that, for any $s > 0$ and for all integer i with $1 \leq i \leq N$

$$\begin{aligned}
& P \left(|e'_{i,N} \Gamma (F_t - \mu_F)| > s \right) \\
&= P \left(\left| \sum_{g=1}^K \Gamma_{(i,g)} (F_{t,g} - \mu_{F,g}) \right| > s \right) \\
&\quad \text{(where } F_{t,g} \text{ and } \mu_{F,g} \text{ denote the } g^{\text{th}} \text{ element of } F_t \text{ and } \mu_F, \text{ respectively)} \\
&\leq P \left(\sum_{g=1}^K |\Gamma_{(i,g)} (F_{t,g} - \mu_{F,g})| > s \right) \\
&\leq P \left(\sum_{g=1}^K |F_{t,g} - \mu_{F,g}| > \frac{s}{\max_{1 \leq i \leq N, 1 \leq g \leq K} |\Gamma_{(i,g)}|} \right) \\
&\leq P \left(\sum_{g=1}^K |F_{t,g} - \mu_{F,g}| > \frac{s}{c^\dagger} \right) \quad \text{(by Assumption (iv))} \\
&\leq P \left(\bigcup_{g=1}^K \left\{ |F_{t,g} - \mu_{F,g}| > \frac{s}{K c^\dagger} \right\} \right) \\
&\leq \sum_{g=1}^K P \left(|F_{t,g} - \mu_{F,g}| > \frac{s}{K c^\dagger} \right) \quad \text{(by Bonferroni inequality)} \\
&\leq K \exp \left\{ 1 - \left(\frac{s}{b_1 K c^\dagger} \right)^{\gamma_1} \right\}
\end{aligned}$$

Note that $\bar{b}_2 \geq \max \left\{ b_1 K \bar{c} [1 + \ln K]^{\frac{1}{\gamma_1}}, b_1 K c^\dagger [1 + \ln K]^{\frac{1}{\gamma_1}} \right\} \geq b_1$ given that $K \geq 1$, $\bar{c} > 1$,

and $c^\dagger > 1$. Hence, if $s > \bar{b}_2$, we further obtain

$$\begin{aligned}
& P(|e'_{i,N}\Gamma(F_t - \mu_F)| > s) \\
& \leq K \exp \left\{ 1 - \left(\frac{s}{b_1 K c^\dagger} \right)^{\gamma_1} \right\} \\
& = K \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \left(\frac{\bar{b}_2}{b_1 K c^\dagger} \right)^{\gamma_1} \right\} \\
& \leq K \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} [1 + \ln K]^{\frac{1}{\gamma_1}} \right)^{\gamma_1} \right\} \quad \left(\text{since } \bar{b}_2 \geq b_1 K c^\dagger [1 + \ln K]^{\frac{1}{\gamma_1}} \right) \\
& = K \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} [1 + \ln K] \right\} \\
& \leq K \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} - \ln K \right\} \quad \left(\text{since } \frac{s}{\bar{b}_2} > 1 \text{ and } \gamma_1 > 0 \right) \\
& = \frac{K}{K} \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
& = \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
& \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right)
\end{aligned}$$

On the other hand, if $s \leq \bar{b}_2$, we have

$$\begin{aligned}
P(|e'_{i,N}\Gamma(F_t - \mu_F)| > s) & \leq 1 \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
& = \exp \{1\} \exp \left\{ - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
& \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right),
\end{aligned}$$

which shows the result for part (a).

To show part (b), note that, by the result given in part (a) of this lemma, there exist constants $\gamma_1 > 0$ and $\bar{b}_2 \geq \max \left\{ b_1 K \bar{c} [1 + \ln K]^{\frac{1}{\gamma_1}}, b_1 K c^\dagger [1 + \ln K]^{\frac{1}{\gamma_1}} \right\} \geq b_1 > 0$, such that $\forall s > 0$, for all integers i and j with $1 \leq i, j \leq N$, and for all $t \in \mathbb{Z}$;

$$P(|e'_{i,N}\Gamma(F_t - \mu_F)| > s) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\},$$

and

$$P(|e'_{j,N}\Gamma(F_t - \mu_F)| > s) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\}$$

It follows that, for any $s > 0$, for all integers i and j with $1 \leq i, j \leq N$, and for all $t \in \mathbb{Z}$;

$$\begin{aligned}
& P \left(|e'_{i,N} \Gamma (F_t - \mu_F) (F_t - \mu_F)' \Gamma' e_{j,N}| > s \right) \\
&= P \left(\left\{ |e'_{i,N} \Gamma (F_t - \mu_F)| |(F_t - \mu_F)' \Gamma' e_{j,N}| > s \right\} \cap \left\{ |(F_t - \mu_F)' \Gamma' e_{j,N}| \leq \sqrt{s} \right\} \right) \\
&\quad + P \left(\left\{ |e'_{i,N} \Gamma (F_t - \mu_F)| |(F_t - \mu_F)' \Gamma' e_{j,N}| > s \right\} \cap \left\{ |(F_t - \mu_F)' \Gamma' e_{j,N}| > \sqrt{s} \right\} \right) \\
&\leq P \left(\sqrt{s} |e'_{i,N} \Gamma (F_t - \mu_F)| > s \right) \\
&\quad + P \left(\left\{ |e'_{i,N} \Gamma (F_t - \mu_F)| |(F_t - \mu_F)' \Gamma' e_{j,N}| > s \right\} \cap \left\{ |(F_t - \mu_F)' \Gamma' e_{j,N}| > \sqrt{s} \right\} \right) \\
&\leq P \left(|e'_{i,N} \Gamma (F_t - \mu_F)| > \sqrt{s} \right) + P \left(|(F_t - \mu_F)' \Gamma' e_{j,N}| > \sqrt{s} \right) \\
&\leq \exp \left\{ 1 - \left(\frac{\sqrt{s}}{\bar{b}_2} \right)^{\gamma_1} \right\} + \exp \left\{ 1 - \left(\frac{\sqrt{s}}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{\bar{b}_2} \right)^{\gamma_1} \right\}.
\end{aligned}$$

Moreover, since, by definition, $\bar{b} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \bar{b}_2^2$, it follows that

$$\frac{\bar{b}}{\bar{b}_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \geq 1$$

This, in turn, implies that, if $s > \bar{b}$,

$$\begin{aligned}
& P \left(|e'_{i,N} \Gamma (F_t - \mu_F) (F_t - \mu_F)' \Gamma' e_{j,N}| > s \right) \\
& \leq 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
& = 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
& = 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \left(\frac{\bar{b}}{\bar{b}_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
& \leq 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} (1 + \ln 2)^{\frac{2}{\gamma_1}} \right)^{\frac{\gamma_1}{2}} \right\} \quad \left(\text{given that } \frac{\bar{b}}{\bar{b}_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \right) \\
& = 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} (1 + \ln 2) \right\} \\
& = 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \ln 2 \right\} \\
& \leq 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} - \ln 2 \right\} \quad \left(\text{since } \frac{s}{\bar{b}} > 1 \text{ and } \frac{\gamma_1}{2} > 0 \right) \\
& = \frac{2}{2} \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \right\} \\
& = \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{by setting } r = \frac{\gamma_1}{2} \right) \\
& \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right)
\end{aligned}$$

On the other hand, if $s \leq \bar{b}$,

$$\begin{aligned}
P \left(|e'_{i,N} \Gamma (F_t - \mu_F) (F_t - \mu_F)' \Gamma' e_{j,N}| > s \right) & \leq 1 \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \\
& = \exp \{1\} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \\
& \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right),
\end{aligned}$$

which shows part (b).

To show part (c). note that, for any $s > 0$, for all integers i and j with $1 \leq i, j \leq N$, and

for all $t \in \mathbb{Z}$;

$$\begin{aligned}
& P(|e'_{i,N}\Gamma(F_t - \mu_F)u_{jt}| > s) \\
= & P(\{|e'_{i,N}\Gamma(F_t - \mu_F)u_{jt}| > s\} \cap \{|u_{jt}| \leq \sqrt{s}\}) \\
& + P(\{|e'_{i,N}\Gamma(F_t - \mu_F)u_{jt}| > s\} \cap \{|u_{jt}| > \sqrt{s}\}) \\
\leq & P(\sqrt{s}|e'_{i,N}\Gamma(F_t - \mu_F)| > s) + P(\{|e'_{i,N}\Gamma(F_t - \mu_F)u_{jt}| > s\} \cap \{|u_{jt}| > \sqrt{s}\}) \\
\leq & P(|e'_{i,N}\Gamma(F_t - \mu_F)| > \sqrt{s}) + P(|u_{jt}| > \sqrt{s}) \\
= & P(|e'_{i,N}\Gamma(F_t - \mu_F)| > \sqrt{s}) + P(|u'_te_{i,N}| > \sqrt{s}) \\
& (\text{noting that, in this case, } \delta = e_{i,N} \text{ and } \|e_{i,N}\|_2 = 1) \\
\leq & \exp\left\{1 - \left(\frac{\sqrt{s}}{\bar{b}_2}\right)^{\gamma_1}\right\} + \exp\left\{1 - \left(\frac{\sqrt{s}}{b_1}\right)^{\gamma_1}\right\} \\
\leq & 2\exp\left\{1 - \left(\frac{\sqrt{s}}{\bar{b}_2}\right)^{\gamma_1}\right\} \quad \left(\text{since } \bar{b}_2 \geq \max\left\{b_1K\bar{c}[1 + \ln K]^{\frac{1}{\gamma_1}}, b_1Kc^\dagger[1 + \ln K]^{\frac{1}{\gamma_1}}\right\} \geq b_1\right. \\
& \left.\text{given that } K \geq 1, \bar{c} > 1, \text{ and } c^\dagger > 1\right).
\end{aligned}$$

If $s > \bar{b}$, the inequality

$$\frac{\bar{b}}{\bar{b}_2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \geq 1$$

further implies that

$$\begin{aligned}
P(|e'_{i,N}\Gamma(F_t - \mu_F)u_{jt}| > s) &\leq 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \left(\frac{\bar{b}}{\bar{b}_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} (1 + \ln 2)^{\frac{2}{\gamma_1}} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\quad \left(\text{given that } \frac{\bar{b}}{\bar{b}_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \right) \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} (1 + \ln 2) \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \ln 2 \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} - \ln 2 \right\} \quad \left(\text{since } \frac{s}{\bar{b}} > 1 \text{ and } \frac{\gamma_1}{2} > 0 \right) \\
&= \frac{2}{2} \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{by setting } r = \frac{\gamma_1}{2} \right) \\
&\leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right)
\end{aligned}$$

On the other hand, if $s \leq \bar{b}$, we have

$$\begin{aligned}
P(|e'_{i,N}\Gamma(F_t - \mu_F)u_{jt}| > s) &\leq 1 \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \\
&= \exp \{1\} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \\
&\leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right).
\end{aligned}$$

To show part (d), note that, for any $s > 0$, for all integers i and j with $1 \leq i, j \leq N$, and for all $t \in \mathbb{Z}$;

$$\begin{aligned}
& P(|u_{it}u_{jt}| > s) \\
&= P(\{|u_{it}u_{jt}| > s\} \cap \{|u_{jt}| \leq \sqrt{s}\} \cup \{|u_{it}u_{jt}| > s\} \cap \{|u_{jt}| > \sqrt{s}\}) \\
&= P(\{|u_{it}u_{jt}| > s\} \cap \{|u_{jt}| \leq \sqrt{s}\}) + P(\{|u_{it}u_{jt}| > s\} \cap \{|u_{jt}| > \sqrt{s}\}) \\
&\leq P(\sqrt{s}|u_{it}| > s) + P(\{|u_{it}u_{jt}| > s\} \cap \{|u_{jt}| > \sqrt{s}\}) \\
&\leq P(|u_{it}| > \sqrt{s}) + P(|u_{jt}| > \sqrt{s}) \\
&= P(|u'_te_{i,N}| > \sqrt{s}) + P(|u'_te_{j,N}| > \sqrt{s}) \quad (\text{noting that } \|e_{i,N}\|_2 = 1 \text{ and } \|e_{j,N}\|_2 = 1) \\
&\leq \exp\left\{1 - \left(\frac{\sqrt{s}}{b_1}\right)^{\gamma_1}\right\} + \exp\left\{1 - \left(\frac{\sqrt{s}}{b_1}\right)^{\gamma_1}\right\} \\
&\leq 2\exp\left\{1 - \left(\frac{\sqrt{s}}{b_1}\right)^{\gamma_1}\right\}.
\end{aligned}$$

If $s > \bar{b}$, the inequality

$$\frac{\bar{b}}{b_1^2} \geq \frac{\bar{b}}{\bar{b}_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \geq 1 \quad (\text{given that } \bar{b}_2 \geq b_1)$$

further implies that

$$\begin{aligned}
P(|u_{it}u_{jt}| > s) &\leq 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_1} \right)^{\gamma_1} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b_1^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \left(\frac{\bar{b}}{b_1^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} (1 + \ln 2)^{\frac{2}{\gamma_1}} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\quad \left(\text{given that } \frac{\bar{b}}{b_1^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \right) \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} (1 + \ln 2) \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \ln 2 \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} - \ln 2 \right\} \quad \left(\text{since } \frac{s}{\bar{b}} > 1 \text{ and } \frac{\gamma_1}{2} > 0 \right) \\
&= \frac{2}{2} \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{by setting } r = \frac{\gamma_1}{2} \right) \\
&\leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right)
\end{aligned}$$

On the other hand, if $s \leq \bar{b}$, we have

$$\begin{aligned}
P(|u_{it}u_{jt}| > s) &\leq 1 \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \\
&= \exp \{1\} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \\
&\leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right).
\end{aligned}$$

To show part (e), note that for any $s > 0$, for all integer i such that $1 \leq i \leq N$, and for

all $t \in \mathbb{Z}$,

$$\begin{aligned}
& P(|e'_{i,N}\Gamma(F_t - \mu_F)\varepsilon_{t+h}| > s) \\
= & P(\{|e'_{i,N}\Gamma(F_t - \mu_F)\varepsilon_{t+h}| > s\} \cap \{|\varepsilon_{t+h}| \leq \sqrt{s}\}) \\
& + P(\{|e'_{i,N}\Gamma(F_t - \mu_F)\varepsilon_{t+h}| > s\} \cap \{|\varepsilon_{t+h}| > \sqrt{s}\}) \\
\leq & P(\sqrt{s}|e'_{i,N}\Gamma(F_t - \mu_F)| > s) + P(\{|e'_{i,N}\Gamma(F_t - \mu_F)\varepsilon_{t+h}| > s\} \cap \{|\varepsilon_{t+h}| > \sqrt{s}\}) \\
\leq & P(|e'_{i,N}\Gamma(F_t - \mu_F)| > \sqrt{s}) + P(|\varepsilon_{t+h}| > \sqrt{s}) \\
\leq & \exp\left\{1 - \left(\frac{\sqrt{s}}{\bar{b}_2}\right)^{\gamma_1}\right\} + \exp\left\{1 - \left(\frac{\sqrt{s}}{b_1}\right)^{\gamma_1}\right\} \\
\leq & 2 \exp\left\{1 - \left(\frac{\sqrt{s}}{\bar{b}_2}\right)^{\gamma_1}\right\} \quad \left(\text{since } \bar{b}_2 \geq \max\left\{b_1 K \bar{c} [1 + \ln K]^{\frac{1}{\gamma_1}}, b_1 K c^\dagger [1 + \ln K]^{\frac{1}{\gamma_1}}\right\} \geq b_1\right. \\
& \quad \left.\text{given that } K \geq 1, \bar{c} > 1, \text{ and } c^\dagger > 1\right)
\end{aligned}$$

Since, by definition, $\bar{b} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \bar{b}_2^2$, it follows that if $s > \bar{b}$, the inequality

$$\frac{\bar{b}}{\bar{b}_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \geq 1$$

further implies that

$$\begin{aligned}
P(|e'_{i,N}\Gamma(F_t - \mu_F)\varepsilon_{t+h}| > s) &\leq 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{\bar{b}_2} \right)^{\gamma_1} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \left(\frac{\bar{b}}{\bar{b}_2^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} (1 + \ln 2)^{\frac{2}{\gamma_1}} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\quad \left(\text{given that } \frac{\bar{b}}{\bar{b}_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \right) \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} (1 + \ln 2) \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \ln 2 \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} - \ln 2 \right\} \quad \left(\text{since } \frac{s}{\bar{b}} > 1 \text{ and } \frac{\gamma_1}{2} > 0 \right) \\
&= \frac{2}{2} \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{by setting } r = \frac{\gamma_1}{2} \right) \\
&\leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right)
\end{aligned}$$

On the other hand, if $s \leq \bar{b}$, we have

$$\begin{aligned}
P(|e'_{i,N}\Gamma(F_t - \mu_F)\varepsilon_{t+h}| > s) &\leq 1 \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \\
&= \exp \{1\} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \\
&\leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right),
\end{aligned}$$

which shows part (e).

To show part (f), note that, for any $s > 0$, for all integer i with $1 \leq i \leq N$, and for all $t \in \mathbb{Z}$;

$$\begin{aligned}
& P(|u_{it}\varepsilon_{t+h}| > s) \\
&= P(\{|u_{it}\varepsilon_{t+h}| > s\} \cap \{|\varepsilon_{t+h}| \leq \sqrt{s}\} \cup \{|u_{it}\varepsilon_{t+h}| > s\} \cap \{|\varepsilon_{t+h}| > \sqrt{s}\}) \\
&= P(\{|u_{it}\varepsilon_{t+h}| > s\} \cap \{|\varepsilon_{t+h}| \leq \sqrt{s}\}) + P(\{|u_{it}\varepsilon_{t+h}| > s\} \cap \{|\varepsilon_{t+h}| > \sqrt{s}\}) \\
&\quad (\text{since } \{|\varepsilon_{t+h}| \leq \sqrt{s}\} \cap \{|\varepsilon_{t+h}| > \sqrt{s}\} = \emptyset) \\
&\leq P(\sqrt{s}|u_{it}| > s) + P(\{|u_{it}\varepsilon_{t+h}| > s\} \cap \{|\varepsilon_{t+h}| > \sqrt{s}\}) \\
&\leq P(|u_{it}| > \sqrt{s}) + P(|\varepsilon_{t+h}| > \sqrt{s}) \\
&= P(|u'_te_{i,N}| > \sqrt{s}) + P(|\varepsilon_{t+h}| > \sqrt{s}) \\
&\quad (\text{noting that, in this case, } \delta = e_{i,N} \text{ and } \|e_{i,N}\|_2 = 1) \\
&\leq \exp\left\{1 - \left(\frac{\sqrt{s}}{b_1}\right)^{\gamma_1}\right\} + \exp\left\{1 - \left(\frac{\sqrt{s}}{b_1}\right)^{\gamma_1}\right\} \\
&\leq 2 \exp\left\{1 - \left(\frac{\sqrt{s}}{b_1}\right)^{\gamma_1}\right\}.
\end{aligned}$$

If $s > \bar{b}$, the inequality

$$\frac{\bar{b}}{b_1^2} \geq \frac{\bar{b}}{\bar{b}_2^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \geq 1 \quad (\text{given that } \bar{b}_2 \geq b_1)$$

further implies that

$$\begin{aligned}
P(|u_{it}\varepsilon_{t+h}| > s) &\leq 2 \exp \left\{ 1 - \left(\frac{\sqrt{s}}{b_1} \right)^{\gamma_1} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{b_1^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \left(\frac{\bar{b}}{b_1^2} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} (1 + \ln 2)^{\frac{2}{\gamma_1}} \right)^{\frac{\gamma_1}{2}} \right\} \\
&\quad \left(\text{given that } \frac{\bar{b}}{b_1^2} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \right) \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} (1 + \ln 2) \right\} \\
&= 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \ln 2 \right\} \\
&\leq 2 \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} - \ln 2 \right\} \quad \left(\text{since } \frac{s}{\bar{b}} > 1 \text{ and } \frac{\gamma_1}{2} > 0 \right) \\
&= \frac{2}{2} \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^{\frac{\gamma_1}{2}} \right\} \\
&= \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{by setting } r = \frac{\gamma_1}{2} \right) \\
&\leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right)
\end{aligned}$$

On the other hand, if $s \leq \bar{b}$, we have

$$\begin{aligned}
P(|u_{it}\varepsilon_{t+h}| > s) &\leq 1 \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \\
&= \exp \{1\} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \\
&\leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\} \quad \left(\text{since } \bar{C} \geq \exp \{1\} \right). \quad \square
\end{aligned}$$

Lemma E-9:

Let $T_h = T - h$, where $h \geq 1$ is a (fixed) finite integer. Suppose that Assumptions (i), (ii),

(iv), and (v) hold. Let γ_* be a positive constant such that

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2}$$

where γ_1 and γ_2 are the positive constants as defined in Assumption (v) and suppose that

$$\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \text{ as } N, T \rightarrow \infty.$$

Under these conditions, the following statements are true.

(a) There exists a constant $C_{\dagger} > 0$, not depending on N and T , such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

(b) There exists a constant $C_{\ddagger} > 0$, not depending on N and T , such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq C_{\ddagger} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right).$$

(c) There exists a constant $\overline{C}_{(1)} > 0$ that does not depend on N and T such that

$$\begin{aligned} & P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\underline{\Lambda}}{\underline{\Delta}_{1,1}} \right) \right\} \Gamma' e_{j,N} \right| \geq \overline{C}_{(1)} \sqrt{\frac{\ln N}{T_h}} \right) \\ &= o \left(\frac{1}{N^2} \right). \end{aligned}$$

(d) There exists constants $\overline{C}_{(2)} > 0$ and $\overline{C}_{(3)} > 0$, not depending on N and T , such that

$$\begin{aligned} & P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) u_{jt} \right| \geq \overline{C}_{(2)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right), \\ & P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \overline{C}_{(3)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right). \end{aligned}$$

(e) There exists a constant $\overline{C}_{(4)} > 0$ that does not depend on N and T such that

$$P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (u_{it} u_{jt} - \overline{\Sigma}_{u,(i,j)}) \right| \geq \overline{C}_{(4)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right).$$

where $\bar{\Sigma}_{u,(i,j)} = e'_{i,N} \bar{\Sigma}_u e_{j,N}$.

(f) There exists a constant $\bar{C}_{(5)} > 0$ that does not depend on N and T such that

$$\begin{aligned} & P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \bar{C}_{(5)} \sqrt{\frac{\ln N}{T_h}} \right) \\ &= o \left(\frac{1}{N^2} \right) .. \end{aligned}$$

(g) There exists constants $\bar{C}_{(6)} > 0$ and $\bar{C}_{(7)} > 0$, not depending on N and T , such that

$$\begin{aligned} & P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| |\bar{u}_{j,T_h}| \geq \bar{C}_{(6)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right), \\ & P \left(\max_{1 \leq i, j \leq N} |\bar{u}_{i,T_h}| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \bar{C}_{(7)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right). \end{aligned}$$

(h) There exists a constant $\bar{C}_{(8)} > 0$ that does not depend on N and T such that

$$P \left(\max_{1 \leq i, j \leq N} |\bar{u}_{i,T_h}| |\bar{u}_{j,T_h}| \geq \bar{C}_{(8)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right).$$

Proof of Lemma E-9:

To show part (a), note first that, by definition, $\mu_F = E[F_t]$ for all $t \in \mathbb{Z}$, from which it follows that

$$E[e'_{i,N} \Gamma (F_t - \mu_F)] = e'_{i,N} \Gamma E(F_t - \mu_F) = 0 \text{ for all } i \text{ and } t.$$

Moreover, by the result given in part (a) of Lemma E-8, there exist constants $\gamma_1 > 0$ and $\bar{b}_2 > 0$, such that for any $s > 0$, for all integer i such that $1 \leq i \leq N$, and for all $t \in \mathbb{Z}$,

$$\begin{aligned} P(|e'_{i,N} \Gamma (F_t - \mu_F)| > s) &\leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \\ &\leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \end{aligned}$$

for some constant $\bar{C} \geq 1$. Next, let $Z_{i,t}^* = e'_{i,N} \Gamma (F_t - \mu_F)$, and let $\underline{V}_t = (\varepsilon_t, F'_t, u'_t)'$ be as defined in part (b) of Assumption (v). Note that $\mathcal{F}_{Z^*,i,-\infty}^t = \sigma(\dots, Z_{i,t-2}^*, Z_{i,t-1}^*, Z_{i,t}^*) \subseteq \mathcal{F}_{-\infty}^t = \sigma(\dots, \underline{V}_{t-2}, \underline{V}_{t-1}, \underline{V}_t)$ and $\mathcal{F}_{Z^*,i,t+m}^\infty = \sigma(Z_{i,t+m}^*, Z_{i,t+m+1}^*, Z_{i,t+m+2}^*, \dots) \subseteq \mathcal{F}_{t+m}^\infty = \sigma(\underline{V}_{t+m}, \underline{V}_{t+m+1}, \underline{V}_{t+m+2}, \dots)$. Hence, it follows by part (b) of Assumption (v) and Theorem 14.1 of Davidson (1994) that $\{Z_{i,t}^*\}_{t \in \mathbb{Z}}$ is an α -mixing process with α coefficient $\alpha_{Z^*,i}(m) =$

$\sup_t \alpha(\mathcal{F}_{Z^*,i,-\infty}^t, \mathcal{F}_{Z^*,i,t+m}^\infty)$ satisfying the rate of decay

$$\begin{aligned} \alpha_{Z^*,i}(m) &= \sup_t \alpha(\mathcal{F}_{Z^*,i,-\infty}^t, \mathcal{F}_{Z^*,i,t+m}^\infty) \\ &\leq \sup_t \alpha(\underline{\mathcal{F}}_{-\infty}^t, \underline{\mathcal{F}}_{t+m}^\infty) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1, \text{ for all } i, \end{aligned}$$

and for some positive constants γ_2 , a and c . Moreover, under Assumption (v), there exists a constant $\gamma \in (0, 1)$ such that

$$\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.$$

Hence, all the assumptions needed to obtain the Bernstein inequality given in Merlevède, Peligrad, and Rio (2011) are fulfilled. It follows from this inequality that there exist positive constants C_1, C_2, C_3, C_4 and C_5 that depend on $a, \bar{b}_2, c, \gamma$, and γ_2 such that for any $s > 0$ and for all integer i with $1 \leq i \leq N$

$$\begin{aligned} &P\left(\left|\frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F)\right| \geq s\right) \\ &= P\left(\left|\sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F)\right| \geq sT_h\right) \\ &\leq T_h \exp\left\{-\frac{(sT_h)^\gamma}{C_1}\right\} + \exp\left\{-\frac{s^2 T_h^2}{C_2(1 + T_h C_3)}\right\} + \exp\left\{-\frac{(sT_h)^2}{C_4 T_h} \exp\left\{\frac{(sT_h)^{\gamma(1-\gamma)}}{C_5 (\ln sT_h)^\gamma}\right\}\right\} \end{aligned}$$

Using Bonferroni's inequality, we further obtain

$$\begin{aligned} &P\left(\max_{1 \leq i \leq N} \left|\frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F)\right| \geq s\right) \\ &\leq N \max_{1 \leq i \leq N} P\left(\left|\frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F)\right| \geq sT_h\right) \\ &\leq NT_h \exp\left\{-\frac{(sT_h)^\gamma}{C_1}\right\} + N \exp\left\{-\frac{s^2 T_h^2}{C_2(1 + T_h C_3)}\right\} \\ &\quad + N \exp\left\{-\frac{(sT_h)^2}{C_4 T_h} \exp\left\{\frac{(sT_h)^{\gamma(1-\gamma)}}{C_5 (\ln sT_h)^\gamma}\right\}\right\} \end{aligned}$$

For some $C_\dagger > 0$, let

$$s = C_\dagger \sqrt{\frac{\ln N}{T_h}}.$$

By choosing $C_\dagger$ to be large enough, we can obtain the following three results, which we will

show in turn:

(I)

$$NT_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} = NT_h \exp \left\{ -\frac{C_1^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(II)

$$N \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(III)

$$N \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = N \exp \left\{ -\frac{C_2^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = O \left(\frac{1}{N^2} \right).$$

To show result (I), first write

$$N^3 T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} = N^3 T_h \exp \left\{ -\frac{C_1^\gamma}{C_1} T_h^\gamma \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma}{2}} \right\} = N^3 T_h \exp \left\{ -\frac{C_1^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\}$$

Taking the log transformation, we have

$$\ln \left[N^3 T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} \right] = 3 \ln N + \ln T_h - \frac{C_1^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}$$

Now, by assumption, $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$, where

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2}.$$

Since

$$\gamma = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} > \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} = \gamma_*$$

it follows that $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$ implies that $(\ln N)^{\frac{2}{\gamma}-1} = o(T)$. This, in turn, implies that

$$\begin{aligned} \frac{3 (\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} &= 3 \left(\frac{T}{T_h} \right)^{\frac{\gamma}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \right)^{\frac{\gamma}{2}} \rightarrow 0 \\ &\left(\text{given that } \frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \rightarrow 0 \text{ and given that} \right. \\ &\left. \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \text{ since } h \text{ is a fixed finite integer} \right), \quad (171) \end{aligned}$$

Making use of expression (171) and the fact that

$$\frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0 \quad (\text{given that } \gamma > 0),$$

we further deduce that

$$\frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} = \frac{3 (\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0,$$

which, in turn, implies that

$$\begin{aligned} N^3 T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} &= \exp \left\{ 3 \ln N + \ln T_h - \frac{C_1^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\} \\ &= \exp \left\{ -\frac{C_1^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \left[1 - \frac{C_1}{C_1^\gamma} \frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \right] \right\} \\ &= o(1) \end{aligned}$$

It follows that

$$N T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} = \frac{1}{N^2} N^3 T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (II), note that

$$\begin{aligned} & N^3 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} \\ &= N^3 \exp \left\{ -\frac{T_h^2}{C_4 T_h} \left(C_\dagger \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma(1-\gamma)}}{C_5 \left(\ln \left[T_h C_\dagger \sqrt{\frac{\ln N}{T_h}} \right] \right)^\gamma} \left(C_\dagger \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma(1-\gamma)} \right) \right\} \\ &= N^3 \exp \left\{ -\frac{C_\dagger^2}{C_4} \ln N \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} C_\dagger^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 (\ln T_h + \ln C_\dagger + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h)^\gamma} \right) \right\} \\ &= N^3 \exp \left\{ -\frac{C_\dagger^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} C_\dagger^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_\dagger + \frac{1}{2} \ln \ln N \right)^\gamma} \right) \right\} \\ &= \exp \left\{ -\ln N \left[\frac{C_\dagger^2}{C_4} \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} C_\dagger^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_\dagger + \frac{1}{2} \ln \ln N \right)^\gamma} \right) - 3 \right] \right\} \\ &= o(1) \end{aligned}$$

This, in turn, implies that

$$\begin{aligned} N \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} &= \frac{1}{N^2} N^3 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} \\ &= o\left(\frac{1}{N^2}\right) \end{aligned}$$

Turning our attention now to showing result (III), we first write

$$\begin{aligned} N^3 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} &= N^3 \exp \left\{ -\frac{T_h^2}{C_2 (1 + T_h C_3)} \left(C_\dagger \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\ &= N^3 \exp \left\{ -\frac{C_\dagger^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned} \ln \left[N^3 \exp \left\{ -\frac{C_\dagger^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \right] &= 3 \ln N - \frac{C_\dagger^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \\ &= 3 \ln N - \frac{C_\dagger^2}{C_2 C_3} \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1}. \end{aligned}$$

Now, let $C_\dagger$ be a positive constant such that $C_\dagger^2 > 3C_2(1 + C_3)$, and set

$$C^* = \frac{C_\dagger^2}{C_2 C_3} - 3.$$

Note that, by this choice of $C_\dagger$ and C^* , we have

$$C^* > \frac{3C_2(1 + C_3)}{C_2 C_3} - 3 = \frac{3}{C_3} > 0$$

It, thus, follows that

$$\begin{aligned}
& N^3 \exp \left\{ -\frac{T_h^2 s^2}{C_2(1+T_h C_3)} \right\} \\
= & N^3 \exp \left\{ -\frac{C_\dagger^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \\
= & \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - \frac{C_\dagger^2}{C_2 C_3} \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - [C^* + 3] \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(\ln N \left[3C_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ -C^* \ln N \left[1 - \frac{3}{C^* C_3} \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
< & \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{3}{C^* C_3} < \frac{C_3 3}{3C_3} = 1 \right) \\
= & o(1)
\end{aligned}$$

This, in turn, implies that

$$N \exp \left\{ -\frac{T_h^2 s^2}{C_2(1+T_h C_3)} \right\} = \left(\frac{1}{N^2} \right) N^3 \exp \left\{ -\frac{T_h^2 s^2}{C_2(1+T_h C_3)} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (III).

Combining the results given in (I)-(III) above, we obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) \right| \geq C_\dagger \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & NT_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} + N \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln T_h s)^\gamma} \right\} \right\} \\
& + N \exp \left\{ -\frac{T_h^2 s^2}{C_2(1+T_h C_3)} \right\} \\
= & o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
= & o \left(\frac{1}{N^2} \right).
\end{aligned}$$

To show part (b), note first that, by part (a) of Assumption (ii), $E[u_{it}] = 0$ for all i and t . Moreover, note that $u_{it} = u'_t e_{i,N}$, where $e_{i,N}$ is the i^{th} elementary vector, i.e., $e_{i,N}$ is an $N \times 1$ vector such that the i^{th} element equals 1 and all other elements are 0. Note further that $\|e_{i,N}\|_2 = 1$; hence, we can set $\delta = e_{i,N}$ in this case and apply part (a) of Assumption (v) to deduce that there exist positive constants γ_1 and b_1 such that for all $t \in \mathbb{Z}$, for all integer i such that $1 \leq i \leq N$, and for all $s > 0$,

$$P(|u_{it}| > s) \leq \exp \left\{ 1 - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\},$$

where $\bar{C}$ is a positive constant such that $\bar{C} \geq \exp\{1\}$. Now, let $\underline{V}_t = (\varepsilon_t, F'_t, u'_t)'$ be defined as in part (b) of Assumption (v), and note that $\underline{\mathcal{F}}_{u,i,-\infty}^t = \sigma(\dots, u_{i,t-2}, u_{i,t-1}, u_{i,t}) \subseteq \underline{\mathcal{F}}_{-\infty}^t = \sigma(\dots, \underline{V}_{t-2}, \underline{V}_{t-1}, \underline{V}_t)$ and $\underline{\mathcal{F}}_{u,i,t+m}^\infty = \sigma(u_{i,t+m}, u_{i,t+m+1}, u_{i,t+m+2}, \dots) \subseteq \underline{\mathcal{F}}_{t+m}^\infty = \sigma(\underline{V}_{t+m}, \underline{V}_{t+m+1}, \underline{V}_{t+m+2}, \dots)$. Hence, by applying part (b) of Assumption (v) and Theorem 14.1 of Davidson (1994), we see that $\{u_{it}\}_{t \in \mathbb{Z}}$ is an α -mixing process with α coefficient satisfying the rate of decay

$$\begin{aligned} \alpha_{u,i}(m) &= \sup_t \alpha(\underline{\mathcal{F}}_{u,i,-\infty}^t, \underline{\mathcal{F}}_{u,i,t+m}^\infty) \\ &\leq \sup_t \alpha(\underline{\mathcal{F}}_{-\infty}^t, \underline{\mathcal{F}}_{t+m}^\infty) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1, \text{ for all } i, \end{aligned}$$

and for some positive constants γ_2 , a , and c . In addition, under Assumption (v), there exists a constant $\gamma \in (0, 1)$ such that

$$\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.$$

Hence, all the assumptions needed to obtain the Bernstein inequality given in Merlevède, Peligrad, and Rio (2011) are fulfilled. It follows from this inequality that there exist positive constants C_1, C_2, C_3, C_4 and C_5 that depend on a, b_1, c, γ , and γ_2 such that for any $s > 0$ and for all integer i with $1 \leq i \leq N$,

$$\begin{aligned} &P(|\bar{u}_{i,T_h}| \geq s) \\ &= P\left(\left|\frac{1}{T_h} \sum_{t=1}^{T_h} u_{it}\right| \geq s\right) \\ &\leq T_h \exp\left\{-\frac{(sT_h)^\gamma}{C_1}\right\} + \exp\left\{-\frac{s^2 T_h^2}{C_2(1 + T_h C_3)}\right\} + \exp\left\{-\frac{(sT_h)^2}{C_4 T_h} \exp\left\{\frac{(sT_h)^{\gamma(1-\gamma)}}{C_5 (\ln sT_h)^\gamma}\right\}\right\} \end{aligned}$$

Using Bonferroni's inequality, we further obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq s \right) \\
& \leq N \max_{1 \leq i \leq N} P \left(\left| \sum_{t=1}^{T_h} u_{it} \right| \geq s T_h \right) \\
& \leq N T_h \exp \left\{ -\frac{(s T_h)^\gamma}{C_1} \right\} + N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} \\
& \quad + N \exp \left\{ -\frac{(s T_h)^2}{C_4 T_h} \exp \left\{ \frac{(s T_h)^{\gamma(1-\gamma)}}{C_5 (\ln s T_h)^\gamma} \right\} \right\}
\end{aligned}$$

For some $C_\dagger > 0$, let

$$s = C_\dagger \sqrt{\frac{\ln N}{T_h}}.$$

By choosing $C_\dagger$ to be large enough, we can obtain the following three results, which we will show in turn:

(I)

$$N T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} = N T_h \exp \left\{ -\frac{C_\dagger^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(II)

$$N \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(III)

$$N \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = N \exp \left\{ -\frac{C_\dagger^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = O \left(\frac{1}{N^2} \right).$$

To show result (I), first write

$$N^3 T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} = N^3 T_h \exp \left\{ -\frac{C_\dagger^\gamma}{C_1} T_h^\gamma \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma}{2}} \right\} = N^3 T_h \exp \left\{ -\frac{C_\dagger^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\}$$

Taking the log transformation, we have

$$\ln \left[N^3 T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} \right] = 3 \ln N + \ln T_h - \frac{C_\dagger^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}$$

Now, by assumption, $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$, where

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2}.$$

Since

$$\gamma = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} > \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} = \gamma_*$$

it follows that $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$ implies that $(\ln N)^{\frac{2}{\gamma}-1} = o(T)$. This, in turn, implies that

$$\begin{aligned} \frac{3(\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} &= 3 \left(\frac{T}{T_h} \right)^{\frac{\gamma}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \right)^{\frac{\gamma}{2}} \rightarrow 0 \\ &\left(\text{given that } \frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \rightarrow 0 \text{ and given that} \right. \\ &\left. \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \text{ since } h \text{ is fixed finite integer} \right). \quad (172) \end{aligned}$$

Making use of expression (172) and the fact that

$$\frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0 \quad (\text{given that } \gamma > 0),$$

we further deduce that

$$\frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} = \frac{3(\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned} N^3 T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} &= \exp \left\{ 3 \ln N + \ln T_h - \frac{C_1^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\} \\ &= \exp \left\{ -\frac{C_1^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \left[1 - \frac{C_1}{C_1^\gamma} \frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \right] \right\} \\ &= o(1) \end{aligned}$$

It follows that

$$N T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} = \frac{1}{N^2} N^3 T_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (II), note that

$$\begin{aligned}
& N^3 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} \\
= & N^3 \exp \left\{ -\frac{T_h^2}{C_4 T_h} \left(C_\dagger \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma(1-\gamma)}}{C_5 \left(\ln \left[T_h C_\dagger \sqrt{\frac{\ln N}{T_h}} \right] \right)^\gamma} \left(C_\dagger \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma(1-\gamma)} \right) \right\} \\
= & N^3 \exp \left\{ -\frac{C_\dagger^2}{C_4} \ln N \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} C_\dagger^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 (\ln T_h + \ln C_\dagger + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h)^\gamma} \right) \right\} \\
= & N^3 \exp \left\{ -\frac{C_\dagger^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} C_\dagger^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_\dagger + \frac{1}{2} \ln \ln N \right)^\gamma} \right) \right\} \\
= & \exp \left\{ -\ln N \left[\frac{C_\dagger^2}{C_4} \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} C_\dagger^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_\dagger + \frac{1}{2} \ln \ln N \right)^\gamma} \right) - 3 \right] \right\} \\
= & o(1)
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
N \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} &= \frac{1}{N^2} N^3 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} \\
&= o\left(\frac{1}{N^2}\right)
\end{aligned}$$

Turning our attention now to showing result (III), we first write

$$\begin{aligned}
N^3 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} &= N^3 \exp \left\{ -\frac{T_h^2}{C_2 (1 + T_h C_3)} \left(C_\dagger \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\
&= N^3 \exp \left\{ -\frac{C_\dagger^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\}
\end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned} \ln \left[N^3 \exp \left\{ -\frac{C_{\dagger}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \right] &= 3 \ln N - \frac{C_{\dagger}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \\ &= 3 \ln N - \frac{C_{\dagger}^2}{C_2 C_3} \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1}. \end{aligned}$$

Now, let $C_{\dagger}^*$ be a positive constant such that $C_{\dagger}^2 > 3C_2(1 + C_3)$, and set

$$C^* = \frac{C_{\dagger}^2}{C_2 C_3} - 3.$$

Note that, by this choice of $C_{\dagger}$ and C^* , we have

$$C^* > \frac{3C_2(1 + C_3)}{C_2 C_3} - 3 = \frac{3}{C_3} > 0$$

It, thus, follows that

$$\begin{aligned} & N^3 \exp \left\{ -\frac{T_h^2 s^2}{C_2(1 + T_h C_3)} \right\} \\ &= N^3 \exp \left\{ -\frac{C_{\dagger}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \\ &= \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - \frac{C_{\dagger}^2}{C_2 C_3} \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - [C^* + 3] \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ \left(\ln N \left[3C_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ -C^* \ln N \left[1 - \frac{3}{C^* C_3} \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &< \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{3}{C^* C_3} < \frac{C_3 3}{3C_3} = 1 \right) \\ &= o(1) \end{aligned}$$

This, in turn, implies that

$$N \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = \left(\frac{1}{N^2} \right) N^3 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (III).

Combining the results given in (I)-(III) above, we obtain

$$\begin{aligned} & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) \\ & \leq NT_h \exp \left\{ -\frac{(T_h s)^\gamma}{C_1} \right\} + N \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma(1-\gamma)}}{C_5 (\ln T_h s)^\gamma} \right\} \right\} \\ & \quad + N \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} \\ & = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\ & = o \left(\frac{1}{N^2} \right). \end{aligned}$$

To show part (c), note first that we can rewrite

$$\begin{aligned} & \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\underline{\Lambda}}{\underline{\lambda}_{1,1}} \right) \right\} \Gamma' e_{j,N} \\ & = \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) (F_t - \mu_F)' \Gamma' e_{j,N} - e'_{i,N} \Gamma \left(\frac{\underline{\Lambda}}{\underline{\lambda}_{1,1}} \right) \Gamma' e_{j,N} \\ & \quad \left(\text{since } \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left(\frac{\underline{\Lambda}}{\underline{\lambda}_{1,1}} \right) \Gamma' e_{j,N} = e'_{i,N} \Gamma \left(\frac{\underline{\Lambda}}{\underline{\lambda}_{1,1}} \right) \Gamma' e_{j,N} \right) \\ & = \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) (F_t - \mu_F)' \Gamma' e_{j,N} \\ & \quad - e'_{i,N} \Gamma \frac{1}{T_h} \sum_{t=1}^{T_h} E [(F_t - \mu_F) (F_t - \mu_F)'] \Gamma' e_{j,N} \\ & \quad \left(\text{since } \frac{1}{T_h} \sum_{t=1}^{T_h} E [(F_t - \mu_F) (F_t - \mu_F)'] = \frac{\underline{\Lambda}}{\underline{\lambda}_{1,1}} \right) \\ & = \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - E [(F_t - \mu_F) (F_t - \mu_F)'] \right\} \Gamma' e_{j,N} \end{aligned}$$

from which it follows that the elements of the sum given above all have expectation equal to zero, i.e.,

$$\begin{aligned} & E \left[e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - E \left[(F_t - \mu_F) (F_t - \mu_F)' \right] \right\} \Gamma' e_{j,N} \right] \\ &= e'_{i,N} \Gamma \left\{ E \left[(F_t - \mu_F) (F_t - \mu_F)' \right] - E \left[(F_t - \mu_F) (F_t - \mu_F)' \right] \right\} \Gamma' e_{j,N} \\ &= 0 \end{aligned}$$

Next, let γ_1 , γ_2 , and γ be positive constants as defined in Assumption (v). Note that, by part (b) of Lemma E-8, there exists a positive constant $\bar{C} \geq \exp \{1\}$ such that, for all integers i and j with $1 \leq i, j \leq N$, for all $t \in Z$, and for any $s > 0$,

$$P \left(\left| e'_{i,N} \Gamma (F_t - \mu_F) (F_t - \mu_F)' \Gamma' e_{j,N} \right| > s \right) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\},$$

where $r = \gamma_1/2 > 0$ and $\bar{b} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \bar{b}_2^2 > 0$ and where $\bar{b}_2$ is also a positive constant which has been previously defined in Lemma E-8. Moreover, let $\gamma_* = \gamma_1 \gamma_2 / (\gamma_1 + 2\gamma_2)$, so that

$$\frac{1}{\gamma_*} = \frac{\gamma_1 + 2\gamma_2}{\gamma_1 \gamma_2} = \frac{2}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r} + \frac{1}{\gamma_2}.$$

Now, the relationship between the positive constants γ_1 , γ_2 , and γ as specified in Assumption (v) implies that

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} = \gamma < 1$$

so that, in particular, $\gamma_* \in (0, 1)$. In addition, let $\underline{Z}_{i,j,t} = e'_{i,N} \Gamma (F_t - \mu_F) (F_t - \mu_F)' \Gamma' e_{j,N}$, and let $\underline{V}_t = (\varepsilon_t, F'_t, u'_t)'$ be as defined in part (b) of Assumption (v). Note that $\mathcal{F}_{\underline{Z},i,j,-\infty}^t = \sigma(\dots, \underline{Z}_{i,j,t-2}, \underline{Z}_{i,j,t-1}, \underline{Z}_{i,j,t}) \subseteq \mathcal{F}_{-\infty}^t = \sigma(\dots, \underline{V}_{t-2}, \underline{V}_{t-1}, \underline{V}_t)$ and $\mathcal{F}_{\underline{Z},i,j,t+m}^\infty = \sigma(\underline{Z}_{i,j,t+m}, \underline{Z}_{i,j,t+m+1}, \underline{Z}_{i,j,t+m+2}, \dots)$. Hence, it follows by part (b) of Assumption (v) and Theorem 14.1 of Davidson (1994) that $\{\underline{Z}_{i,j,t}\}_{t \in Z}$ is an α -mixing process with α coefficient $\alpha_{\underline{Z},i,j}(m) = \sup_t \alpha(\mathcal{F}_{\underline{Z},i,j,-\infty}^t, \mathcal{F}_{\underline{Z},i,j,t+m}^\infty)$ satisfying the rate of decay

$$\begin{aligned} \alpha_{\underline{Z},i,j}(m) &= \sup_t \alpha(\mathcal{F}_{\underline{Z},i,j,-\infty}^t, \mathcal{F}_{\underline{Z},i,j,t+m}^\infty) \\ &\leq \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1; \text{ for all } i, j; \end{aligned}$$

and for some positive constants γ_2 , a and c .

Hence, all the conditions needed to obtain the Bernstein inequality given in Merlevède, Peligrad, and Rio (2011) are fulfilled. It then follows from applying the Bernstein inequality for weakly dependent processes, originally obtained by Merlevède, Peligrad, and Rio (2011), and restated here as Lemma C-1, that there exist constants C_1, C_2, C_3, C_4 and C_5 depending on $a, \bar{b}, c, \gamma_*$ and γ_2 (where a and c are positive constants defined in part (b) of Assumption

(v)) such that for any $s > 0$ and for all integers i and j with $1 \leq i, j \leq N$,

$$\begin{aligned}
& P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\underline{\Lambda}}{\underline{\Delta}_{1,1}} \right) \right\} \Gamma' e_{j,N} \right| \geq s \right) \\
&= P \left(\left| \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\underline{\Lambda}}{\underline{\Delta}_{1,1}} \right) \right\} \Gamma' e_{j,N} \right| \geq T_h s \right) \\
&= P \left(\left| \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - E [(F_t - \mu_F) (F_t - \mu_F)'] \right\} \Gamma' e_{j,N} \right| \geq T_h s \right) \\
&\leq T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} + \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} + \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_* (1 - \gamma_*)}}{C_5 (\ln T_h s)^{\gamma_*}} \right\} \right\}
\end{aligned}$$

Applying Bonferroni's inequality, we further obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\underline{\Lambda}}{\underline{\Delta}_{1,1}} \right) \right\} \Gamma' e_{j,N} \right| \geq s \right) \\
&= N^2 \max_{1 \leq i, j \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\underline{\Lambda}}{\underline{\Delta}_{1,1}} \right) \right\} \Gamma' e_{j,N} \right| \geq s \right) \\
&= N^2 \max_{1 \leq i, j \leq N} P \left(\left| \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - E [(F_t - \mu_F) (F_t - \mu_F)'] \right\} \Gamma' e_{j,N} \right| \geq T_h s \right) \\
&\leq N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} + N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_* (1 - \gamma_*)}}{C_5 (\ln T_h s)^{\gamma_*}} \right\} \right\} \\
&\quad + N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\}
\end{aligned}$$

For some $\overline{C}_{(1)} > 0$, let

$$s = \overline{C}_{(1)} \sqrt{\frac{\ln N}{T_h}}.$$

and, note that, by assumption, $(\ln N)^{\frac{2}{\gamma_*} - 1} = o(T)$. By choosing $\overline{C}_{(1)}$ to be large enough, we can obtain the following three results, which we will show in turn:

(I)

$$N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} = N^2 T_h \exp \left\{ -\frac{\overline{C}_{(1)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(II)

$$N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} = o \left(\frac{1}{N^2} \right),$$

(III)

$$N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = N^2 \exp \left\{ -\frac{\overline{C}_{(1)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = o \left(\frac{1}{N^2} \right).$$

To show result (I), first write

$$\begin{aligned} N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} &= N^4 T_h \exp \left\{ -\frac{\overline{C}_{(1)}^{\gamma_*}}{C_1} T_h^{\gamma_*} \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma_*}{2}} \right\} \\ &= N^4 T_h \exp \left\{ -\frac{\overline{C}_{(1)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \end{aligned}$$

Taking the log transformation, we have

$$\ln \left[N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} \right] = 4 \ln N + \ln T_h - \frac{\overline{C}_{(1)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}$$

Since

$$\frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0 \quad (\text{given that } \gamma_* > 0)$$

and since

$$\begin{aligned} \frac{4 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} &= 4 \left(\frac{T}{T_h} \right)^{\frac{\gamma_*}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \right)^{\frac{\gamma_*}{2}} \rightarrow 0 \quad \left(\text{given the assumption } \frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \right. \\ &\quad \left. \text{and given that } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ since } h \text{ is a fixed finite integer} \right), \end{aligned}$$

we see that

$$\frac{4 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} = \frac{4 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned}
N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} &= \exp \left\{ 4 \ln N + \ln T_h - \frac{\overline{C}_{(1)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \\
&= \exp \left\{ -\frac{\overline{C}_{(1)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \left[1 - \frac{C_1}{\overline{C}_{(1)}^{\gamma_*}} \frac{4 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \right] \right\} \\
&= o(1)
\end{aligned}$$

It follows that

$$N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} = \frac{1}{N^2} N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (II), note that

$$\begin{aligned}
&N^4 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} \\
&= N^4 \exp \left\{ -\frac{T_h^2}{C_4 T_h} \left(\overline{C}_{(1)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma_*(1-\gamma_*)}}{C_5 \left(\ln \left[T_h \overline{C}_{(1)} \sqrt{\frac{\ln N}{T_h}} \right] \right)^{\gamma_*}} \left(\overline{C}_{(1)} \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma_*(1-\gamma_*)} \right) \right\} \\
&= N^4 \exp \left\{ -\frac{\overline{C}_{(1)}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} \overline{C}_{(1)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 (\ln T_h + \ln \overline{C}_{(1)} + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h)^{\gamma_*}} \right) \right\} \\
&= N^4 \exp \left\{ -\frac{\overline{C}_{(1)}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} \overline{C}_{(1)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln \overline{C}_{(1)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) \right\} \\
&= \exp \left\{ -\ln N \left[\frac{\overline{C}_{(1)}^2}{C_4} \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} \overline{C}_{(1)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln \overline{C}_{(1)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) - 4 \right] \right\} \\
&= o(1)
\end{aligned}$$

This, in turn, implies that

$$N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

Turning our attention now to showing result (III), we first write

$$\begin{aligned} N^4 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} &= N^4 \exp \left\{ -\frac{T_h^2}{C_2 (1 + T_h C_3)} \left(\overline{C}_{(1)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\ &= N^4 \exp \left\{ -\frac{\overline{C}_{(1)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned} \ln \left[N^4 \exp \left\{ -\frac{\overline{C}_{(1)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \right] &= 4 \ln N - \frac{\overline{C}_{(1)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \\ &= 4 \ln N - \frac{\overline{C}_{(1)}^2}{C_2 C_3} \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \end{aligned}$$

Now, let $\overline{C}_{(1)}$ be a positive constant $\overline{C}_{(1)}^2 > 4C_2(1 + C_3)$, and set

$$C^* = \frac{\overline{C}_{(1)}^2}{C_2 C_3} - 4.$$

Note that, by this choice of $\overline{C}_{(1)}$ and C^* , we have

$$C^* > \frac{4C_2(1 + C_3)}{C_2 C_3} - 4 = \frac{4}{C_3} > 0$$

It, thus, follows that

$$\begin{aligned}
& N^4 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} \\
= & N^4 \exp \left\{ -\frac{\overline{C}_{(1)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \\
= & \exp \left\{ \left(4 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - \frac{\overline{C}_{(1)}^2}{C_2 C_3} \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(4 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - [C^* + 4] \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(\ln N \left[4 C_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ -C^* \ln N \left[1 - \frac{4}{C^* C_3} \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
< & \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{4}{C^* C_3} < \frac{C_3 4}{4 C_3} = 1 \right) \\
= & o(1)
\end{aligned}$$

This, in turn, implies that

$$N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = \left(\frac{1}{N^2} \right) N^4 \exp \left\{ -\frac{\overline{C}_{(1)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (III)

Combining the results given in (I)-(III) above, we obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\underline{\Lambda}}{\underline{\Delta}_{1,1}} \right) \right\} \Gamma' e_{j,N} \right| \geq \overline{C}_{(1)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq N^2 \max_{1 \leq i, j \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\underline{\Lambda}}{\underline{\Delta}_{1,1}} \right) \right\} \Gamma' e_{j,N} \right| \geq \overline{C}_{(1)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} + N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} \\
& \quad + N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_* (1 - \gamma_*)}}{C_5 (\ln T_h s)^{\gamma_*}} \right\} \right\} \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

To show part (d), first note that, by part (a) of Assumption (ii),

$$E [e'_{i,N} \Gamma (F_t - \mu_F) u_{jt}] = e'_{i,N} \Gamma E [(F_t - \mu_F) u_{jt}] = 0 \text{ for all } i, j, \text{ and } t.$$

Next, note that by part (c) of Lemma E-8, there exist constants $\gamma_1 > 0$ and $\bar{b} > 0$ and a positive constant $\overline{C} \geq \exp \{1\}$ such that, $\forall s > 0$, for all integers i and j with $1 \leq i, j \leq N$, and for all $t \in \mathbb{Z}$,

$$P (|e'_{i,N} \Gamma (F_t - \mu_F) u_{jt}| > s) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \leq \overline{C} \exp \left\{ -\left(\frac{s}{\bar{b}} \right)^r \right\}$$

where $r = \gamma_1/2 > 0$. Moreover, let $\gamma_* = \gamma_1 \gamma_2 / (\gamma_1 + 2\gamma_2)$, so that

$$\frac{1}{\gamma_*} = \frac{\gamma_1 + 2\gamma_2}{\gamma_1 \gamma_2} = \frac{2}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r} + \frac{1}{\gamma_2}.$$

Now, the relationship between the positive constants γ_1 , γ_2 , and γ as specified in Assumption (v) further implies that

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} = \gamma < 1$$

so that, in particular, $\gamma_* \in (0, 1)$. In addition, let $V_{i,j,t}^* = e'_{i,N} \Gamma (F_t - \mu_F) u_{jt}$, and let $\underline{V}_t = (\varepsilon_t, F'_t, u'_t)'$ be as defined in part (b) of Assumption (v). Note that $\mathcal{F}_{V^*, i, j - \infty}^t = \sigma (\dots, V_{i,j,t-2}^*, V_{i,j,t-1}^*, V_{i,j,t}^*) \subseteq \mathcal{F}_{-\infty}^t = \sigma (\dots, \underline{V}_{t-2}, \underline{V}_{t-1}, \underline{V}_t)$ and $\mathcal{F}_{V^*, i, j, t+m}^\infty = \sigma (V_{i,j,t+m}^*, V_{i,j,t+m+1}^*, V_{i,j,t+m+2}^*, \dots)$. Hence, it follows by part (b) of Assumption (v) and

Theorem 14.1 of Davidson (1994) that $\{V_{i,j,t}^*\}_{t \in \mathbb{Z}}$ is an α -mixing process with α coefficient $\alpha_{V^*,i,j}(m) = \sup_t \alpha(\mathcal{F}_{V^*,i,j,-\infty}^t, \mathcal{F}_{V^*,i,j,t+m}^\infty)$ satisfying the rate of decay

$$\begin{aligned} \alpha_{V^*,i,j}(m) &= \sup_t \alpha(\mathcal{F}_{V^*,i,j,-\infty}^t, \mathcal{F}_{V^*,i,j,t+m}^\infty) \\ &\leq \sup_t \alpha(\underline{\mathcal{F}}_{-\infty}^t, \underline{\mathcal{F}}_{t+m}^\infty) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1; \text{ for all } i, j; \end{aligned}$$

and for some positive constants γ_2 , a and c .

Hence, all the assumptions needed to apply the Bernstein inequality given in Merlevède, Peligrad, and Rio (2011) (restated here as Lemma C-1) are fulfilled. Application of this Bernstein inequality then implies that there exist positive constants C_1, C_2, C_3, C_4 and C_5 depending on $a, \bar{b}, c, \gamma_*$, and γ_2 such that for any $s > 0$ and for all integers i and j with $1 \leq i, j \leq N$,

$$\begin{aligned} &P\left(\left|\frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) u_{jt}\right| \geq s\right) \\ &= P\left(\left|\sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) u_{jt}\right| \geq T_h s\right) \\ &\leq T_h \exp\left\{-\frac{(sT_h)^{\gamma_*}}{C_1}\right\} + \exp\left\{-\frac{s^2 T_h^2}{C_2(1 + T_h C_3)}\right\} + \exp\left\{-\frac{(sT_h)^2}{C_4 T_h} \exp\left\{\frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln sT_h)^{\gamma_*}}\right\}\right\} \end{aligned}$$

Using Bonferroni's inequality, we further obtain

$$\begin{aligned} &P\left(\max_{1 \leq i, j \leq N} \left|\frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) u_{jt}\right| \geq s\right) \\ &\leq N^2 \max_{1 \leq i, j \leq N} P\left(\left|\sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) u_{jt}\right| \geq T_h s\right) \\ &\leq N^2 T_h \exp\left\{-\frac{(T_h s)^{\gamma_*}}{C_1}\right\} + N^2 \exp\left\{-\frac{(T_h s)^2}{C_4 T_h} \exp\left\{\frac{(T_h s)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}}\right\}\right\} \\ &\quad + N^2 \exp\left\{-\frac{T_h^2 s^2}{C_2(1 + T_h C_3)}\right\} \end{aligned}$$

For some $\bar{C}_{(2)} > 0$, let

$$s = \bar{C}_{(2)} \sqrt{\frac{\ln N}{T_h}}.$$

and, note that, by assumption, $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$. By choosing $\bar{C}_{(2)}$ to be large enough, we

can obtain the following three results, which we will show in turn:

(I)

$$N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} = N^2 T_h \exp \left\{ -\frac{\overline{C}_{(2)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(II)

$$N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(III)

$$N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = N^2 \exp \left\{ -\frac{\overline{C}_{(2)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = o \left(\frac{1}{N^2} \right).$$

To show result (I), first write

$$\begin{aligned} N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} &= N^4 T_h \exp \left\{ -\frac{\overline{C}_{(2)}^{\gamma_*}}{C_1} T_h^{\gamma_*} \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma_*}{2}} \right\} \\ &= N^4 T_h \exp \left\{ -\frac{\overline{C}_{(2)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \end{aligned}$$

Taking the log transformation, we have

$$\ln \left[N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} \right] = 4 \ln N + \ln T_h - \frac{\overline{C}_{(2)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}$$

Since

$$\frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0 \quad (\text{given that } \gamma_* > 0)$$

and since

$$\begin{aligned} \frac{4 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} &= 4 \left(\frac{T}{T_h} \right)^{\frac{\gamma_*}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \right)^{\frac{\gamma_*}{2}} \rightarrow 0 \quad \left(\text{given the assumption } \frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \right. \\ &\quad \left. \text{and given that } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \text{ since } h \text{ is a fixed finite integer} \right), \end{aligned}$$

we see that

$$\frac{4 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} = \frac{4 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned}
N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} &= \exp \left\{ 4 \ln N + \ln T_h - \frac{\overline{C}_{(2)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \\
&= \exp \left\{ -\frac{\overline{C}_{(2)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \left[1 - \frac{C_1}{\overline{C}_{(2)}^{\gamma_*}} \frac{4 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \right] \right\} \\
&= o(1)
\end{aligned}$$

It follows that

$$N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} = \frac{1}{N^2} N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (II), note that

$$\begin{aligned}
&N^4 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} \\
&= N^4 \exp \left\{ -\frac{T_h^2}{C_4 T_h} \left(\overline{C}_{(2)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma_*(1-\gamma_*)}}{C_5 \left(\ln \left[T_h \overline{C}_{(2)} \sqrt{\frac{\ln N}{T_h}} \right] \right)^{\gamma_*}} \left(\overline{C}_{(2)} \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma_*(1-\gamma_*)} \right) \right\} \\
&= N^4 \exp \left\{ -\frac{\overline{C}_{(2)}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} \overline{C}_{(2)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 (\ln T_h + \ln \overline{C}_{(2)} + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h)^{\gamma_*}} \right) \right\} \\
&= N^4 \exp \left\{ -\frac{\overline{C}_{(2)}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} \overline{C}_{(2)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln \overline{C}_{(2)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) \right\} \\
&= \exp \left\{ -\ln N \left[\frac{\overline{C}_{(2)}^2}{C_4} \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} \overline{C}_{(2)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln \overline{C}_{(2)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) - 4 \right] \right\} \\
&= o(1)
\end{aligned}$$

This, in turn, implies that

$$N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

Turning our attention now to showing result (III), we first write

$$\begin{aligned} N^4 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} &= N^4 \exp \left\{ -\frac{T_h^2}{C_2 (1 + T_h C_3)} \left(\overline{C}_{(2)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\ &= N^4 \exp \left\{ -\frac{\overline{C}_{(2)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned} \ln \left[N^4 \exp \left\{ -\frac{\overline{C}_{(2)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \right] &= 4 \ln N - \frac{\overline{C}_{(2)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \\ &= 4 \ln N - \frac{\overline{C}_{(2)}^2}{C_2 C_3} \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \end{aligned}$$

Now, let $\overline{C}_{(2)}$ be a positive constant such that $\overline{C}_{(2)}^2 > 4C_2 (1 + C_3)$, and set

$$C^* = \frac{\overline{C}_{(2)}^2}{C_2 C_3} - 4.$$

Note that, by this choice of $\overline{C}_{(2)}$ and C^* , we have

$$C^* > \frac{4C_2 (1 + C_3)}{C_2 C_3} - 4 = \frac{4}{C_3} > 0$$

It, thus, follows that

$$\begin{aligned}
& N^4 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} \\
= & N^4 \exp \left\{ -\frac{\overline{C}_{(2)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \\
= & \exp \left\{ \left(4 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - \frac{\overline{C}_{(2)}^2}{C_2 C_3} \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(4 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - [C^* + 4] \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(\ln N \left[4 C_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ -C^* \ln N \left[1 - \frac{4}{C^* C_3} \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
< & \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{4}{C^* C_3} < \frac{C_3 4}{4 C_3} = 1 \right) \\
= & o(1)
\end{aligned}$$

This, in turn, implies that

$$N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = \left(\frac{1}{N^2} \right) N^4 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (III)

Combining the results given in (I)-(III) above, we obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) u_{jt} \right| \geq \overline{C}_{(2)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq N^2 \max_{1 \leq i, j \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) u_{jt} \right| \geq \overline{C}_{(2)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} + N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} \\
& \quad + N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_* (1 - \gamma_*)}}{C_5 (\ln T_h s)^{\gamma_*}} \right\} \right\} \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

Moreover, by the same argument we can also show that there exists a constant $\overline{C}_{(3)} > 0$ such that

$$P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \overline{C}_{(3)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right).$$

To show part (e), first note that, by Assumption (ii),

$$E [u_{it} u_{jt} - \Sigma_{u,t,(i,j)}] = 0 \text{ for all } i, j, \text{ and } t;$$

where $\Sigma_{u,t,(i,j)}$ denotes the $(i, j)^{th}$ element of $\Sigma_{u,t}$. Next, note that by part (d) of Lemma E-8, there exist constants $\gamma_1 > 0$ and $\bar{b} > 0$ and a positive constant $\overline{C} \geq \exp \{1\}$ such that, $\forall s > 0$, for all $t \in \mathbb{Z}$, and for all integers i and j with $1 \leq i, j \leq N$

$$P (|u_{it} u_{jt}| > s) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \leq \overline{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\}$$

where $r = \gamma_1/2 > 0$. Moreover, let $\gamma_* = \gamma_1 \gamma_2 / (\gamma_1 + 2\gamma_2)$, so that

$$\frac{1}{\gamma_*} = \frac{\gamma_1 + 2\gamma_2}{\gamma_1 \gamma_2} = \frac{2}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r} + \frac{1}{\gamma_2}.$$

Now, the relationship between the positive constants γ_1 , γ_2 , and γ specified in Assumption (v) further implies that

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} = \gamma < 1$$

so that, in particular, $\gamma_* \in (0, 1)$. In addition, let $V_{i,j,t}^\dagger = u_{it}u_{jt}$, and let $\underline{V}_t = (\varepsilon_t, F'_t, u'_t)'$ be as defined in part (b) of Assumption (v). Note that $\mathcal{F}_{V^\dagger, i, j, -\infty}^t = \sigma \left(\dots, V_{i,j,t-2}^\dagger, V_{i,j,t-1}^\dagger, V_{i,j,t}^\dagger \right) \subseteq \mathcal{F}_{-\infty}^t = \sigma \left(\dots, \underline{V}_{t-2}, \underline{V}_{t-1}, \underline{V}_t \right)$ and $\mathcal{F}_{V^\dagger, i, j, t+m}^\infty = \sigma \left(V_{i,j,t+m}^\dagger, V_{i,j,t+m+1}^\dagger, V_{i,j,t+m+2}^\dagger, \dots \right) \subseteq \mathcal{F}_{t+m}^\infty = \sigma \left(\underline{V}_{t+m}, \underline{V}_{t+m+1}, \underline{V}_{t+m+2}, \dots \right)$. Hence, it follows by part (b) of Assumption (v) and Theorem 14.1 of Davidson (1994) that $\left\{ V_{i,j,t}^\dagger \right\}_{t \in \mathbb{Z}}$ is an α -mixing process with α coefficient $\alpha_{V^\dagger, i, j}(m) = \sup_t \alpha \left(\mathcal{F}_{V^\dagger, i, j, -\infty}^t, \mathcal{F}_{V^\dagger, i, j, t+m}^\infty \right)$ satisfying the rate of decay

$$\begin{aligned} \alpha_{V^\dagger, i, j}(m) &= \sup_t \alpha \left(\mathcal{F}_{V^\dagger, i, j, -\infty}^t, \mathcal{F}_{V^\dagger, i, j, t+m}^\infty \right) \\ &\leq \sup_t \alpha \left(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty \right) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1; \text{ for all } i, j; \end{aligned}$$

and for some positive constants γ_2 , a and c .

Hence, all the assumptions needed to apply the Bernstein inequality given in Merlevède, Peligrad, and Rio (2011) (restated here as Lemma C-1) are fulfilled. Application of this Bernstein inequality then implies that there exist positive constants C_1, C_2, C_3, C_4 and C_5 depending on $a, \bar{b}, c, \gamma_*$, and γ_2 such that for any $s > 0$ and for all integers i and j with $1 \leq i, j \leq N$,

$$\begin{aligned} &P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (u_{it}u_{jt} - \bar{\Sigma}_{u,(i,j)}) \right| \geq s \right) \\ &= P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (u_{it}u_{jt} - \Sigma_{u,t,(i,j)}) \right| \geq s \right) \left(\text{since } \frac{1}{T_h} \sum_{t=1}^{T_h} \bar{\Sigma}_{u,(i,j)} = \bar{\Sigma}_{u,(i,j)} = \frac{1}{T_h} \sum_{t=1}^{T_h} \Sigma_{u,t,(i,j)} \right) \\ &= P \left(\left| \sum_{t=1}^{T_h} (u_{it}u_{jt} - \Sigma_{u,t,(i,j)}) \right| \geq T_h s \right) \\ &\leq T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} + \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_* (1-\gamma_*)}}{C_5 (\ln s T_h)^{\gamma_*}} \right\} \right\} \end{aligned}$$

Using Bonferroni's inequality, we further obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (u_{it} u_{jt} - \bar{\Sigma}_{u,(i,j)}) \right| \geq s \right) \\
& \leq N^2 \max_{1 \leq i, j \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (u_{it} u_{jt} - \bar{\Sigma}_{u,(i,j)}) \right| \geq s \right) \\
& = N^2 \max_{1 \leq i, j \leq N} P \left(\left| \sum_{t=1}^{T_h} (u_{it} u_{jt} - \Sigma_{u,t,(i,j)}) \right| \geq T_h s \right) \\
& \leq N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} + N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} \\
& \quad + N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\}
\end{aligned}$$

For some $\bar{C}_{(4)} > 0$, let

$$s = \bar{C}_{(4)} \sqrt{\frac{\ln N}{T_h}}.$$

and, note that, by assumption, $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$. By choosing $\bar{C}_{(4)}$ to be large enough, we can obtain the following three results, which we will show in turn:

(I)

$$N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} = N^2 T_h \exp \left\{ -\frac{\bar{C}_{(4)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(II)

$$N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(III)

$$N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = N^2 \exp \left\{ -\frac{\bar{C}_{(4)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = o \left(\frac{1}{N^2} \right).$$

To show result (I), first write

$$\begin{aligned}
N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} &= N^4 T_h \exp \left\{ -\frac{\bar{C}_{(4)}^{\gamma_*}}{C_1} T_h^{\gamma_*} \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma_*}{2}} \right\} \\
&= N^4 T_h \exp \left\{ -\frac{\bar{C}_{(4)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\}
\end{aligned}$$

Taking the log transformation, we have

$$\ln \left[N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} \right] = 4 \ln N + \ln T_h - \frac{\overline{C}_{(4)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}$$

Since

$$\frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0 \quad (\text{given that } \gamma_* > 0)$$

and since

$$\begin{aligned} \frac{4 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} &= 4 \left(\frac{T}{T_h} \right)^{\frac{\gamma_*}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \right)^{\frac{\gamma_*}{2}} \rightarrow 0 \quad \left(\text{given the assumption } \frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \right. \\ &\quad \left. \text{and given that } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \text{ since } h \text{ is a fixed finite integer} \right), \end{aligned}$$

we see that

$$\frac{4 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} = \frac{4 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned} N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} &= \exp \left\{ 4 \ln N + \ln T_h - \frac{\overline{C}_{(4)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \\ &= \exp \left\{ -\frac{\overline{C}_{(4)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \left[1 - \frac{C_1}{\overline{C}_{(4)}^{\gamma_*}} \frac{4 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \right] \right\} \\ &= o(1) \end{aligned}$$

It follows that

$$N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} = \frac{1}{N^2} N^4 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (II), note that

$$\begin{aligned}
& N^4 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} \\
= & N^4 \exp \left\{ -\frac{T_h^2}{C_4 T_h} \left(\bar{C}_{(4)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma_*(1-\gamma_*)}}{C_5 \left(\ln \left[T_h \bar{C}_{(4)} \sqrt{\frac{\ln N}{T_h}} \right] \right)^{\gamma_*}} \left(\bar{C}_{(4)} \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma_*(1-\gamma_*)} \right) \right\} \\
= & N^4 \exp \left\{ -\frac{\bar{C}_{(4)}^2}{C_4} \ln N \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} \bar{C}_{(4)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 (\ln T_h + \ln \bar{C}_{(4)} + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h)^{\gamma_*}} \right) \right\} \\
= & N^4 \exp \left\{ -\frac{\bar{C}_{(4)}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} \bar{C}_{(4)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln \bar{C}_{(4)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) \right\} \\
= & \exp \left\{ -\ln N \left[\frac{\bar{C}_{(4)}^2}{C_4} \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} \bar{C}_{(4)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln \bar{C}_{(4)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) - 4 \right] \right\} \\
= & o(1)
\end{aligned}$$

This, in turn, implies that

$$N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} = o\left(\frac{1}{N^2}\right)$$

Turning our attention now to showing result (III), we first write

$$\begin{aligned}
N^4 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} &= N^4 \exp \left\{ -\frac{T_h^2}{C_2 (1 + T_h C_3)} \left(\bar{C}_{(4)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\
&= N^4 \exp \left\{ -\frac{\bar{C}_{(4)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\}
\end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned}
\ln \left[N^4 \exp \left\{ -\frac{\bar{C}_{(4)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \right] &= 4 \ln N - \frac{\bar{C}_{(4)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \\
&= 4 \ln N - \frac{\bar{C}_{(4)}^2}{C_2 C_3} \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1}
\end{aligned}$$

Now, let $\overline{C}_{(4)}$ be a positive constant such that $\overline{C}_{(4)}^2 > 4C_2(1 + C_3)$, and set

$$C^* = \frac{\overline{C}_{(4)}^2}{C_2 C_3} - 4.$$

Note that, by this choice of $\overline{C}_{(4)}$ and C^* , we have

$$C^* > \frac{4C_2(1 + C_3)}{C_2 C_3} - 4 = \frac{4}{C_3} > 0$$

It, thus, follows that

$$\begin{aligned} & N^4 \exp \left\{ -\frac{T_h^2 s^2}{C_2(1 + T_h C_3)} \right\} \\ = & N^4 \exp \left\{ -\frac{\overline{C}_{(4)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \\ = & \exp \left\{ \left(4 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - \frac{\overline{C}_{(4)}^2}{C_2 C_3} \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ = & \exp \left\{ \left(4 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - [C^* + 4] \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ = & \exp \left\{ \left(\ln N \left[4C_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ = & \exp \left\{ -C^* \ln N \left[1 - \frac{4}{C^* C_3} \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ < & \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{4}{C^* C_3} < \frac{C_3 4}{4C_3} = 1 \right) \\ = & o(1) \end{aligned}$$

This, in turn, implies that

$$N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2(1 + T_h C_3)} \right\} = \left(\frac{1}{N^2} \right) N^4 \exp \left\{ -\frac{T_h^2 s^2}{C_2(1 + T_h C_3)} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (III)

Combining the results given in (I)-(III) above, we obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (u_{it} u_{jt} - \bar{\Sigma}_{u, (i, j)}) \right| \geq \bar{C}_{(4)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq N^2 \max_{1 \leq i, j \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (u_{it} u_{jt} - \bar{\Sigma}_{u, (i, j)}) \right| \geq \bar{C}_{(4)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq N^2 T_h \exp \left\{ -\frac{(T_h s)^{\gamma_*}}{C_1} \right\} + N^2 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} \\
& \quad + N^2 \exp \left\{ -\frac{(T_h s)^2}{C_4 T_h} \exp \left\{ \frac{(T_h s)^{\gamma_* (1 - \gamma_*)}}{C_5 (\ln T_h s)^{\gamma_*}} \right\} \right\} \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

To show part (f), note first that by part (a) of this lemma, there exists a constant $C_{\dagger} > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i, N} \Gamma (F_t - \mu_F) \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right).$$

Now, set $\overline{C}_{(5)} = C_{\dagger}^2$, and we have

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \overline{C}_{(5)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \overline{C}_{(5)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\leq P \left(\left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \geq \sqrt{\overline{C}_{(5)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right. \\
&\quad \left. \cup \left\{ \max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \sqrt{\overline{C}_{(5)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \geq \sqrt{\overline{C}_{(5)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \\
&\quad + P \left(\max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \sqrt{\overline{C}_{(5)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \quad (\text{by Bonferroni inequality}) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \geq \sqrt{\overline{C}_{(5)}} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad + P \left(\max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \sqrt{\overline{C}_{(5)}} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad \left(\text{since } \sqrt{\frac{\ln N}{T_h}} < \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \text{ given that } \frac{\ln N}{T_h} = \frac{\ln N}{T-h} < 1 \text{ for all } N, T \text{ sufficiently large} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad + P \left(\max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) \quad \left(\text{since } \sqrt{\overline{C}_{(5)}} = C_{\dagger} \right).
\end{aligned}$$

It then follows, by part (a) of this lemma, that for all N and T sufficiently large,

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \overline{C}_{(5)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad + P \left(\max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

To show part (g), note first that by part (a) of this lemma, there exists a constant $C_{\dagger} > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

and, by part (b) of this lemma, there exists a constant $C_{\ddagger} > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq C_{\ddagger} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right).$$

Now, set $\overline{C}_{(6)} = \max \{C_{\dagger}^2, C_{\ddagger}^2\}$, and note that

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| |\overline{u}_{j, T_h}| \geq \overline{C}_{(6)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{jt} \right| \geq \overline{C}_{(6)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\leq P \left(\left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq \sqrt{\overline{C}_{(6)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right. \\
&\quad \left. \cup \left\{ \max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{jt} \right| \geq \sqrt{\overline{C}_{(6)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq \sqrt{\overline{C}_{(6)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \\
&\quad + P \left(\max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{jt} \right| \geq \sqrt{\overline{C}_{(6)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \quad (\text{by Bonferroni inequality}) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq \sqrt{\overline{C}_{(6)}} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad + P \left(\max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{jt} \right| \geq \sqrt{\overline{C}_{(6)}} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad \left(\text{since } \sqrt{\frac{\ln N}{T_h}} < \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \text{ given that } \frac{\ln N}{T_h} = \frac{\ln N}{T-h} < 1 \text{ for all } N, T \text{ sufficiently large} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad + P \left(\max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{jt} \right| \geq C_{\ddagger} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad \left(\text{since } \sqrt{\overline{C}_{(6)}} = \max \{C_{\dagger}, C_{\ddagger}\} \text{ so that } \sqrt{\overline{C}_{(6)}} \geq C_{\dagger} \text{ and } \sqrt{\overline{C}_{(6)}} \geq C_{\ddagger} \right).
\end{aligned}$$

It then follows, by parts (a) and (b) of this lemma, that for all N and T sufficiently large,

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| |\bar{u}_{j,T_h}| \geq \bar{C}_{(6)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(\max_{1 \leq j \leq N} |\bar{u}_{j,T_h}| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

Moreover, by the same argument, we can also show that there exists a constant $\bar{C}_{(7)} > 0$ such that

$$P \left(\max_{1 \leq i, j \leq N} |\bar{u}_{i,T_h}| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \bar{C}_{(7)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right).$$

Finally, to show part (h), note again that by part (b) of this lemma, there exists a constant $C_{\dagger} > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right).$$

Now, set $\overline{C}_{(8)} = C_{\dagger}^2$, and note that

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} |\overline{u}_{i, T_h}| |\overline{u}_{j, T_h}| \geq \overline{C}_{(8)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{jt} \right| \geq \overline{C}_{(8)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\leq P \left(\left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq \sqrt{\overline{C}_{(8)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \cup \left\{ \max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{jt} \right| \geq \sqrt{\overline{C}_{(8)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq \sqrt{\overline{C}_{(8)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \\
&\quad + P \left(\max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{jt} \right| \geq \sqrt{\overline{C}_{(8)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \quad (\text{by Bonferroni inequality}) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq \sqrt{\overline{C}_{(8)}} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(\max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{jt} \right| \geq \sqrt{\overline{C}_{(8)}} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad \left(\text{since } \sqrt{\frac{\ln N}{T_h}} < \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \text{ given that } \frac{\ln N}{T_h} = \frac{\ln N}{T-h} < 1 \text{ for all } N, T \text{ sufficiently large} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(\max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{jt} \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad \left(\text{since } \sqrt{\overline{C}_{(8)}} = C_{\dagger} \right).
\end{aligned}$$

It then follows, by part (b) of this lemma, that for all N and T sufficiently large,

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} |\overline{u}_{i, T_h}| |\overline{u}_{j, T_h}| \geq \overline{C}_{(8)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(\max_{1 \leq j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{jt} \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
&= o \left(\frac{1}{N^2} \right). \quad \square
\end{aligned}$$

Lemma E-10: Suppose that Assumptions (i), (ii), (iv), and (v) hold. Let γ_* be a positive

constant such that

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2}$$

where γ_1 and γ_2 are the positive constants defined in Assumption (v) and suppose that

$$\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \text{ as } N, T \rightarrow \infty.$$

Under these conditions, there exist a constant $\overline{B}_1 > 0$ not depending on (N, T) such that

$$P \left(\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \geq \overline{B}_1 \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right), \text{ where } T_h = T - h.$$

or

$$\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| = O_p \left(\sqrt{\frac{\ln N}{T}} \right).$$

Proof of Lemma E-10: Note first that

$$\begin{aligned}
& \widehat{\Sigma}_X - \overline{\Sigma}_X \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} (X_t - \overline{X}_{T_h}) (X_t - \overline{X}_{T_h})' - \overline{\Sigma}_X \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} (X_t - \mu_X + \mu_X - \overline{X}_{T_h}) (X_t - \mu_X + \mu_X - \overline{X}_{T_h})' - \overline{\Sigma}_X \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} (X_t - \mu_X) (X_t - \mu_X)' - \overline{\Sigma}_X - \frac{1}{T_h} \sum_{t=1}^{T_h} (X_t - \mu_X) (\overline{X}_{T_h} - \mu_X)' \\
&\quad - \frac{1}{T_h} \sum_{t=1}^{T_h} (\overline{X}_{T_h} - \mu_X) (X_t - \mu_X)' + \frac{1}{T_h} \sum_{t=1}^{T_h} (\overline{X}_{T_h} - \mu_X) (\overline{X}_{T_h} - \mu_X)' \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} (X_t - \mu_X) (X_t - \mu_X)' - \overline{\Sigma}_X - \frac{1}{T_h} \sum_{t=1}^{T_h} (X_t - \mu_X) \frac{1}{T_h} \sum_{s=1}^{T_h} (X_s - \mu_X)' \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\underline{\Lambda}}{\underline{\Delta}_{1,1}} \right) \right\} \Gamma' \\
&\quad + \frac{1}{T_h} \sum_{t=1}^{T_h} \Gamma (F_t - \mu_F) u_t' + \frac{1}{T_h} \sum_{t=1}^{T_h} u_t (F_t - \mu_F)' \Gamma' + \frac{1}{T_h} \sum_{t=1}^{T_h} (u_t u_t' - \overline{\Sigma}_u) \\
&\quad - \frac{1}{T_h} \sum_{s=1}^{T_h} \Gamma (F_s - \mu_F) \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' - \frac{1}{T_h} \sum_{s=1}^{T_h} \Gamma (F_s - \mu_F) \frac{1}{T_h} \sum_{t=1}^{T_h} u_t' \\
&\quad - \frac{1}{T_h} \sum_{s=1}^{T_h} u_s \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' - \frac{1}{T_h} \sum_{s=1}^{T_h} u_s \frac{1}{T_h} \sum_{t=1}^{T_h} u_t'
\end{aligned}$$

so that

$$\begin{aligned}
& \widehat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\underline{\Lambda}}{\underline{\Delta}_{1,1}} \right) \right\} \Gamma' e_{j,N} + \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) u'_t e_{j,N} \\
&+ \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} u_t (F_t - \mu_F)' \Gamma' e_{j,N} + \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} (u_t u'_t - \bar{\Sigma}_u) e_{j,N} \\
&- \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} - \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \frac{1}{T_h} \sum_{t=1}^{T_h} u'_t e_{j,N} \\
&- \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} u_s \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} - \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} u_s \frac{1}{T_h} \sum_{t=1}^{T_h} u'_t e_{j,N}
\end{aligned}$$

Hence, applying the triangle inequality and taking the maximum over all $i, j \in \{1, \dots, N\}$; we obtain

$$\begin{aligned}
& \max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \bar{\Sigma}_{X,(i,j)} \right| \\
&\leq \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\underline{\Lambda}}{\underline{\Delta}_{1,1}} \right) \right\} \Gamma' e_{j,N} \right| \\
&+ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) u_{jt} \right| + \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \\
&+ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (u_{it} u_{jt} - e'_{i,N} \bar{\Sigma}_u e_{j,N}) \right| \\
&+ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \\
&+ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| |\bar{u}_{j,T_h}| + \max_{1 \leq i, j \leq N} |\bar{u}_{i,T_h}| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \\
&+ \max_{1 \leq i, j \leq N} |\bar{u}_{i,T_h}| |\bar{u}_{j,T_h}| \tag{173}
\end{aligned}$$

where $\bar{\Sigma}_{u,(i,j)} = e'_{i,N} \bar{\Sigma}_u e_{j,N}$, $\bar{u}_{i,T_h} = T_h^{-1} \sum_{s=1}^{T_h} u_{is}$, and $\bar{u}_{j,T_h} = T_h^{-1} \sum_{t=1}^{T_h} u_{jt}$. Now, let $\bar{B}_1 > 0$ be a constant such that

$$\bar{B}_1 \geq 8 \max \{ \bar{C}_{(1)}, \bar{C}_{(2)}, \bar{C}_{(3)}, \bar{C}_{(4)}, \bar{C}_{(5)}, \bar{C}_{(6)}, \bar{C}_{(7)}, \bar{C}_{(8)} \}$$

where $\bar{C}_{(1)}$, $\bar{C}_{(2)}$, $\bar{C}_{(3)}$, $\bar{C}_{(4)}$, $\bar{C}_{(5)}$, $\bar{C}_{(6)}$, $\bar{C}_{(7)}$, and $\bar{C}_{(8)}$ are constants as defined in parts

(c), (d), (e), (f), (g), and (h) of Lemma E-9. Given expression (173) and given the above

definition of the constant $\overline{B}_1$, we can write

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \geq \overline{B}_1 \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\Lambda}{\underline{\lambda}_{1,1}} \right) \right\} \Gamma' e_{j,N} \right| \right. \\
& + \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) u_{jt} \right| + \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \\
& + \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (u_{it} u_{jt} - \overline{\Sigma}_{u,(i,j)}) \right| \\
& + \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \\
& + \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| |\overline{u}_{j,T_h}| + \max_{1 \leq i, j \leq N} |\overline{u}_{i,T_h}| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \\
& \left. + \max_{1 \leq i, j \leq N} |\overline{u}_{i,T_h}| |\overline{u}_{j,T_h}| \geq \overline{B}_1 \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\left\{ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\Lambda}{\underline{\lambda}_{1,1}} \right) \right\} \Gamma' e_{j,N} \right| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \right. \\
& \cup \left\{ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) u_{jt} \right| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \cup \left\{ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \cup \left\{ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (u_{it} u_{jt} - e'_{i,N} \overline{\Sigma}_u e_{j,N}) \right| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \cup \left\{ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \cup \left\{ \max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| |\overline{u}_{j,T_h}| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \cup \left\{ \max_{1 \leq i, j \leq N} |\overline{u}_{i,T_h}| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \left. \cup \left\{ \max_{1 \leq i, j \leq N} |\overline{u}_{i,T_h}| |\overline{u}_{j,T_h}| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \right)
\end{aligned}$$

It then follows, by applying the Bonferroni inequality, that

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X, (i, j)} - \overline{\Sigma}_{X, (i, j)} \right| \geq \overline{B}_1 \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i, N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\Lambda}{\Delta_{1,1}} \right) \right\} \Gamma' e_{j, N} \right| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i, N} \Gamma (F_t - \mu_F) u_{jt} \right| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_t - \mu_F)' \Gamma' e_{j, N} \right| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (u_{it} u_{jt} - e'_{i, N} \overline{\Sigma}_u e_{j, N}) \right| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i, N} \Gamma (F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j, N} \right| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i, N} \Gamma (F_s - \mu_F) \right| |\overline{u}_{j, T_h}| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} |\overline{u}_{i, T_h}| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j, N} \right| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} |\overline{u}_{i, T_h}| |\overline{u}_{j, T_h}| \geq \frac{\overline{B}_1}{8} \sqrt{\frac{\ln N}{T_h}} \right)
\end{aligned}$$

Moreover, note that $\overline{B}_1 \geq 8 \max \{ \overline{C}_{(1)}, \overline{C}_{(2)}, \overline{C}_{(3)}, \overline{C}_{(4)}, \overline{C}_{(5)}, \overline{C}_{(6)}, \overline{C}_{(7)}, \overline{C}_{(8)} \}$ implies that $\overline{B}_1/8 \geq \overline{C}_{(k)}$ for $k = 1, 2, \dots, 8$ and applying the results given in parts (c), (d), (e), (f),

(g), and (h) of Lemma E-9, we further deduce that

$$\begin{aligned}
& P \left(\max_{1 \leq i, j \leq N} \left| \widehat{\Sigma}_{X,(i,j)} - \overline{\Sigma}_{X,(i,j)} \right| \geq \overline{B}_1 \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_t - \mu_F)' - \left(\frac{\Lambda}{\Delta_{1,1}} \right) \right\} \Gamma' e_{j,N} \right| \geq \overline{C}_{(1)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) u_{jt} \right| \geq \overline{C}_{(2)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \overline{C}_{(3)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (u_{it} u_{jt} - e'_{i,N} \overline{\Sigma}_u e_{j,N}) \right| \geq \overline{C}_{(4)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \overline{C}_{(5)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| |\overline{u}_{j,T_h}| \geq \overline{C}_{(6)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} |\overline{u}_{i,T_h}| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \Gamma' e_{j,N} \right| \geq \overline{C}_{(7)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i, j \leq N} |\overline{u}_{i,T_h}| |\overline{u}_{j,T_h}| \geq \overline{C}_{(8)} \sqrt{\frac{\ln N}{T_h}} \right) \\
= & o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
= & o \left(\frac{1}{N^2} \right). \quad \square
\end{aligned}$$

Lemma E-11: Suppose that Assumption S1-1 holds under Scenario 1. Let $T_h = T - h$, where $h \geq 1$ is a (fixed) finite integer; and suppose in addition that Assumptions (i), (ii), (iv), and (v) hold. Furthermore, suppose that $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$, where

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < 1$$

and where γ_1 and γ_2 are positive constants as defined in Assumption (v). Under these conditions, the following statements are true:

(a) There exists a constant $C_\Delta > 0$ such that

$$P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_\Delta \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

(b) There exists a constant $C_{(1)} > 0$ such that

$$\begin{aligned} & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \left(e'_{i,N} \Gamma (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - e'_{i,N} \Gamma_{\lambda,1} \frac{\Lambda_1}{\sqrt{\Delta_{1,1}}} \right) \varphi_1 \right| \geq C_{(1)} \sqrt{\frac{\ln N}{T_h}} \right) \\ &= o \left(\frac{1}{N^2} \right) \end{aligned}$$

(c) There exists a constant $C_{(2)} > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{(2)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

(d) There exists a constant $C_{(3)} > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{(3)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

(e) There exists a constant $C_{(4)} > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{(4)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

Proof of Lemma E-11:

To show part (a), note first that, by definition $\mu_{F,1} = E[F_{1,t}]$, so that

$$E \left[(F_{1,t} - \mu_{F,1})' \varphi_1 \right] = E \left[(F_{1,t} - \mu_{F,1})' \right] \varphi_1 = 0.$$

Moreover, under the result given in part (a) of Lemma E-6, there exist constants $\gamma_1 > 0$ and $\bar{b}_2 > 0$, such that for any $s > 0$ and for all $t \in \mathbb{Z}$,

$$\begin{aligned} P \left(\left| (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) &\leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \\ &\leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}_2} \right)^{\gamma_1} \right\} \end{aligned}$$

for some constant $\bar{C} \geq 1$. Next, let $Z_t^\# = (F_{1,t} - \mu_{F,1})' \varphi_1$, and let $\underline{V}_t = (\varepsilon_t, F_t', u_t')'$ be as defined in part (b) of Assumption (v). Note that $\mathcal{F}_{Z^\#-\infty}^t = \sigma(\dots, Z_{t-2}^\#, Z_{t-1}^\#, Z_t^\#) \subseteq \mathcal{F}_{-\infty}^t = \sigma(\dots, \underline{V}_{t-2}, \underline{V}_{t-1}, \underline{V}_t)$ and $\mathcal{F}_{Z^\# t+m}^\infty = \sigma(Z_{t+m}^\#, Z_{t+m+1}^\#, Z_{t+m+2}^\#, \dots) \subseteq \mathcal{F}_{t+m}^\infty = \sigma(\underline{V}_{t+m}, \underline{V}_{t+m+1}, \underline{V}_{t+m+2}, \dots)$. Hence, it follows by part (b) of Assumption (v) and Theorem 14.1 of Davidson (1994) that $\{Z_t^\#\}_{t \in \mathbb{Z}}$ is an α -mixing process with α coefficient $\alpha_{Z^\#}(m) = \sup_t \alpha(\mathcal{F}_{Z^\#-\infty}^t, \mathcal{F}_{Z^\# t+m}^\infty)$ satisfying the rate of decay

$$\begin{aligned} \alpha_{Z^\#}(m) &= \sup_t \alpha(\mathcal{F}_{Z^\#-\infty}^t, \mathcal{F}_{Z^\# t+m}^\infty) \\ &\leq \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1 \end{aligned}$$

and for some positive constants γ_2 , a and c . In addition, under Assumption (v), there exists a constant $\gamma \in (0, 1)$ such that

$$\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.$$

Hence, all the assumptions needed to obtain the Bernstein inequality given in Merlevède, Peligrad, and Rio (2011) are fulfilled. It follows from this inequality that there exist positive constants C_1, C_2, C_3, C_4 and C_5 that depend on $a, \bar{b}_2, c, \gamma$, and γ_2 such that for, any $s > 0$,

$$\begin{aligned} &P\left(\left|\frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1\right| \geq s\right) \\ &= P\left(\left|\sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1\right| \geq sT_h\right) \\ &\leq T_h \exp\left\{-\frac{(sT_h)^\gamma}{C_1}\right\} + \exp\left\{-\frac{s^2 T_h^2}{C_2(1 + T_h C_3)}\right\} + \exp\left\{-\frac{(sT_h)^2}{C_4 T_h} \exp\left\{\frac{(sT_h)^{\gamma(1-\gamma)}}{C_5 (\ln sT_h)^\gamma}\right\}\right\} \end{aligned}$$

For some $C_\Delta > 0$, let

$$s = C_\Delta \sqrt{\frac{\ln N}{T_h}}.$$

By choosing C_Δ to be large enough, we can obtain the following three results, which we will show in turn:

(I)

$$T_h \exp\left\{-\frac{(sT_h)^\gamma}{C_1}\right\} = T_h \exp\left\{-\frac{C_\Delta^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}\right\} = o\left(\frac{1}{N^2}\right).$$

(II)

$$\exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_5 (\ln [sT_h])^\gamma} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(III)

$$\exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} = \exp \left\{ -\frac{C_\Delta^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = o \left(\frac{1}{N^2} \right).$$

To show result (I), first write

$$N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_1} \right\} = N^2 T_h \exp \left\{ -\frac{C_\Delta^\gamma}{C_1} T_h^\gamma \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma}{2}} \right\} = N^2 T_h \exp \left\{ -\frac{C_\Delta^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\}$$

Taking the log transformation, we have

$$\ln \left[N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_1} \right\} \right] = 2 \ln N + \ln T_h - \frac{C_\Delta^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}$$

Now, by assumption, $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$, where

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2}.$$

Since, given Assumption (v), we have

$$1 > \gamma = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} > \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} = \gamma_* > 0;$$

it follows that $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$ implies that $(\ln N)^{\frac{2}{\gamma}-1} = o(T)$. Moreover, since

$$\frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0 \quad (\text{given that } \gamma > 0)$$

and since

$$\begin{aligned} \frac{2(\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} &= 2 \left(\frac{T}{T_h} \right)^{\frac{\gamma}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \right)^{\frac{\gamma}{2}} \rightarrow 0 \quad \left(\text{given that } \frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \rightarrow 0 \text{ and} \right. \\ &\quad \left. \text{given that } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \text{ since } h \text{ is a fixed finite integer} \right), \end{aligned}$$

we see that

$$\frac{2 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} = \frac{2 (\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned} N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_1} \right\} &= \exp \left\{ 2 \ln N + \ln T_h - \frac{C_\Delta^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\} \\ &= \exp \left\{ -\frac{C_\Delta^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \left[1 - \frac{C_1}{C_\Delta^\gamma} \frac{2 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \right] \right\} \\ &= o(1) \end{aligned}$$

It follows that

$$T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_1} \right\} = \frac{1}{N^2} N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (II), note that

$$\begin{aligned} & N^2 \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_5 (\ln [sT_h])^\gamma} \right\} \right\} \\ &= N^2 \exp \left\{ -\frac{T_h^2}{C_4 T_h} \left(C_\Delta \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma(1-\gamma)}}{C_5 \left(\ln \left[T_h C_\Delta \sqrt{\frac{\ln N}{T_h}} \right] \right)^\gamma} \left(C_\Delta \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma(1-\gamma)} \right) \right\} \\ &= N^2 \exp \left\{ -\frac{C_\Delta^2}{C_4} (\ln N) \exp \left\{ \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} C_\Delta^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 \left(\ln T_h + \ln C_\Delta + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h \right)^\gamma} \right) \right\} \right\} \\ &= N^2 \exp \left\{ -\frac{C_\Delta^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} C_\Delta^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_\Delta + \frac{1}{2} \ln \ln N \right)^\gamma} \right) \right\} \\ &= \exp \left\{ -\ln N \left[\frac{C_\Delta^2}{C_4} \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} C_\Delta^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_\Delta + \frac{1}{2} \ln \ln N \right)^\gamma} \right) - 2 \right] \right\} \\ &= o(1) \end{aligned}$$

This, in turn, implies that

$$\begin{aligned} \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} &= \frac{1}{N^2} N^2 \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_5 (\ln [sT_h])^\gamma} \right\} \right\} \\ &= o \left(\frac{1}{N^2} \right) \end{aligned}$$

Turning our attention now to showing result (III), we first write

$$\begin{aligned} N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} &= N^2 \exp \left\{ -\frac{T_h^2}{C_2 (1 + T_h C_3)} \left(C_\Delta \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\ &= N^2 \exp \left\{ -\frac{C_\Delta^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned} \ln \left[N^2 \exp \left\{ -\frac{C_\Delta^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \right] &= 2 \ln N - \frac{C_\Delta^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \\ &= 2 \ln N - \frac{C_\Delta^2}{C_2 C_3} \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \end{aligned}$$

Now, let C_Δ be a positive constant such that $C_\Delta^2 > 2C_2 (1 + C_3)$, and set

$$C^* = \frac{C_\Delta^2}{C_2 C_3} - 2.$$

Note that, by this choice of C_Δ and C^* , we have

$$C^* > \frac{2C_2 (1 + C_3)}{C_2 C_3} - 2 = \frac{2}{C_3} > 0$$

It, thus, follows that

$$\begin{aligned}
& N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} \\
= & N^2 \exp \left\{ -\frac{C_\Delta^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \\
= & \exp \left\{ \left(2 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - \frac{C_\Delta^2}{C_2 C_3} \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(2 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - [C^* + 2] \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(\ln N \left[2 C_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ -C^* \ln N \left[1 - \frac{2}{C^* C_3} \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
< & \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{2}{C^* C_3} < \frac{C_3 2}{2 C_3} = 1 \right) \\
= & o(1)
\end{aligned}$$

This, in turn, implies that

$$\exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} = \left(\frac{1}{N^2} \right) N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (III).

Combining the results given in (I)-(III) above, we obtain

$$\begin{aligned}
& P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_\Delta \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & T_h \exp \left\{ -\frac{(s T_h)^\gamma}{C_1} \right\} + \exp \left\{ -\frac{(s T_h)^2}{C_4 T_h} \exp \left\{ \frac{(s T_h)^{\gamma(1-\gamma)}}{C_5 (\ln s T_h)^\gamma} \right\} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} \\
= & o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
= & o \left(\frac{1}{N^2} \right).
\end{aligned}$$

To show part (b), note first that, if we let $\varphi = (\varphi'_1, \varphi'_2)'$ where $\varphi_2 = 0$; it is easy to see

that we can rewrite

$$\begin{aligned}
& \frac{1}{T_h} \sum_{t=1}^{T_h} \left(e'_{i,N} \Gamma (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - e'_{i,N} \Gamma_{\lambda,1} \frac{\Lambda_1}{\sqrt{\Delta_{1,1}}} \right) \varphi_1 \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) \begin{pmatrix} (F_{1,t} - \mu_{F,1})' & (F_{2,t} - \mu_{F,2})' \end{pmatrix} \begin{pmatrix} \varphi_1 \\ 0 \end{pmatrix} \\
&\quad - e'_{i,N} \frac{\begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix}}{\sqrt{\Delta_{1,1}}} \begin{pmatrix} \Lambda_1/\sqrt{\Delta_{1,1}} & 0 \\ 0 & \Lambda_2/\sqrt{\Delta_{1,1}} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ 0 \end{pmatrix} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) (F_t - \mu_F)' \varphi - e'_{i,N} \Gamma (\underline{\Lambda}/\Delta_{1,1}) \varphi \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) (F_t - \mu_F)' \varphi \\
&\quad - e'_{i,N} \Gamma \frac{1}{T_h} \sum_{t=1}^{T_h} E [(F_t - \mu_F) (F_t - \mu_F)'] \varphi \quad \left(\text{since } \frac{1}{T_h} \sum_{t=1}^{T_h} E [(F_t - \mu_F) (F_t - \mu_F)'] = \frac{\underline{\Lambda}}{\Delta_{1,1}} \right) \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - E [(F_t - \mu_F) (F_{1,t} - \mu_{F,1})'] \right\} \varphi_1 \quad (\text{since } \varphi_2 = 0)
\end{aligned}$$

from which it follows that the elements of the sum given above all have expectation equal to zero, i.e.,

$$\begin{aligned}
& E \left[e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - E [(F_t - \mu_F) (F_{1,t} - \mu_{F,1})'] \right\} \varphi_1 \right] \\
&= e'_{i,N} \Gamma \left\{ E [(F_t - \mu_F) (F_{1,t} - \mu_{F,1})'] - E [(F_t - \mu_F) (F_{1,t} - \mu_{F,1})'] \right\} \varphi_1 \\
&= 0 \text{ for all } i \text{ and } t.
\end{aligned}$$

Next, let γ_1 , γ_2 , and γ be positive constants as defined in Assumption (v). Note that, by part (b) of Lemma E-6, there exists a positive constant $\overline{C} \geq \exp \{1\}$ such that, for any $s > 0$, for all integer i such that $1 \leq i \leq N$, and for all $t \in \mathbb{Z}$

$$P \left(\left| e'_{i,N} \Gamma (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \leq \overline{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\}$$

where $r = \gamma_1/2 > 0$ and $\bar{b} \geq (1 + \ln 2)^{\frac{2}{\gamma_1}} \bar{b}_2^2 > 0$ and where $\bar{b}_2$ is also a positive constant which has been previously defined in Lemma E-6. Next, let $\gamma_* = \gamma_1 \gamma_2 / (\gamma_1 + 2\gamma_2)$, so that

$$\frac{1}{\gamma_*} = \frac{\gamma_1 + 2\gamma_2}{\gamma_1 \gamma_2} = \frac{2}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r} + \frac{1}{\gamma_2}.$$

Now, the relationship between the positive constants γ_1 , γ_2 , and γ as specified in Assumption (v) implies that

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} = \gamma < 1$$

so that, in particular, $\gamma_* \in (0, 1)$. In addition, let $Z_{i,t}^\% =$

$e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - E \left[(F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \right] \right\} \varphi_1$, and let $\underline{V}_t = (\varepsilon_t, F'_t, u'_t)'$ be as defined in part (b) of Assumption (v). Note that $\mathcal{F}_{Z^\%, i-\infty}^t = \sigma(\dots, Z_{i,t-2}^\%, Z_{i,t-1}^\%, Z_{i,t}^\%) \subseteq \mathcal{F}_{-\infty}^t = \sigma(\dots, \underline{V}_{t-2}, \underline{V}_{t-1}, \underline{V}_t)$ and $\mathcal{F}_{Z^\%, i+t+m}^\infty = \sigma(Z_{i,t+m}^\%, Z_{i,t+m+1}^\%, Z_{i,t+m+2}^\%, \dots) \subseteq \mathcal{F}_{t+m}^\infty = \sigma(\underline{V}_{t+m}, \underline{V}_{t+m+1}, \underline{V}_{t+m+2}, \dots)$. Hence, it follows by part (b) of Assumption (v) and Theorem 14.1 of Davidson (1994) that $\{Z_{i,t}^\%\}_{t \in \mathbb{Z}}$ is an α -mixing process with α coefficient $\alpha_{Z^\%, i}(m) = \sup_t \alpha(\mathcal{F}_{Z^\%, i-\infty}^t, \mathcal{F}_{Z^\%, i+t+m}^\infty)$ satisfying the rate of decay

$$\begin{aligned} \alpha_{Z^\%, i}(m) &= \sup_t \alpha(\mathcal{F}_{Z^\%, i-\infty}^t, \mathcal{F}_{Z^\%, i+t+m}^\infty) \\ &\leq \sup_t \alpha(\underline{\mathcal{F}}_{-\infty}^t, \underline{\mathcal{F}}_{t+m}^\infty) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1, \text{ for all } i, \end{aligned}$$

and for some positive constants γ_2 , a and c .

Hence, all the assumptions needed to apply the Bernstein inequality given in Lemma C-1 are fulfilled. Making use of this Bernstein inequality then implies that there exist positive constants C_1, C_2, C_3, C_4 and C_5 depending on $a, \bar{b}, c, \gamma_*$, and γ_2 such that, for any $s > 0$ and for all integer i with $1 \leq i \leq N$,

$$\begin{aligned} &P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} \left(e'_{i,N} \Gamma (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - e'_{i,N} \Gamma_{\lambda,1} \frac{\underline{\Delta}_1}{\sqrt{\underline{\Delta}_{1,1}}} \right) \varphi_1 \right| \geq s \right) \\ &= P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - E \left[(F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \right] \right\} \varphi_1 \right| \geq s \right) \\ &= P \left(\left| \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - E \left[(F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \right] \right\} \varphi_1 \right| \geq T_h s \right) \\ &\leq T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} + \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_* (1-\gamma_*)}}{C_5 (\ln sT_h)^{\gamma_*}} \right\} \right\} \end{aligned}$$

Using Bonferroni's inequality, we further obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \left(e'_{i,N} \Gamma (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - e'_{i,N} \Gamma_{\lambda,1} \frac{\underline{\Lambda}_1}{\sqrt{\Delta_{1,1}}} \right) \varphi_1 \right| \geq s \right) \\
& \leq N \max_{1 \leq i \leq N} P \left(\left| \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - E \left[(F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \right] \right\} \varphi_1 \right| \geq s T_h \right) \\
& \leq N T_h \exp \left\{ -\frac{(s T_h)^{\gamma_*}}{C_1} \right\} + N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} \\
& \quad + N \exp \left\{ -\frac{(s T_h)^2}{C_4 T_h} \exp \left\{ \frac{(s T_h)^{\gamma_* (1 - \gamma_*)}}{C_5 (\ln s T_h)^{\gamma_*}} \right\} \right\}
\end{aligned}$$

Now, for some $C_{(1)} > 0$, let

$$s = C_{(1)} \sqrt{\frac{\ln N}{T_h}}.$$

By choosing $C_{(1)}$ to be large enough, we can obtain the following three results, which we will show in turn:

(I)

$$N T_h \exp \left\{ -\frac{(s T_h)^{\gamma_*}}{C_1} \right\} = N T_h \exp \left\{ -\frac{C_{(1)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(II)

$$N \exp \left\{ -\frac{(s T_h)^2}{C_4 T_h} \exp \left\{ \frac{(s T_h)^{\gamma_* (1 - \gamma_*)}}{C_5 (\ln [s T_h])^{\gamma_*}} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(III)

$$N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} = N \exp \left\{ -\frac{C_{(1)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = o \left(\frac{1}{N^2} \right).$$

To show result (I), first write

$$\begin{aligned}
N^3 T_h \exp \left\{ -\frac{(s T_h)^{\gamma_*}}{C_1} \right\} &= N^3 T_h \exp \left\{ -\frac{C_{(1)}^{\gamma_*}}{C_1} T_h^{\gamma_*} \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma_*}{2}} \right\} \\
&= N^3 T_h \exp \left\{ -\frac{C_{(1)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\}
\end{aligned}$$

Taking the log transformation, we have

$$\ln \left[N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} \right] = 3 \ln N + \ln T_h - \frac{C_{(1)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}$$

Given that

$$\frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0 \quad (\text{since } \gamma_* > 0)$$

and given that

$$\begin{aligned} \frac{3 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} &= 3 \left(\frac{T}{T_h} \right)^{\frac{\gamma_*}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \right)^{\frac{\gamma_*}{2}} \rightarrow 0 \\ &\left(\text{given that } \frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \text{ by assumption and} \right. \\ &\left. \text{given that } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \text{ since } h \text{ is a fixed finite integer} \right) \end{aligned}$$

we see that

$$\frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} = \frac{3 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned} N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} &= \exp \left\{ 3 \ln N + \ln T_h - \frac{C_{(1)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \\ &= \exp \left\{ -\frac{C_{(1)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \left[1 - \frac{C_1}{C_{(1)}^{\gamma_*}} \frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \right] \right\} \\ &= o(1) \end{aligned}$$

It follows that

$$N T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} = \frac{1}{N^2} N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (II), note that

$$\begin{aligned}
& N^3 \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} \\
&= N^3 \exp \left\{ -\frac{T_h^2}{C_4 T_h} \left(C_{(1)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma_*(1-\gamma_*)} \left(C_{(1)} \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma_*(1-\gamma_*)}}{C_5 \left(\ln \left[T_h C_{(1)} \sqrt{\frac{\ln N}{T_h}} \right] \right)^{\gamma_*}} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{C_{(1)}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} C_{(1)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 (\ln T_h + \ln C_{(1)} + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h)^{\gamma_*}} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{C_{(1)}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} C_{(1)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_{(1)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) \right\} \\
&= \exp \left\{ -\ln N \left[\frac{C_{(1)}^2}{C_4} \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} C_{(1)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_{(1)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) - 3 \right] \right\} \\
&= o(1)
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
& N \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} \\
&= \left(\frac{1}{N^2} \right) N^3 \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} \\
&= O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right)
\end{aligned}$$

Turning our attention now to showing result (III), we first write

$$\begin{aligned}
N^3 \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} &= N^3 \exp \left\{ -\frac{T_h^2}{C_2 (1 + T_h C_3)} \left(C_{(1)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\
&= N^3 \exp \left\{ -\frac{C_{(1)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\}
\end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned} \ln \left[N^3 \exp \left\{ -\frac{C_{(1)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \right] &= 3 \ln N - \frac{C_{(1)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \\ &= 3 \ln N - \frac{C_{(1)}^2}{C_2 C_3} \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \end{aligned}$$

Now, let $C_{(1)}$ be a positive constant such that $C_{(1)}^2 > 3C_2(1 + C_3)$, and set

$$C^* = \frac{C_{(1)}^2}{C_2 C_3} - 3.$$

Note that, by this choice of $C_{(1)}$ and C^* , we have

$$C^* > \frac{3C_2(1 + C_3)}{C_2 C_3} - 3 = \frac{3}{C_3} > 0$$

It, thus, follows that

$$\begin{aligned} & N^3 \exp \left\{ -\frac{s^2 T_h^2}{C_2(1 + T_h C_3)} \right\} \\ &= N^3 \exp \left\{ -\frac{C_{(1)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \\ &= \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - \frac{C_{(1)}^2}{C_2 C_3} \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - [C^* + 3] \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ \left(\ln N \left[3C_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ -C^* \ln N \left[1 - \frac{3}{C^* C_3} \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &< \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{3}{C^* C_3} < \frac{C_3 3}{3C_3} = 1 \right) \\ &= o(1) \end{aligned}$$

This, in turn, implies that

$$\begin{aligned} N \exp \left\{ -\frac{sT_h^2}{C_2(1+T_hC_3)} \right\} &= \left(\frac{1}{N^2} \right) N^3 \exp \left\{ -\frac{sT_h^2}{C_2(1+T_hC_3)} \right\} \\ &= O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right) \end{aligned}$$

which shows result (III)

Combining the results given in (I)-(III) above, we obtain

$$\begin{aligned} &P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \left(e'_{i,N} \Gamma(F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - e'_{i,N} \Gamma_{\lambda,1} \frac{\underline{\Lambda}_1}{\sqrt{\underline{\Lambda}_{1,1}}} \right) \varphi_1 \right| \geq C_{(1)} \sqrt{\frac{\ln N}{T_h}} \right) \\ &\leq N \max_{1 \leq i \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma \left\{ (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - E \left[(F_t - \mu_F) (F_{1,t} - \mu_{F,1})' \right] \right\} \varphi_1 \right| \right. \\ &\quad \left. \geq C_{(1)} \sqrt{\frac{\ln N}{T_h}} \right) \\ &\leq NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} + N \exp \left\{ -\frac{s^2 T_h^2}{C_2(1+T_hC_3)} \right\} \\ &\quad + N \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln sT_h)^\gamma} \right\} \right\} \\ &= o \left(\frac{1}{N^2} \right) \end{aligned}$$

which shows the desired result for part (b).

To show part (c), first note that, by part (a) of Assumption (ii),

$$E \left[u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right] = E \left[u_{it} (F_{1,t} - \mu_{F,1})' \right] \varphi_1 = 0 \text{ for all } i \text{ and } t.$$

Next, note that by part (c) of Lemma E-6, there exist constants $\gamma_1 > 0$ and $\bar{b} > 0$ and a positive constant $\bar{C} \geq \exp \{1\}$ such that, $\forall s > 0$, for all integer i with $1 \leq i \leq N$, and for all $t \in \mathbb{Z}$

$$P \left(\left| u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| > s \right) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \leq \bar{C} \exp \left\{ -\left(\frac{s}{\bar{b}} \right)^r \right\},$$

where $r = \gamma_1/2 > 0$. Moreover, let $\gamma_* = \gamma_1\gamma_2/(\gamma_1 + 2\gamma_2)$, so that

$$\frac{1}{\gamma_*} = \frac{\gamma_1 + 2\gamma_2}{\gamma_1\gamma_2} = \frac{2}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r} + \frac{1}{\gamma_2}.$$

Now, the relationship between the positive constants γ_1 , γ_2 , and γ as specified in Assumption (v) implies that

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} = \gamma < 1$$

so that, in particular, $\gamma_* \in (0, 1)$. In addition, let $V_{i,t}^\# = u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1$, and let $\underline{V}_t = (\varepsilon_t, F_t', u_t')'$ be as defined in part (b) of Assumption (v). Note that $\mathcal{F}_{V^\#, i-\infty}^t = \sigma(\dots, V_{i,t-2}^\#, V_{i,t-1}^\#, V_{i,t}^\#) \subseteq \mathcal{F}_{-\infty}^t = \sigma(\dots, \underline{V}_{t-2}, \underline{V}_{t-1}, \underline{V}_t)$ and $\mathcal{F}_{V^\#, i+t+m}^\infty = \sigma(V_{i,t+m}^\#, V_{i,t+m+1}^\#, V_{i,t+m+2}^\#, \dots)$. Hence, it follows by part (b) of Assumption (v) and Theorem 14.1 of Davidson (1994) that $\{V_{i,t}^\#\}_{t \in \mathbb{Z}}$ is an α -mixing process with α coefficient $\alpha_{V^\#, i}(m) = \sup_t \alpha(\mathcal{F}_{V^\#, i-\infty}^t, \mathcal{F}_{V^\#, i+t+m}^\infty)$ satisfying the rate of decay

$$\begin{aligned} \alpha_{V^\#, i}(m) &= \sup_t \alpha(\mathcal{F}_{V^\#, i-\infty}^t, \mathcal{F}_{V^\#, i+t+m}^\infty) \\ &\leq \sup_t \alpha(\mathcal{F}_{-\infty}^t, \mathcal{F}_{t+m}^\infty) \\ &\leq a \exp(-cm^{\gamma_2}) \text{ for all } m \geq 1, \text{ for all } i, \end{aligned}$$

and for some positive constants γ_2 , a and c .

Hence, all the assumptions needed to apply the Bernstein inequality given in Merlevède, Peligrad, and Rio (2011) (restated here as Lemma C-1) are fulfilled. Application of this Bernstein inequality then implies that there exist positive constants C_1, C_2, C_3, C_4 and C_5 depending on $a, \bar{b}, c, \gamma_*$, and γ_2 such that, for any $s > 0$ and for all integer i with $1 \leq i \leq N$,

$$\begin{aligned} &P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq s \right) \\ &P \left(\left| \sum_{t=1}^{T_h} u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq T_h s \right) \\ &\leq T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} + \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_* (1-\gamma_*)}}{C_5 (\ln sT_h)^{\gamma_*}} \right\} \right\} \end{aligned}$$

Using Bonferroni's inequality, we further obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq s \right) \\
& \leq N \max_{1 \leq i \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq s \right) \\
& \leq NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} + N \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln sT_h)^{\gamma_*}} \right\} \right\} \\
& \quad + N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\}
\end{aligned}$$

For some $C_{(2)} > 0$, let

$$s = C_{(2)} \sqrt{\frac{\ln N}{T_h}}.$$

By choosing $C_{(2)}$ to be large enough, we can obtain the following three results, which we will show in turn:

(I)

$$NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} = NT_h \exp \left\{ -\frac{C_{(2)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(II)

$$N \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(III)

$$N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} = N \exp \left\{ -\frac{C_{(2)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = O \left(\frac{1}{N^2} \right).$$

To show result (I), first write

$$\begin{aligned}
N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} &= N^3 T_h \exp \left\{ -\frac{C_{(2)}^{\gamma_*}}{C_1} T_h^{\gamma_*} \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma_*}{2}} \right\} \\
&= N^3 T_h \exp \left\{ -\frac{C_{(2)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\}
\end{aligned}$$

Taking the log transformation, we have

$$\ln \left[N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} \right] = 3 \ln N + \ln T_h - \frac{C_{(2)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}$$

Since

$$\frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0 \quad (\text{given that } \gamma_* > 0)$$

and since

$$\begin{aligned} \frac{3 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} &= 3 \left(\frac{T}{T_h} \right)^{\frac{\gamma_*}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \right)^{\frac{\gamma_*}{2}} \rightarrow 0 \\ &\left(\text{given that, by assumption, } \frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \right. \\ &\left. \text{given that } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \text{ since } h \text{ is a fixed finite integer} \right), \end{aligned}$$

we see that

$$\frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} = \frac{3 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned} N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} &= \exp \left\{ 3 \ln N + \ln T_h - \frac{C_{(2)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \\ &= \exp \left\{ -\frac{C_{(2)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \left[1 - \frac{C_1}{C_{(3)}^{\gamma_*}} \frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \right] \right\} \\ &= o(1) \end{aligned}$$

It follows that

$$N T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} = \frac{1}{N^2} N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (II), note that

$$\begin{aligned}
& N^3 \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} \\
&= N^3 \exp \left\{ -\frac{T_h^2}{C_4 T_h} \left(C_{(2)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma_*(1-\gamma_*)} \left(C_{(2)} \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma_*(1-\gamma_*)}}{C_5 \left(\ln \left[T_h C_{(2)} \sqrt{\frac{\ln N}{T_h}} \right] \right)^{\gamma_*}} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{C_{(2)}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} C_{(2)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 (\ln T_h + \ln C_{(2)} + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h)^{\gamma_*}} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{C_{(2)}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} C_{(2)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_{(2)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) \right\} \\
&= \exp \left\{ -\ln N \left[\frac{C_{(2)}^2}{C_4} \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} C_{(2)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_{(2)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) - 3 \right] \right\} \\
&= o(1)
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
N \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} &= \frac{1}{N^2} N^3 \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} \\
&= o\left(\frac{1}{N^2}\right)
\end{aligned}$$

Turning our attention now to showing result (III), we first write

$$\begin{aligned}
N^3 \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} &= N^3 \exp \left\{ -\frac{T_h^2}{C_2 (1 + T_h C_3)} \left(C_{(2)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\
&= N^3 \exp \left\{ -\frac{C_{(2)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\}
\end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned} \ln \left[N^3 \exp \left\{ -\frac{C_{(2)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \right] &= 3 \ln N - \frac{C_{(2)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \\ &= 3 \ln N - \frac{C_{(2)}^2}{C_2 C_3} \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \end{aligned}$$

Now, let $C_{(2)}$ be a positive constant such that $C_{(2)}^2 > 3C_2(1 + C_3)$, and set

$$C^* = \frac{C_{(2)}^2}{C_2 C_3} - 3.$$

Note that, by this choice of $C_{(2)}$ and C^* , we have

$$C^* > \frac{3C_2(1 + C_3)}{C_2 C_3} - 3 = \frac{3}{C_3} > 0$$

It, thus, follows that

$$\begin{aligned} & N^3 \exp \left\{ -\frac{s^2 T_h^2}{C_2(1 + T_h C_3)} \right\} \\ &= N^3 \exp \left\{ -\frac{C_{(2)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \\ &= \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - \frac{C_{(2)}^2}{C_2 C_3} \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - [C^* + 3] \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ \left(\ln N \left[3C_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ -C^* \ln N \left[1 - \frac{3}{C^* C_3} \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &< \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{3}{C^* C_3} < \frac{C_3 3}{3C_3} = 1 \right) \\ &= o(1) \end{aligned}$$

This, in turn, implies that

$$N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} = \left(\frac{1}{N^2} \right) N^3 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (III).

Combining the results given in (I)-(III) above, we obtain

$$\begin{aligned} & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{(2)} \sqrt{\frac{\ln N}{T_h}} \right) \\ & \leq N \max_{1 \leq i \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{(2)} \sqrt{\frac{\ln N}{T_h}} \right) \\ & \leq N T_h \exp \left\{ -\frac{(s T_h)^{\gamma_*}}{C_1} \right\} + N \exp \left\{ -\frac{(s T_h)^2}{C_4 T_h} \exp \left\{ \frac{(s T_h)^{\gamma_* (1 - \gamma_*)}}{C_5 (\ln s T_h)^{\gamma_*}} \right\} \right\} \\ & \quad + N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} \\ & = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\ & = o \left(\frac{1}{N^2} \right). \end{aligned}$$

To show part (d), first set $C_{(3)} = \max \{C_{\dagger}^2, C_{\Delta}^2\}$, where $C_{\dagger} > 0$ is the constant given in part (a) of Lemma E-9 and where $C_{\Delta} > 0$ is the constant given in part (a) of this lemma. It then follows by part (a) of Lemma E-9 and (a) of this lemma that, for all N and T sufficiently

large,

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{(3)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq P \left(\left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq \sqrt{C_{(3)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right. \\
& \quad \left. \cup \left\{ \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq \sqrt{C_{(3)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq \sqrt{C_{(3)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \\
& \quad + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq \sqrt{C_{(3)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \quad (\text{by Bonferroni inequality}) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq \sqrt{C_{(3)}} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq \sqrt{C_{(3)}} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad \left(\text{since } \sqrt{\frac{\ln N}{T_h}} < \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \text{ given that } \frac{\ln N}{T_h} = \frac{\ln N}{T-h} < 1 \text{ for all } N, T \text{ sufficiently large} \right) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{\Delta} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad (\text{since } \sqrt{C_{(3)}} = \max \{C_{\dagger}, C_{\Delta}\}) \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \quad (\text{by part (a) of Lemma E-9 and (a) of this lemma}) \\
& = o \left(\frac{1}{N^2} \right)
\end{aligned}$$

To show part (e), first set $C_{(4)} = \max \{C_{\dagger}^2, C_{\Delta}^2\}$, where $C_{\dagger} > 0$ is the constant given in part (b) of Lemma E-9 and where $C_{\Delta} > 0$ is the constant given in part (a) of this lemma.

Note that

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} |\bar{u}_{i,T_h}| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{(4)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{(4)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\leq P \left(\left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq \sqrt{C_{(4)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right. \\
&\quad \left. \cup \left\{ \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq \sqrt{C_{(4)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq \sqrt{C_{(4)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \\
&\quad + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq \sqrt{C_{(4)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \\
&\quad \text{(by Bonferroni inequality)} \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq \sqrt{C_{(4)}} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq \sqrt{C_{(4)}} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad \left(\text{since } \sqrt{\frac{\ln N}{T_h}} < \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \text{ given that } \frac{\ln N}{T_h} = \frac{\ln N}{T-h} < 1 \text{ for all } N, T \text{ sufficiently large} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{\Delta} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad \text{(since } C_{(4)} = \max \{C_{\dagger}^2, C_{\Delta}^2\} \text{ implies that } \sqrt{C_{(4)}} \geq C_{\dagger} \text{ and } \sqrt{C_{(4)}} \geq C_{\Delta} \text{).}
\end{aligned}$$

It then follows, by part (a) of this lemma and (b) of Lemma E-9, that for all N and T

sufficiently large,

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{(4)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{\Delta} \sqrt{\frac{\ln N}{T_h}} \right) \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right). \quad \square
\end{aligned}$$

Lemma E-12: Suppose that Assumption S2-1 holds under Scenario 2, where $\kappa = K$. Let $T_h = T - h$, where $h \geq 1$ is a (fixed) finite integer; and suppose in addition that Assumptions (i), (ii), (iv), and (v) hold. Furthermore, suppose that $(\ln N)^{\frac{2}{\gamma_*} - 1} = o(T)$, where

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < 1$$

and where γ_1 and γ_2 are positive constants as defined in Assumption (v). Under these conditions, the following statements are true:

(a) There exists a constant $C_{\Delta} > 0$ such that

$$P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \varphi \right| \geq C_{\Delta} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

(b) There exists a constant $C_{(1)} > 0$ such that

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \left(e'_{i,N} \Gamma (F_t - \mu_F) (F_t - \mu_F)' - e'_{i,N} \Gamma_{\lambda} \frac{\Lambda}{\sqrt{\Delta_{1,1}}} \right) \varphi \right| \geq C_{(1)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& = o \left(\frac{1}{N^2} \right)
\end{aligned}$$

(c) There exists a constant $C_{(2)} > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_t - \mu_F)' \varphi \right| \geq C_{(2)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

(d) There exists a constant $C_{(3)} > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \varphi \right| \geq C_{(3)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

(e) There exists a constant $C_{(4)} > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_t - \mu_F)' \varphi \right| \geq C_{(4)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

Proof of Lemma E-12: The results given in parts (a)-(e) of this lemma can be shown using essentially the same argument as that given, respectively, in the proof of parts (a)-(e) of Lemma E-11, except that we apply Assumption S2-1 instead of Assumption S1-1 and also apply parts (a)-(c) of Lemma E-7 instead of parts (a)-(c) of Lemma E-6. In addition, we need to change the notations used in the proof of Lemma E-11 by setting

$$\kappa = K, F_{1,t} = F_t, \mu_{F,1} = \mu_F, \Gamma_{\lambda,1} = \Gamma_\lambda, \varphi_1 = \varphi, \text{ and } \underline{\Lambda}_1 = \underline{\Lambda};$$

since, under Scenario 2 considered here, all available factors are relevant for the purpose of forecasting and there is no reason to partition $F_t = (F'_{1,t}, F'_{2,t})'$, $\mu_F = (\mu'_{F,1}, \mu'_{F,2})'$, $\Gamma_\lambda = (\Gamma_{\lambda,1} \quad \Gamma_{\lambda,2})$, $\varphi = (\varphi'_1, \varphi'_2)'$, and

$$\underline{\Lambda} = \begin{pmatrix} \underline{\Lambda}_1 & 0 \\ 0 & \underline{\Lambda}_2 \end{pmatrix}. \quad \square$$

Lemma E-13: Let $T_h = T - h$, where $h \geq 1$ is a (fixed) finite integer. Suppose that Assumptions (ii), (iii), (iv), and (v) hold and suppose that $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$, where

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < 1$$

and where γ_1 and γ_2 are positive constants as defined in Assumption (v). Under these conditions, the following statements are true as $N, T \rightarrow \infty$:

(a) There exists a constant $C_{(5)} > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) \varepsilon_{t+h} \right| \geq C_{(5)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

(b) There exists a constant $C_{\nabla} > 0$ such that

$$P \left(|\bar{\varepsilon}_{T_h}| \geq C_{\nabla} \sqrt{\frac{\ln N}{T_h}} \right) = P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} \right| \geq C_{\nabla} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

(c) There exists a constant $C_{(6)} > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \varepsilon_{t+h} \right| \geq C_{(6)} \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right)$$

(d) There exists a constant $C_{(7)} > 0$ such that

$$P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| |\bar{\varepsilon}_{T_h}| \geq C_{(7)} \sqrt{\frac{\ln N}{T_h}} \right), \text{ where } \bar{\varepsilon}_{T_h} = \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h}$$

(e) There exists a constant $C_{(8)} > 0$ such that

$$\begin{aligned} & P \left(\max_{1 \leq i \leq N} |\bar{u}_{i,T_h}| |\bar{\varepsilon}_{T_h}| \geq C_{(8)} \sqrt{\frac{\ln N}{T_h}} \right) \\ &= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} \right| \geq C_{(8)} \sqrt{\frac{\ln N}{T_h}} \right) \\ &= o \left(\frac{1}{N^2} \right) \end{aligned}$$

Proof of Lemma E-13:

To show part (a), first note that, by Assumption (iii),

$$E [e'_{i,N} \Gamma (F_t - \mu_F) \varepsilon_{t+h}] = e'_{i,N} \Gamma E [(F_t - \mu_F) \varepsilon_{t+h}] = 0 \text{ for all } i \text{ and } t.$$

Next, note that by part (e) of Lemma E-8, there exist constants $\gamma_1 > 0$ and $\bar{b} > 0$ and a positive constant $\bar{C} \geq \exp \{1\}$ such that $\forall s > 0$, for all integer i with $1 \leq i \leq N$, and for all $t \in \mathbb{Z}$,

$$P \left(|e'_{i,N} \Gamma (F_t - \mu_F) \varepsilon_{t+h}| > s \right) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\}$$

where $r = \gamma_1/2 > 0$. Moreover, let $\gamma_* = \gamma_1 \gamma_2 / (\gamma_1 + 2\gamma_2)$, so that

$$\frac{1}{\gamma_*} = \frac{\gamma_1 + 2\gamma_2}{\gamma_1 \gamma_2} = \frac{2}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r} + \frac{1}{\gamma_2}.$$

Now, the relationship between the positive constants γ_1 , γ_2 , and γ as specified in Assumption (v) implies that

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} = \gamma < 1$$

which, in turn, implies that $\gamma_* \in (0, 1)$. In addition, let $\tau = t + h$, and we can rewrite $e'_{i,N} \Gamma(F_t - \mu_F) \varepsilon_{t+h} = e'_{i,N} \Gamma(F_{\tau-h} - \mu_F) \varepsilon_\tau = \underline{W}_{i,\tau}$. Next, let $\underline{V}_t = (\varepsilon_t, F'_t, u'_t)'$ be as defined in part (b) of Assumption (v), and note that $\underline{\mathcal{F}}_{\underline{W},i,-\infty}^\tau = \sigma(\dots, \underline{W}_{i,\tau-2}, \underline{W}_{i,\tau-1}, \underline{W}_{i,\tau}) \subseteq \underline{\mathcal{F}}_{-\infty}^{t+h} = \sigma(\dots, V_{t+h-2}, V_{t+h-1}, V_{t+h})$ and $\underline{\mathcal{F}}_{\underline{W},i,\tau+m}^\infty = \sigma(\underline{W}_{i,\tau+m}, \underline{W}_{i,\tau+m+1}, \underline{W}_{i,\tau+m+2}, \dots) \subseteq \underline{\mathcal{F}}_{t+m}^\infty = \sigma(\underline{V}_{t+m}, \underline{V}_{t+m+1}, \underline{V}_{t+m+2}, \dots)$. Hence, given that $h \geq 1$ is a fixed (finite) integer, it follows by part (b) of Assumption (v) and Theorem 14.1 of Davidson (1994) that $\{\underline{W}_{i,\tau}\}_{\tau \in \mathbb{Z}}$ is an α -mixing process with α coefficient given by $\alpha_{\underline{W},i}(m) = \sup_\tau \alpha(\underline{\mathcal{F}}_{\underline{W},i,-\infty}^\tau, \underline{\mathcal{F}}_{\underline{W},i,\tau+m}^\infty)$. To analyze the rate of decay of $\alpha_{\underline{W},i}(m)$, we first consider the case where $m \geq h + 1$. In this case, note that

$$\begin{aligned} & \alpha_{\underline{W},i}(m) \\ &= \sup_\tau \alpha(\underline{\mathcal{F}}_{\underline{W},i,-\infty}^\tau, \underline{\mathcal{F}}_{\underline{W},i,\tau+m}^\infty) \\ &\leq \sup_t \alpha(\underline{\mathcal{F}}_{-\infty}^{t+h}, \underline{\mathcal{F}}_{t+m}^\infty) \\ &\leq a \exp\{-c(m-h)^{\gamma_2}\} \text{ for all integers } m \geq h+1 \text{ and for all } i \\ &\quad \text{(by part (b) of Assumption 4-3)} \\ &= a \exp\left\{-cm^{\gamma_2} \left(1 - \frac{h}{m}\right)^{\gamma_2}\right\} \\ &\leq a \exp\left\{-cm^{\gamma_2} \left(1 - \frac{h}{h+1}\right)^{\gamma_2}\right\} \quad (\text{since } m \geq h+1) \\ &= a \exp\left\{-cm^{\gamma_2} \left(\frac{1}{h+1}\right)^{\gamma_2}\right\} \\ &\leq a \exp\{-c^\# m^{\gamma_2}\} \text{ where } c^\# = c \left(\frac{1}{h+1}\right)^{\gamma_2} > 0 \text{ given that } h \text{ is a finite positive integer} \end{aligned} \tag{174}$$

Next, consider the case where $m \leq h$. In this case, we have

$$\begin{aligned} \alpha_{\underline{W},i}(m) &= \sup_\tau \alpha(\underline{\mathcal{F}}_{\underline{W},i,-\infty}^\tau, \underline{\mathcal{F}}_{\underline{W},i,\tau+m}^\infty) \\ &\leq 1 \\ &= \exp\{c^\# m^{\gamma_2}\} \exp\{-c^\# m^{\gamma_2}\} \\ &\leq \exp\{c^\# h^{\gamma_2}\} \exp\{-c^\# m^{\gamma_2}\} \\ &= a_* \exp\{-c^\# m^{\gamma_2}\} \end{aligned} \tag{175}$$

where $a_* = \exp\{c^\# h^{\gamma_2}\} > 0$ since $c^\# > 0$ and $h \geq 1$. Moreover, note that $a_* =$

$\exp \{c^\# h^{\gamma_2}\} < \infty$ since $h < \infty$ and since

$$c^\# = c \left(\frac{1}{h+1} \right)^{\gamma_2} < \infty$$

Let $a^\# = \max \{a, a_*\}$, and it follows by combining expressions (174) and (??) that

$$\alpha_{\underline{W}, i}(m) \leq a^\# \exp \{-c^\# m^{\gamma_2}\} \text{ for all } m \geq 1 \text{ and for all } i.$$

Therefore, all the assumptions needed to apply the Bernstein inequality given in Merlevède, Peligrad, and Rio (2011) (restated here as Lemma C-1) are fulfilled. Application of this Bernstein inequality then implies that there exist positive constants C_1, C_2, C_3, C_4 and C_5 depending on $a^\#, \bar{b}, c^\#, \gamma_*$, and γ_2 such that for any $s > 0$ and for all integer i with $1 \leq i \leq N$,

$$\begin{aligned} & P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) \varepsilon_{t+h} \right| \geq s \right) \\ &= P \left(\left| \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) \varepsilon_{t+h} \right| \geq T_h s \right) \\ &\leq T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} + \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_* (1-\gamma_*)}}{C_5 (\ln sT_h)^{\gamma_*}} \right\} \right\} \end{aligned}$$

Using Bonferroni's inequality, we further obtain

$$\begin{aligned} & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) \varepsilon_{t+h} \right| \geq s \right) \\ &\leq N \max_{1 \leq i \leq N} P \left(\left| \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) \varepsilon_{t+h} \right| \geq T_h s \right) \\ &\leq NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} + N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} \\ &\quad + N \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_* (1-\gamma_*)}}{C_5 (\ln sT_h)^{\gamma_*}} \right\} \right\} \end{aligned}$$

For some $C_{(5)} > 0$, let

$$s = C_{(5)} \sqrt{\frac{\ln N}{T_h}}.$$

By choosing $C_{(5)}$ to be large enough, we can obtain the following three results, which we will show in turn:

(I)

$$NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} = NT_h \exp \left\{ -\frac{C_{(5)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(II)

$$N \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(III)

$$N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} = N \exp \left\{ -\frac{C_{(5)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = O \left(\frac{1}{N^2} \right).$$

To show result (I), first write

$$\begin{aligned} N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} &= N^3 T_h \exp \left\{ -\frac{C_{(5)}^{\gamma_*}}{C_1} T_h^{\gamma_*} \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma_*}{2}} \right\} \\ &= N^3 T_h \exp \left\{ -\frac{C_{(5)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \end{aligned}$$

Taking the log transformation, we have

$$\ln \left[N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} \right] = 3 \ln N + \ln T_h - \frac{C_{(5)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}$$

Since

$$\frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0 \quad (\text{given that } \gamma_* > 0)$$

and since

$$\begin{aligned} \frac{3 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} &= 3 \left(\frac{T}{T_h} \right)^{\frac{\gamma_*}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \right)^{\frac{\gamma_*}{2}} \rightarrow 0 \\ &\left(\text{given that, by assumption, } \frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \text{ and given} \right. \\ &\left. \text{that } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \text{ since } h \text{ is a fixed finite integer} \right), \end{aligned}$$

we see that

$$\frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} = \frac{3 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned}
N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} &= \exp \left\{ 3 \ln N + \ln T_h - \frac{C_{(5)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \\
&= \exp \left\{ -\frac{C_{(5)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \left[1 - \frac{C_1}{C_{(5)}^{\gamma_*}} \frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \right] \right\} \\
&= o(1)
\end{aligned}$$

It follows that

$$N T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} = \frac{1}{N^2} N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (II), note that

$$\begin{aligned}
&N^3 \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} \\
&= N^3 \exp \left\{ -\frac{T_h^2}{C_4 T_h} \left(C_{(5)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma_*(1-\gamma_*)} \left(C_{(5)} \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma_*(1-\gamma_*)}}{C_5 \left(\ln \left[T_h C_{(5)} \sqrt{\frac{\ln N}{T_h}} \right] \right)^{\gamma_*}} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{C_{(5)}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} C_{(5)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 (\ln T_h + \ln C_{(5)} + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h)^{\gamma_*}} \right) \right\} \\
&= N^3 \exp \left\{ -\frac{C_{(5)}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} C_{(5)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_{(5)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) \right\} \\
&= \exp \left\{ -\ln N \left[\frac{C_{(5)}^2}{C_4} \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} C_{(5)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_{(5)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) - 3 \right] \right\} \\
&= o(1)
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
N \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} &= \frac{1}{N^2} N^3 \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} \\
&= o \left(\frac{1}{N^2} \right)
\end{aligned}$$

Turning our attention now to showing result (III), we first write

$$\begin{aligned} N^3 \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} &= N^3 \exp \left\{ -\frac{T_h^2}{C_2 (1 + T_h C_3)} \left(C_{(5)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\ &= N^3 \exp \left\{ -\frac{C_{(5)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned} \ln \left[N^3 \exp \left\{ -\frac{C_{(5)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \right] &= 3 \ln N - \frac{C_{(5)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \\ &= 3 \ln N - \frac{C_{(5)}^2}{C_2 C_3} \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \end{aligned}$$

Now, let $C_{(5)}$ be a positive constant such that $C_{(5)}^2 > 3C_2 (1 + C_3)$, and set

$$C^* = \frac{C_{(5)}^2}{C_2 C_3} - 3.$$

Note that, by this choice of $C_{(5)}$ and C^* , we have

$$C^* > \frac{3C_2 (1 + C_3)}{C_2 C_3} - 3 = \frac{3}{C_3} > 0$$

It, thus, follows that

$$\begin{aligned}
& N^3 \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} \\
= & N^3 \exp \left\{ -\frac{C_{(5)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \\
= & \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - \frac{C_{(5)}^2}{C_2 C_3} \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - [C^* + 3] \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ \left(\ln N \left[3 C_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
= & \exp \left\{ -C^* \ln N \left[1 - \frac{3}{C^* C_3} \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\
< & \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{3}{C^* C_3} < \frac{C_3 3}{3 C_3} = 1 \right) \\
= & o(1)
\end{aligned}$$

This, in turn, implies that

$$N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} = \left(\frac{1}{N^2} \right) N^3 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (III).

Combining the results given in (I)-(III) above, we obtain

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) \varepsilon_{t+h} \right| \geq C_{(5)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq N \max_{1 \leq i \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) \varepsilon_{t+h} \right| \geq C_{(5)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} + N \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln sT_h)^{\gamma_*}} \right\} \right\} \\
& \quad + N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

To show part (b), note that, given that h is a fixed finite integer, we can let $\tau = t + h$ and, without loss of generality, write $\varepsilon_{t+h} = \varepsilon_\tau$. Now, by Assumption (iii), $E[\varepsilon_\tau] = 0$ for all $\tau \in \mathbb{Z}$. Moreover, by part (a) of Assumption (v), there exist positive constants γ_1 and b_1 such that, for any $s > 0$,

$$\sup_{\tau \in \mathbb{Z}} P(|\varepsilon_\tau| > s) \leq \exp \left\{ 1 - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{b_1} \right)^{\gamma_1} \right\},$$

where $\bar{C}$ is a positive constant such that $\bar{C} \geq \exp\{1\}$. Next, let $\underline{V}_\tau = (\varepsilon_\tau, F'_\tau, u'_\tau)'$ be as defined in part (b) of Assumption (v), and note that $\underline{\mathcal{F}}_{\varepsilon, -\infty}^\tau = \sigma(\dots, \varepsilon_{\tau-2}, \varepsilon_{\tau-1}, \varepsilon_\tau) \subseteq \underline{\mathcal{F}}_{-\infty}^\tau = \sigma(\dots, \underline{V}_{\tau-2}, \underline{V}_{\tau-1}, \underline{V}_\tau)$ and $\underline{\mathcal{F}}_{\varepsilon, \tau+m}^\infty = \sigma(\varepsilon_{\tau+m}, \varepsilon_{\tau+m+1}, \varepsilon_{\tau+m+2}, \dots) \subseteq \underline{\mathcal{F}}_{\tau+m}^\infty = \sigma(\underline{V}_{\tau+m}, \underline{V}_{\tau+m+1}, \underline{V}_{\tau+m+2}, \dots)$. It follows by part (b) of Assumption (v) and Theorem 14.1 of Davidson (1994) that $\{\varepsilon_{t+h}\}_{t \in \mathbb{Z}} \equiv \{\varepsilon_\tau\}_{\tau \in \mathbb{Z}}$ is an α -mixing process with α coefficient satisfying the rate of decay

$$\begin{aligned}
\alpha_\varepsilon(m) &= \sup_{\tau} \alpha(\underline{\mathcal{F}}_{\varepsilon, -\infty}^\tau, \underline{\mathcal{F}}_{\varepsilon, \tau+m}^\infty) \\
&\leq \sup_{\tau} \alpha(\underline{\mathcal{F}}_{-\infty}^\tau, \underline{\mathcal{F}}_{\tau+m}^\infty) \\
&\leq a \exp(-cm^{\gamma_2}) \text{ for any } m \geq 1,
\end{aligned}$$

and for some positive constants γ_2 , a and c . Furthermore, under Assumption (v), there exists a constant $\gamma \in (0, 1)$ such that

$$\frac{1}{\gamma} = \frac{1}{\gamma_1} + \frac{1}{\gamma_2}.$$

Thefore, all the assumptions needed to obtain the Bernstein-type inequality given in Merlevède, Peligrad, and Rio (2011) (restated here as Lemma C-1) are fulfilled. Applying this Bernstein-type inequality then implies that there exist constants C_1, C_2, C_3, C_4 and C_5 depending on a, b_1, c, γ , and γ_2 such that, for any $s > 0$,

$$\begin{aligned} & P(|\bar{\varepsilon}_{T_h}| \geq s) \\ = & P\left(\left|\frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h}\right| \geq s\right) \\ \leq & T_h \exp\left\{-\frac{(sT_h)^\gamma}{C_1}\right\} + \exp\left\{-\frac{s^2 T_h^2}{C_2(1+T_h C_3)}\right\} + \exp\left\{-\frac{(sT_h)^2}{C_4 T_h} \exp\left\{\frac{(sT_h)^{\gamma(1-\gamma)}}{C_5 (\ln sT_h)^\gamma}\right\}\right\}. \end{aligned}$$

For some $C_\nabla > 0$, let

$$s = C_\nabla \sqrt{\frac{\ln N}{T_h}}.$$

By choosing C_∇ to be large enough, we can obtain the following three results, which we will show in turn:

(I)

$$T_h \exp\left\{-\frac{(sT_h)^\gamma}{C_1}\right\} = T_h \exp\left\{-\frac{C_\nabla^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}\right\} = o\left(\frac{1}{N^2}\right).$$

(II)

$$\exp\left\{-\frac{(sT_h)^2}{C_4 T_h} \exp\left\{\frac{(sT_h)^{\gamma(1-\gamma)}}{C_5 (\ln [sT_h])^\gamma}\right\}\right\} = o\left(\frac{1}{N^2}\right)$$

(III)

$$\exp\left\{-\frac{s^2 T_h^2}{C_2(1+T_h C_3)}\right\} = \exp\left\{-\frac{C_\nabla^2 \ln N}{C_2\left(\frac{1}{T_h} + C_3\right)}\right\} = O\left(\frac{1}{N^2}\right).$$

To show result (I), first write

$$N^2 T_h \exp\left\{-\frac{(sT_h)^\gamma}{C_1}\right\} = N^2 T_h \exp\left\{-\frac{C_\nabla^\gamma}{C_1} T_h^\gamma \left(\frac{\ln N}{T_h}\right)^{\frac{\gamma}{2}}\right\} = N^2 T_h \exp\left\{-\frac{C_\nabla^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}\right\}$$

Taking the log transformation, we have

$$\ln \left[N^2 T_h \exp\left\{-\frac{(sT_h)^\gamma}{C_1}\right\} \right] = 2 \ln N + \ln T_h - \frac{C_\nabla^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}$$

Now, by assumption, $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$, where

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2}.$$

Since, given Assumption (v),

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} = \gamma < 1$$

it follows that $(\ln N)^{\frac{2}{\gamma_*}-1} = o(T)$ implies that $(\ln N)^{\frac{2}{\gamma}-1} = o(T)$. Moreover, since

$$\frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0 \quad (\text{given that } \gamma > 0)$$

and since

$$\begin{aligned} \frac{2(\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} &= 2 \left(\frac{T}{T_h} \right)^{\frac{\gamma}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \right)^{\frac{\gamma}{2}} \rightarrow 0 \quad \left(\text{given that } \frac{(\ln N)^{\frac{2}{\gamma}-1}}{T} \rightarrow 0 \text{ and} \right. \\ &\quad \left. \text{given that } \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \text{ since } h \text{ is a fixed finite integer} \right), \end{aligned}$$

we see that

$$\frac{2 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} = \frac{2(\ln N)^{1-\frac{\gamma}{2}}}{T_h^{\frac{\gamma}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \rightarrow 0,$$

which, in turn, implies that

$$\begin{aligned} N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_1} \right\} &= \exp \left\{ 2 \ln N + \ln T_h - \frac{C_\nabla^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \right\} \\ &= \exp \left\{ -\frac{C_\nabla^\gamma}{C_1} T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}} \left[1 - \frac{C_1}{C_\nabla^\gamma} \frac{2 \ln N + \ln T_h}{T_h^{\frac{\gamma}{2}} (\ln N)^{\frac{\gamma}{2}}} \right] \right\} \\ &= o(1) \end{aligned}$$

It further follows that

$$T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_1} \right\} = \frac{1}{N^2} N^2 T_h \exp \left\{ -\frac{(sT_h)^\gamma}{C_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (II), note that

$$\begin{aligned}
& N^2 \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_5 (\ln [sT_h])^\gamma} \right\} \right\} \\
&= N^2 \exp \left\{ -\frac{T_h^2}{C_4 T_h} \left(C_\nabla \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma(1-\gamma)} \left(C_\nabla \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma(1-\gamma)}}{C_5 \left(\ln \left[T_h C_\nabla \sqrt{\frac{\ln N}{T_h}} \right] \right)^\gamma} \right) \right\} \\
&= N^2 \exp \left\{ -\frac{C_\nabla^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} C_\nabla^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 \left(\ln T_h + \ln C_\nabla + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h \right)^\gamma} \right) \right\} \\
&= N^2 \exp \left\{ -\frac{C_\nabla^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} C_\nabla^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_\nabla + \frac{1}{2} \ln \ln N \right)^\gamma} \right) \right\} \\
&= \exp \left\{ -\ln N \left[\frac{C_\nabla^2}{C_4} \exp \left(\frac{T_h^{\frac{\gamma(1-\gamma)}{2}} C_\nabla^{\gamma(1-\gamma)} (\ln N)^{\frac{\gamma(1-\gamma)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_\nabla + \frac{1}{2} \ln \ln N \right)^\gamma} \right) - 2 \right] \right\} \\
&= o(1)
\end{aligned}$$

This, in turn, implies that

$$\begin{aligned}
\exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_5 (\ln [T_h s])^\gamma} \right\} \right\} &= \left(\frac{1}{N^2} \right) N^2 \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma(1-\gamma)}}{C_5 (\ln [sT_h])^\gamma} \right\} \right\} \\
&= O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right)
\end{aligned}$$

Turning our attention now to showing result (III), we first write

$$\begin{aligned}
N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} &= N^2 \exp \left\{ -\frac{T_h^2}{C_2 (1 + T_h C_3)} \left(C_\nabla \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\
&= N^2 \exp \left\{ -\frac{C_\nabla^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\}
\end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned} \ln \left[N^2 \exp \left\{ -\frac{C_{\nabla}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \right] &= 2 \ln N - \frac{C_{\nabla}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \\ &= 2 \ln N - \frac{C_{\nabla}^2}{C_2 C_3} \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \end{aligned}$$

Now, let C_{∇} be a positive constant such that $C_{\nabla}^2 > 2C_2(1 + C_3)$, and set

$$C^* = \frac{C_{\nabla}^2}{C_2 C_3} - 2.$$

Note that, by choosing C_{∇} and C^* in this way, we have

$$C^* > \frac{2C_2(1 + C_3)}{C_2 C_3} - 2 = \frac{2}{C_3} > 0$$

It, thus, follows that

$$\begin{aligned} & N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_2(1 + T_h C_3)} \right\} \\ &= N^2 \exp \left\{ -\frac{C_{\nabla}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \\ &= \exp \left\{ \left(2 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - \frac{C_{\nabla}^2}{C_2 C_3} \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ \left(2 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - [C^* + 2] \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ \left(\ln N \left[2C_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ -C^* \ln N \left[1 - \frac{2}{C^* C_3} \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &< \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{2}{C^* C_3} < \frac{C_3 2}{2C_3} = 1 \right) \\ &= o(1) \end{aligned}$$

This, in turn, implies that

$$\exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} = \left(\frac{1}{N^2} \right) N^2 \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (III).

Combining the results given in (I)-(III) above, we obtain

$$\begin{aligned} & P \left(|\bar{\varepsilon}_{T_h}| \geq C_{\nabla} \sqrt{\frac{\ln N}{T_h}} \right) \\ &= P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} \right| \geq C_{\nabla} \sqrt{\frac{\ln N}{T_h}} \right) \\ &\leq T_h \exp \left\{ -\frac{(s T_h)^{\gamma}}{C_1} \right\} + \exp \left\{ -\frac{(s T_h)^2}{C_4 T_h} \exp \left\{ \frac{(s T_h)^{\gamma(1-\gamma)}}{C_5 (\ln s T_h)^{\gamma}} \right\} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} \\ &= o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\ &= o \left(\frac{1}{N^2} \right). \end{aligned}$$

To show part (c), first note that, by Assumption (iii),

$$E [u_{it} \varepsilon_{t+h}] = 0 \text{ for all } i \text{ and } t.$$

Next, note that by part (f) of Lemma E-8, there exist constants $\gamma_1 > 0$ and $\bar{b} > 0$ and a positive constant $\bar{C} \geq \exp \{1\}$ such that $\forall s > 0$, for all $1 \leq i \leq N$, and for all $t \in \mathbb{Z}$,

$$P (|u_{it} \varepsilon_{t+h}| > s) \leq \exp \left\{ 1 - \left(\frac{s}{\bar{b}} \right)^r \right\} \leq \bar{C} \exp \left\{ - \left(\frac{s}{\bar{b}} \right)^r \right\}$$

where $r = \gamma_1/2 > 0$. Moreover, let $\gamma_* = \gamma_1 \gamma_2 / (\gamma_1 + 2\gamma_2)$, so that

$$\frac{1}{\gamma_*} = \frac{\gamma_1 + 2\gamma_2}{\gamma_1 \gamma_2} = \frac{2}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{r} + \frac{1}{\gamma_2}.$$

Now, the relationship between the positive constants γ_1 , γ_2 , and γ as specified in Assumption (v) implies that

$$0 < \gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2} < \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} = \gamma < 1$$

which, in turn, implies that $\gamma_* \in (0, 1)$. In addition, let $\tau = t+h$, and we can rewrite $u_{it} \varepsilon_{t+h} = u_{i,\tau-h} \varepsilon_{\tau} = \tilde{V}_{i,\tau}$. Next, let $\underline{V}_t = (\varepsilon_t, F'_t, u'_t)'$ be as defined in part (b) of Assumption (v), and note that $\mathcal{F}_{\tilde{V},i,-\infty}^{\tau} = \sigma \left(\dots, \tilde{V}_{i,\tau-2}, \tilde{V}_{i,\tau-1}, \tilde{V}_{i,\tau} \right) \subseteq \mathcal{F}_{-\infty}^{t+h} = \sigma \left(\dots, \underline{V}_{t+h-2}, \underline{V}_{t+h-1}, \underline{V}_{t+h} \right)$

and $\mathcal{F}_{\tilde{V},i,\tau+m}^\infty = \sigma(\tilde{V}_{i,\tau+m}, \tilde{V}_{i,\tau+m+1}, \tilde{V}_{i,\tau+m+2}, \dots) \subseteq \mathcal{F}_{t+m}^\infty = \sigma(\underline{V}_{t+m}, \underline{V}_{t+m+1}, \underline{V}_{t+m+2}, \dots)$. Hence, given that $h \geq 1$ is a fixed (finite) integer, it follows by part (b) of Assumption (v) and Theorem 14.1 of Davidson (1994) that $\{\tilde{V}_{i,\tau}\}_{\tau \in \mathbb{Z}}$ is an α -mixing process with α coefficient given by $\alpha_{\tilde{V},i}(m) = \sup_{\tau} \alpha(\mathcal{F}_{\tilde{V},i,-\infty}^\tau, \mathcal{F}_{\tilde{V},i,\tau+m}^\infty)$. To analyze the rate of decay of $\alpha_{\tilde{V},i}(m)$, we first consider the case where $m \geq h+1$. In this case, note that

$$\begin{aligned}
& \alpha_{\tilde{V},i}(m) \\
&= \sup_{\tau} \alpha(\mathcal{F}_{\tilde{V},i,-\infty}^\tau, \mathcal{F}_{\tilde{V},i,\tau+m}^\infty) \\
&\leq \sup_t \alpha(\underline{\mathcal{F}}_{-\infty}^{t+h}, \underline{\mathcal{F}}_{t+m}^\infty) \\
&\leq a \exp\{-c(m-h)^{\gamma_2}\} \text{ for all integers } m \geq h+1 \text{ and for all } i \\
&\quad \text{(by part (b) of Assumption 4-3)} \\
&= a \exp\left\{-cm^{\gamma_2} \left(1 - \frac{h}{m}\right)^{\gamma_2}\right\} \\
&\leq a \exp\left\{-cm^{\gamma_2} \left(1 - \frac{h}{h+1}\right)^{\gamma_2}\right\} \quad (\text{since } m \geq h+1) \\
&= a \exp\left\{-cm^{\gamma_2} \left(\frac{1}{h+1}\right)^{\gamma_2}\right\} \\
&\leq a \exp\{-c^\# m^{\gamma_2}\} \text{ where } c^\# = c \left(\frac{1}{h+1}\right)^{\gamma_2} > 0 \text{ given that } h \text{ is a finite positive int} \quad (176)
\end{aligned}$$

Next, consider the case where $m \leq h$. In this case, we have

$$\begin{aligned}
\alpha_{\tilde{V},i}(m) &= \sup_{\tau} \alpha(\underline{\mathcal{F}}_{\underline{W},i,-\infty}^\tau, \underline{\mathcal{F}}_{\underline{W},i,\tau+m}^\infty) \\
&\leq 1 \\
&= \exp\{c^\# m^{\gamma_2}\} \exp\{-c^\# m^{\gamma_2}\} \\
&\leq \exp\{c^\# h^{\gamma_2}\} \exp\{-c^\# m^{\gamma_2}\} \\
&= a_* \exp\{-c^\# m^{\gamma_2}\} \quad (177)
\end{aligned}$$

where $a_* = \exp\{c^\# h^{\gamma_2}\} > 0$ since $c^\# > 0$ and $h \geq 1$. Moreover, note that $a_* = \exp\{c^\# h^{\gamma_2}\} < \infty$ since $h < \infty$ and since

$$c^\# = c \left(\frac{1}{h+1}\right)^{\gamma_2} < \infty$$

Let $a^\# = \max\{a, a_*\}$, and it follows by combining expressions (176) and (177) that

$$\alpha_{\tilde{V},i}(m) \leq a^\# \exp\{-c^\# m^{\gamma_2}\} \text{ for all } m \geq 1 \text{ and for all } i.$$

Therefore, all the assumptions needed to apply the Bernstein inequality given in Merlevède, Peligrad, and Rio (2011) (restated here as Lemma C-1) are fulfilled. Application of this Bernstein inequality then implies that there exist positive constants C_1, C_2, C_3, C_4 and C_5 depending on $a^\#, \bar{b}, c^\#, \gamma_*$, and γ_2 such that, for any $s > 0$ and for all integer i with $1 \leq i \leq N$,

$$\begin{aligned} & P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \varepsilon_{t+h} \right| \geq s \right) \\ &= P \left(\left| \sum_{t=1}^{T_h} u_{it} \varepsilon_{t+h} \right| \geq T_h s \right) \\ &\leq T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} + \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} + \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_* (1-\gamma_*)}}{C_5 (\ln sT_h)^{\gamma_*}} \right\} \right\} \end{aligned}$$

Using Bonferroni's inequality, we further obtain

$$\begin{aligned} P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \varepsilon_{t+h} \right| \geq s \right) &\leq N \max_{1 \leq i \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \varepsilon_{t+h} \right| \geq s \right) \\ &\leq NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} + N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} \\ &\quad + N \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_* (1-\gamma_*)}}{C_5 (\ln sT_h)^{\gamma_*}} \right\} \right\} \end{aligned}$$

For some $C_{(6)} > 0$, let

$$s = C_{(6)} \sqrt{\frac{\ln N}{T_h}}.$$

By choosing $C_{(6)}$ to be large enough, we can obtain the following three results, which we will show in turn:

(I)

$$NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} = NT_h \exp \left\{ -\frac{C_{(6)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} = o \left(\frac{1}{N^2} \right).$$

(II)

$$N \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_* (1-\gamma_*)}}{C_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} = o \left(\frac{1}{N^2} \right)$$

(III)

$$N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} = N \exp \left\{ -\frac{C_{(6)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} = O \left(\frac{1}{N^2} \right).$$

To show result (I), first write

$$\begin{aligned} N^3 T_h \exp \left\{ -\frac{(s T_h)^{\gamma_*}}{C_1} \right\} &= N^3 T_h \exp \left\{ -\frac{C_{(6)}^{\gamma_*}}{C_1} T_h^{\gamma_*} \left(\frac{\ln N}{T_h} \right)^{\frac{\gamma_*}{2}} \right\} \\ &= N^3 T_h \exp \left\{ -\frac{C_{(6)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \end{aligned}$$

Taking the log transformation, we have

$$\ln \left[N^3 T_h \exp \left\{ -\frac{(s T_h)^{\gamma_*}}{C_1} \right\} \right] = 3 \ln N + \ln T_h - \frac{C_{(6)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}$$

Since

$$\frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0 \quad (\text{given that } \gamma_* > 0)$$

and since

$$\begin{aligned} \frac{3 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} &= 3 \left(\frac{T}{T_h} \right)^{\frac{\gamma_*}{2}} \left(\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \right)^{\frac{\gamma_*}{2}} \rightarrow 0 \\ &\left(\text{given that, by assumption, } \frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \text{ and given that} \right. \\ &\left. \frac{T}{T_h} = \frac{T}{T-h} \rightarrow 1 \text{ as } T \rightarrow \infty \text{ since } h \text{ is a fixed finite integer} \right), \end{aligned}$$

we see that

$$\frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} = \frac{3 (\ln N)^{1-\frac{\gamma_*}{2}}}{T_h^{\frac{\gamma_*}{2}}} + \frac{\ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \rightarrow 0$$

which, in turn, implies that

$$\begin{aligned} N^3 T_h \exp \left\{ -\frac{(s T_h)^{\gamma_*}}{C_1} \right\} &= \exp \left\{ 3 \ln N + \ln T_h - \frac{C_{(6)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \right\} \\ &= \exp \left\{ -\frac{C_{(6)}^{\gamma_*}}{C_1} T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}} \left[1 - \frac{C_1}{C_{(6)}^{\gamma_*}} \frac{3 \ln N + \ln T_h}{T_h^{\frac{\gamma_*}{2}} (\ln N)^{\frac{\gamma_*}{2}}} \right] \right\} \\ &= o(1) \end{aligned}$$

It follows that

$$NT_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} = \frac{1}{N^2} N^3 T_h \exp \left\{ -\frac{(sT_h)^{\gamma_*}}{C_1} \right\} = O \left(\frac{1}{N^2} \right) o(1) = o \left(\frac{1}{N^2} \right).$$

Next, to show result (II), note that

$$\begin{aligned} & N^3 \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} \\ = & N^3 \exp \left\{ -\frac{T_h^2}{C_4 T_h} \left(C_{(6)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \exp \left(\frac{T_h^{\gamma_*(1-\gamma_*)} \left(C_{(6)} \sqrt{\frac{\ln N}{T_h}} \right)^{\gamma_*(1-\gamma_*)}}{C_5 \left(\ln \left[T_h C_{(6)} \sqrt{\frac{\ln N}{T_h}} \right] \right)^{\gamma_*}} \right) \right\} \\ = & N^3 \exp \left\{ -\frac{C_{(6)}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} C_{(6)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\ln T_h + \ln C_{(6)} + \frac{1}{2} \ln \ln N - \frac{1}{2} \ln T_h \right)^{\gamma_*}} \right) \right\} \\ = & N^3 \exp \left\{ -\frac{C_{(6)}^2}{C_4} (\ln N) \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} C_{(6)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_{(6)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) \right\} \\ = & \exp \left\{ -\ln N \left[\frac{C_{(6)}^2}{C_4} \exp \left(\frac{T_h^{\frac{\gamma_*(1-\gamma_*)}{2}} C_{(6)}^{\gamma_*(1-\gamma_*)} (\ln N)^{\frac{\gamma_*(1-\gamma_*)}{2}}}{C_5 \left(\frac{1}{2} \ln T_h + \ln C_{(6)} + \frac{1}{2} \ln \ln N \right)^{\gamma_*}} \right) - 3 \right] \right\} \\ = & o(1) \end{aligned}$$

This, in turn, implies that

$$\begin{aligned} N \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [T_h s])^{\gamma_*}} \right\} \right\} &= \frac{1}{N^2} N^3 \exp \left\{ -\frac{(sT_h)^2}{C_4 T_h} \exp \left\{ \frac{(sT_h)^{\gamma_*(1-\gamma_*)}}{C_5 (\ln [sT_h])^{\gamma_*}} \right\} \right\} \\ &= o \left(\frac{1}{N^2} \right) \end{aligned}$$

Turning our attention now to showing result (III), we first write

$$\begin{aligned} N^3 \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} &= N^3 \exp \left\{ -\frac{T_h^2}{C_2 (1 + T_h C_3)} \left(C_{(6)} \sqrt{\frac{\ln N}{T_h}} \right)^2 \right\} \\ &= N^3 \exp \left\{ -\frac{C_{(6)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \end{aligned}$$

Taking log transformation of the right-hand side, we get

$$\begin{aligned} \ln \left[N^3 \exp \left\{ -\frac{C_{(6)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \right] &= 3 \ln N - \frac{C_{(6)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \\ &= 3 \ln N - \frac{C_{(6)}^2}{C_2 C_3} \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \end{aligned}$$

Now, let $C_{(6)}$ be a positive constant $C_{(6)}^2 > 3C_2(1 + C_3)$, and set

$$C^* = \frac{C_{(6)}^2}{C_2 C_3} - 3.$$

Note that, by choosing $C_{(6)}$ and C^* in this way, we have

$$C^* > \frac{3C_2(1 + C_3)}{C_2 C_3} - 3 = \frac{3}{C_3} > 0$$

It, thus, follows that

$$\begin{aligned} & N^3 \exp \left\{ -\frac{s^2 T_h^2}{C_2(1 + T_h C_3)} \right\} \\ &= N^3 \exp \left\{ -\frac{C_{(6)}^2 \ln N}{C_2 \left(\frac{1}{T_h} + C_3 \right)} \right\} \\ &= \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - \frac{C_{(6)}^2}{C_2 C_3} \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ \left(3 \ln N \left[1 + C_3^{-1} \frac{1}{T_h} \right] - [C^* + 3] \ln N \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ \left(\ln N \left[3C_3^{-1} \frac{1}{T_h} - C^* \right] \right) \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &= \exp \left\{ -C^* \ln N \left[1 - \frac{3}{C^* C_3} \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \\ &< \exp \left\{ -C^* \ln N \left[1 - \frac{1}{T_h} \right] \left[1 + C_3^{-1} \frac{1}{T_h} \right]^{-1} \right\} \left(\text{since } \frac{3}{C^* C_3} < \frac{C_3 3}{3C_3} = 1 \right) \\ &= o(1) \end{aligned}$$

This, in turn, implies that

$$N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} = \left(\frac{1}{N^2} \right) N^3 \exp \left\{ -\frac{T_h^2 s^2}{C_2 (1 + T_h C_3)} \right\} = o \left(\frac{1}{N^2} \right).$$

which shows result (III).

Combining the results given in (I)-(III) above, we obtain

$$\begin{aligned} & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \varepsilon_{t+h} \right| \geq C_{(6)} \sqrt{\frac{\ln N}{T_h}} \right) \\ & \leq N \max_{1 \leq i \leq N} P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \varepsilon_{t+h} \right| \geq C_{(6)} \sqrt{\frac{\ln N}{T_h}} \right) \\ & \leq N T_h \exp \left\{ -\frac{(s T_h)^{\gamma_*}}{C_1} \right\} + N \exp \left\{ -\frac{(s T_h)^2}{C_4 T_h} \exp \left\{ \frac{(s T_h)^{\gamma_* (1-\gamma_*)}}{C_5 (\ln s T_h)^{\gamma_*}} \right\} \right\} \\ & \quad + N \exp \left\{ -\frac{s^2 T_h^2}{C_2 (1 + T_h C_3)} \right\} \\ & = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\ & = o \left(\frac{1}{N^2} \right). \end{aligned}$$

To show part (d), first set $C_{(7)} = \max \{C_{\dagger}^2, C_{\nabla}^2\}$, where $C_{\dagger} > 0$ is the constant given in

part (a) of Lemma E-9 and where $C_{\nabla} > 0$ is the constant given part (b) of this lemma.

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| |\bar{\varepsilon}_{T_h}| \geq C_{(\tau)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq P \left(\left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq \sqrt{C_{(\tau)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \cup \left\{ |\bar{\varepsilon}_{T_h}| \geq \sqrt{C_{(\tau)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq \sqrt{C_{(\tau)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) + P \left(|\bar{\varepsilon}_{T_h}| \geq \sqrt{C_{(\tau)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \\
& \quad (\text{by Bonferroni inequality}) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq \sqrt{C_{(\tau)}} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(|\bar{\varepsilon}_{T_h}| \geq \sqrt{C_{(\tau)}} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad \left(\text{since } \sqrt{\frac{\ln N}{T_h}} < \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \text{ given that } \frac{\ln N}{T_h} = \frac{\ln N}{T-h} < 1 \text{ for all } N, T \text{ sufficiently large} \right) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(|\bar{\varepsilon}_{T_h}| \geq C_{\nabla} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \quad (\text{since } C_{(\tau)} = \max \{C_{\dagger}^2, C_{\nabla}^2\} \text{ implies that } \sqrt{C_{(\tau)}} \geq C_{\dagger} \text{ and } \sqrt{C_{(\tau)}} \geq C_{\nabla}).
\end{aligned}$$

It then follows, by part (a) of Lemma E-9 and part (b) of this lemma, that for all N and T sufficiently large,

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| |\bar{\varepsilon}_{T_h}| \geq C_{(\tau)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& \leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(|\bar{\varepsilon}_{T_h}| \geq C_{\nabla} \sqrt{\frac{\ln N}{T_h}} \right) \\
& = o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
& = o \left(\frac{1}{N^2} \right).
\end{aligned}$$

To show part (e), first set $C_{(8)} = \max \{C_{\dagger}^2, C_{\nabla}^2\}$, where $C_{\dagger} > 0$ is the constant given in

part (b) of Lemma E-9 and where $C_{\nabla} > 0$ is the constant given in part (b) of this lemma.

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} |\bar{u}_{i,T_h}| |\bar{\varepsilon}_{T_h}| \geq C_{(8)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} \right| \geq C_{(8)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\leq P \left(\left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq \sqrt{C_{(8)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \cup \left\{ \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} \right| \geq \sqrt{C_{(8)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right\} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq \sqrt{C_{(8)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} \right| \geq \sqrt{C_{(8)}} \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \right) \\
&\quad (\text{by Bonferroni inequality}) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq \sqrt{C_{(8)}} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} \right| \geq \sqrt{C_{(8)}} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad \left(\text{since } \sqrt{\frac{\ln N}{T_h}} < \left[\frac{\ln N}{T_h} \right]^{\frac{1}{4}} \text{ given that } \frac{\ln N}{T_h} = \frac{\ln N}{T-h} < 1 \text{ for all } N, T \text{ sufficiently large} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} \right| \geq C_{\nabla} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\quad (\text{since } C_{(8)} = \max \{C_{\dagger}^2, C_{\nabla}^2\} \text{ implies that } \sqrt{C_{(8)}} \geq C_{\dagger} \text{ and } \sqrt{C_{(8)}} \geq C_{\nabla}).
\end{aligned}$$

It then follows, by part (b) of Lemma E-9 and and part (b) of this lemma, that for all N and T sufficiently large,

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} \right| \geq C_{(8)} \sqrt{\frac{\ln N}{T_h}} \right) \\
&\leq P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \right| \geq C_{\dagger} \sqrt{\frac{\ln N}{T_h}} \right) + P \left(\left| \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} \right| \geq C_{\nabla} \sqrt{\frac{\ln N}{T_h}} \right) \\
&= o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
&= o \left(\frac{1}{N^2} \right). \quad \square
\end{aligned}$$

Lemma E-14: Let

$$\hat{\sigma}_{xy,i} = \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \bar{X}_{i,T_h}] y_{t+h} \text{ where } T_h = T - h,$$

where $h \geq 1$ is a (fixed) finite integer. Suppose that Assumptions (i), (ii), (iii), (iv) and (v) hold. Let γ_* be a positive constant such that

$$\gamma_* = \frac{\gamma_1 \gamma_2}{\gamma_1 + 2\gamma_2}$$

where γ_1 and γ_2 are the positive constants defined in Assumption (v) and suppose that

$$\frac{(\ln N)^{\frac{2}{\gamma_*}-1}}{T} \rightarrow 0 \text{ as } N, T \rightarrow \infty.$$

Under these conditions, consider the following two scenarios.

- (a) **Scenario 1 ($1 \leq \kappa < K$ Case):** Suppose in addition that Assumption S1-1 is satisfied. Then, there exists a constant $\overline{B}_2 > 0$ such that, for all N and T sufficiently large,

$$P \left(\max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i}| \geq \overline{B}_2 \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right).$$

or

$$\max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i}| = O_p \left(\sqrt{\frac{\ln N}{T}} \right).$$

- (b) **Scenario 2 ($\kappa = K$ Case):** Suppose in addition that Assumptions S2-1 is satisfied. Then, there exists a constant $\overline{B}_2 > 0$ such that, for all N and T sufficiently large,

$$P \left(\max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i}| \geq \overline{B}_2 \sqrt{\frac{\ln N}{T_h}} \right) = o \left(\frac{1}{N^2} \right).$$

or

$$\max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i}| = O_p \left(\sqrt{\frac{\ln N}{T}} \right).$$

Proof of Lemma E-14:

To show the result for part (a), note first that, for $i = 1, \dots, N$; we can write

$$\begin{aligned}
& \widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \overline{X}_{i,T_h}] y_{t+h} - \overline{\sigma}_{Xy,i} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \overline{X}_{i,T_h}] \left\{ \alpha + (F_{1,t} - \mu_{F,1})' \varphi_1 + \varepsilon_{t+h} \right\} - \overline{\sigma}_{Xy,i} \\
&\quad (\text{given expression (142)}) \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \overline{X}_{i,T_h}] (F_{1,t} - \mu_{F,1})' \varphi_1 - e'_{i,N} \Gamma_{\lambda,1} \frac{\underline{\Lambda}_1}{\sqrt{\underline{\Lambda}_{1,1}}} \varphi_1 + \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \overline{X}_{i,T_h}] \varepsilon_{t+h} \\
&\quad \left(\text{since } \overline{\sigma}_{Xy,i} = e'_{i,N} \Gamma_{\lambda,1} \frac{\underline{\Lambda}_1}{\sqrt{\underline{\Lambda}_{1,1}}} \varphi_1 \text{ with } e_{i,N} \text{ being the } i^{th} \text{ elementary vector} \right) \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} \left([X_{it} - \overline{X}_{i,T_h}] (F_{1,t} - \mu_{F,1})' - e'_{i,N} \Gamma_{\lambda,1} \frac{\underline{\Lambda}_1}{\sqrt{\underline{\Lambda}_{1,1}}} \right) \varphi_1 + \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \overline{X}_{i,T_h}] \varepsilon_{t+h} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} \left[(X_{it} - \mu_{X,i} + \mu_{X,i} - \overline{X}_{i,T_h}) (F_{1,t} - \mu_{F,1})' - e'_{i,N} \Gamma_{\lambda,1} \frac{\underline{\Lambda}_1}{\sqrt{\underline{\Lambda}_{1,1}}} \right] \varphi_1 \\
&\quad + \frac{1}{T_h} \sum_{t=1}^{T_h} [X_{it} - \mu_{X,i} + \mu_{X,i} - \overline{X}_{i,T_h}] \varepsilon_{t+h} \\
&= \frac{1}{T_h} \sum_{t=1}^{T_h} \left(e'_{i,N} \Gamma (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - e'_{i,N} \Gamma_{\lambda,1} \frac{\underline{\Lambda}_1}{\sqrt{\underline{\Lambda}_{1,1}}} \right) \varphi_1 + \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \\
&\quad - \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 - \frac{1}{T_h} \sum_{s=1}^{T_h} u_{is} \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \\
&\quad + \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) \varepsilon_{t+h} + \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \varepsilon_{t+h} - \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} \\
&\quad - \frac{1}{T_h} \sum_{s=1}^{T_h} u_{is} \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h}
\end{aligned}$$

. Hence, applying the triangle inequality and taking the max over all $i \in \{1, \dots, N\}$; we

obtain

$$\begin{aligned}
& \max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i}| \\
\leq & \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \left(e'_{i,N} \Gamma(F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - e'_{i,N} \Gamma_{\lambda,1} \frac{\underline{\Delta}_1}{\sqrt{\underline{\Delta}_{1,1}}} \right) \varphi_1 \right| \\
& + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \\
& + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \\
& + \max_{1 \leq i \leq N} |\overline{u}_{i,T_h}| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) \varepsilon_{t+h} \right| \\
& + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \varepsilon_{t+h} \right| + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| |\overline{\varepsilon}_{T_h}| \\
& + \max_{1 \leq i \leq N} |\overline{u}_{i,T_h}| |\overline{\varepsilon}_{T_h}| \tag{178}
\end{aligned}$$

where $\overline{u}_{i,T_h} = T_h^{-1} \sum_{s=1}^{T_h} u_{is}$ and $\overline{\varepsilon}_{T_h} = T_h^{-1} \sum_{t=1}^{T_h} \varepsilon_{t+h}$. Now, let $\overline{B}_2 > 0$ be a constant such that

$$\overline{B}_2 \geq 8 \max \{C_{(1)}, C_{(2)}, C_{(3)}, C_{(4)}, C_{(5)}, C_{(6)}, C_{(7)}, C_{(8)}\}$$

where $C_{(1)}$, $C_{(2)}$, $C_{(3)}$, $C_{(4)}$, $C_{(5)}$, $C_{(6)}$, $C_{(7)}$, and $C_{(8)}$ are constants defined previously in parts (b), (c), (d), and (e) of Lemma E-11 and in parts (a), (c), (d), and (e) of Lemma E-13. Given

expression (178) and given the above definition of the constant $\overline{B}_2$, we can further write

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} |\widehat{\sigma}_{Xy,i} - \overline{\sigma}_{Xy,i}| \geq \overline{B}_2 \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \left(e'_{i,N} \Gamma(F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - e'_{i,N} \Gamma_{\lambda,1} \frac{\Lambda_1}{\sqrt{\Lambda_{1,1}}} \right) \varphi_1 \right| \right. \\
& + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \\
& + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \\
& + \max_{1 \leq i \leq N} |\overline{u}_{i,T_h}| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) \varepsilon_{t+h} \right| \\
& + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \varepsilon_{t+h} \right| + \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| |\overline{\varepsilon}_{T_h}| \\
& + \max_{1 \leq i \leq N} |\overline{u}_{i,T_h}| |\overline{\varepsilon}_{T_h}| \geq \overline{B}_2 \sqrt{\frac{\ln N}{T_h}} \Bigg) \\
\leq & P \left(\left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \left(e'_{i,N} \Gamma(F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - e'_{i,N} \Gamma_{\lambda,1} \frac{\Lambda_1}{\sqrt{\Lambda_{1,1}}} \right) \varphi_1 \right| \geq \frac{\overline{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \right. \\
& \cup \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq \frac{\overline{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \cup \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq \frac{\overline{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \cup \left\{ \max_{1 \leq i \leq N} |\overline{u}_{i,T_h}| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq \frac{\overline{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \cup \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) \varepsilon_{t+h} \right| \geq \frac{\overline{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \cup \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \varepsilon_{t+h} \right| \geq \frac{\overline{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \cup \left\{ \max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| |\overline{\varepsilon}_{T_h}| \geq \frac{\overline{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \\
& \cup \left\{ \max_{1 \leq i \leq N} |\overline{u}_{i,T_h}| |\overline{\varepsilon}_{T_h}| \geq \frac{\overline{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right\} \Bigg)
\end{aligned}$$

It then follows, by applying the Bonferroni inequality, that

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \geq \bar{B}_2 \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \left(e'_{i,N} \Gamma(F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - e'_{i,N} \Gamma_{\lambda,1} \frac{\Lambda_1}{\sqrt{\Lambda_{1,1}}} \right) \varphi_1 \right| \geq \frac{\bar{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq \frac{\bar{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq \frac{\bar{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} |\bar{u}_{i,T_h}| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq \frac{\bar{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma(F_t - \mu_F) \varepsilon_{t+h} \right| \geq \frac{\bar{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \varepsilon_{t+h} \right| \geq \frac{\bar{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma(F_s - \mu_F) \right| |\bar{\varepsilon}_{T_h}| \geq \frac{\bar{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} |\bar{u}_{i,T_h}| |\bar{\varepsilon}_{T_h}| \geq \frac{\bar{B}_2}{8} \sqrt{\frac{\ln N}{T_h}} \right)
\end{aligned}$$

Finally, note that $\bar{B}_2 \geq 8 \max \{C_{(1)}, C_{(2)}, C_{(3)}, C_{(4)}, C_{(5)}, C_{(6)}, C_{(7)}, C_{(8)}\}$ implies that $\bar{B}_2/8 \geq C_{(k)}$ for $k \in \{1, 2, \dots, 8\}$. Hence, by applying the results given in parts (b), (c), (d), and (e)

of Lemma E-11 and also in parts (a), (c), (d), and (e) of Lemma E-13, we have

$$\begin{aligned}
& P \left(\max_{1 \leq i \leq N} |\hat{\sigma}_{Xy,i} - \bar{\sigma}_{Xy,i}| \geq \bar{B}_2 \sqrt{\frac{\ln N}{T_h}} \right) \\
\leq & P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} \left(e'_{i,N} \Gamma (F_t - \mu_F) (F_{1,t} - \mu_{F,1})' - e'_{i,N} \Gamma_{\lambda,1} \frac{\underline{\Lambda}_1}{\sqrt{\underline{\Lambda}_{1,1}}} \right) \varphi_1 \right| \geq C_{(1)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{(2)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{(3)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} |\bar{u}_{i,T_h}| \left| \frac{1}{T_h} \sum_{t=1}^{T_h} (F_{1,t} - \mu_{F,1})' \varphi_1 \right| \geq C_{(4)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} e'_{i,N} \Gamma (F_t - \mu_F) \varepsilon_{t+h} \right| \geq C_{(5)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{t=1}^{T_h} u_{it} \varepsilon_{t+h} \right| \geq C_{(6)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} \left| \frac{1}{T_h} \sum_{s=1}^{T_h} e'_{i,N} \Gamma (F_s - \mu_F) \right| |\bar{\varepsilon}_{T_h}| \geq C_{(7)} \sqrt{\frac{\ln N}{T_h}} \right) \\
& + P \left(\max_{1 \leq i \leq N} |\bar{u}_{i,T_h}| |\bar{\varepsilon}_{T_h}| \geq C_{(8)} \sqrt{\frac{\ln N}{T_h}} \right) \\
= & o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) + o \left(\frac{1}{N^2} \right) \\
= & o \left(\frac{1}{N^2} \right).
\end{aligned}$$

The result in part (b) can be shown using essentially the same argument as that given in the proof of part (a), except that we apply Assumption S2-1 instead of Assumption S1-1 and also apply parts (b), (c), (d), and (e) of Lemma E-12 instead of parts (b), (c), (d), and (e) of Lemma E-11. In addition, we need to change the notations used in the proof of part (a) above by setting

$$\kappa = K, F_{1,t} = F_t, \mu_{F,1} = \mu_F, \Gamma_{\lambda,1} = \Gamma_\lambda, \varphi_1 = \varphi, \text{ and } \underline{\Lambda}_1 = \underline{\Lambda};$$

since, under Scenario 2 considered here, all available factors are relevant for the purpose of forecasting and there is no reason to partition $F_t = (F'_{1,t}, F'_{2,t})'$, $\mu_F = (\mu'_{F,1}, \mu'_{F,2})'$, $\Gamma_\lambda =$

$(\Gamma_{\lambda,1} \quad \Gamma_{\lambda,2})$, $\varphi = (\varphi'_1, \varphi'_2)'$, and

$$\underline{\Lambda} = \begin{pmatrix} \underline{\Lambda}_1 & 0 \\ 0 & \underline{\Lambda}_2 \end{pmatrix}. \quad \square$$

Lemma E-15: Suppose that Assumptions (i)-(iii), given at the beginning of Appendix E, are satisfied.

- (a) **Scenario 1 (The $1 \leq \kappa < K$ Case):** Suppose that Assumptions S1-1 and S1-2 hold. Then, there exists a positive constant C_1 such that

$$\left\| \frac{\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 = \frac{\bar{\sigma}'_{Xy} \bar{\sigma}_{Xy}}{\lambda_1} \leq C_1.$$

- (b) **Scenario 2 (The $\kappa = K$ Case):** Suppose that Assumptions S2-1 and S2-2 hold. Then, there exists a positive constant C_2 such that

$$\left\| \frac{\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 = \frac{\bar{\sigma}'_{Xy} \bar{\sigma}_{Xy}}{\lambda_1} \leq C_2.$$

Proof of Lemma E-15: To show the result under Scenario 1, note that

$$\begin{aligned}
& \left\| \frac{\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 \\
&= \frac{\bar{\sigma}_{Xy}' \bar{\sigma}_{Xy}}{\lambda_1} \\
&= \frac{\varphi_1' \underline{\Lambda}_1 \Gamma'_{\lambda,1} \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\lambda_1 \underline{\lambda}_{1,1}} \\
&\quad \left(\text{since, in this case, } \bar{\sigma}_{Xy} = \Gamma_1 \frac{\underline{\Lambda}_1}{\underline{\lambda}_{1,1}} \varphi_1 = \Gamma_{\lambda,1} \frac{\underline{\Lambda}_1}{\sqrt{\underline{\lambda}_{1,1}}} \varphi_1, \text{ where } \Gamma_{\lambda,1} = \frac{\Gamma_1}{\sqrt{\underline{\lambda}_{1,1}}} \right) \\
&= \frac{\varphi_1' \underline{\Lambda}_1^2 \varphi_1}{\lambda_1 \underline{\lambda}_{1,1}} \quad (\text{since } \Gamma'_{\lambda,1} \Gamma_{\lambda,1} = I_\kappa) \\
&\leq \frac{\underline{\lambda}_{1,1}^2}{\lambda_1 \underline{\lambda}_{1,1}} \|\varphi_1\|_2^2 \quad (\text{given that, under Scenario 1, } \underline{\lambda}_{1,1} = \lambda_{\max}(\underline{\Lambda}_1)) \\
&= \frac{\underline{\lambda}_{1,1}}{\lambda_1} \|\varphi_1\|_2^2 \\
&\leq c^* \|\varphi_1\|_2^2 \quad (\text{since part (a) of Lemma D-3 implies that there exists a constant } c^* > 1 \\
&\quad \text{such that } 0 < \frac{1}{c^*} \leq \frac{\underline{\lambda}_{1,1}}{\lambda_1} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large}) \\
&\leq \kappa \bar{c}^2 c^* \quad (\text{given that under Assumption S1-1, there exists a constant } \\
&\quad \bar{c} > 1 \text{ such that } \max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \leq \bar{c} < \infty) \\
&\leq C_1 < \infty \quad (\text{for some positive constant } C_1 \geq \kappa \bar{c}^2 c^*)
\end{aligned}$$

The result under Scenario 2 can be shown in the same way as follows

$$\begin{aligned}
& \left\| \frac{\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 \\
&= \frac{\bar{\sigma}'_{Xy} \bar{\sigma}_{Xy}}{\lambda_1} \\
&= \frac{\varphi' \underline{\Lambda} \Gamma'_\lambda \Gamma_\lambda \underline{\Lambda} \varphi}{\lambda_1 \underline{\lambda}_{1,1}} \quad \left(\text{since, in this case, } \bar{\sigma}_{Xy} = \Gamma \frac{\underline{\Lambda}}{\underline{\lambda}_{1,1}} \varphi = \Gamma_\lambda \frac{\underline{\Lambda}}{\sqrt{\underline{\lambda}_{1,1}}} \varphi, \text{ where } \Gamma_\lambda = \frac{\Gamma}{\sqrt{\underline{\lambda}_{1,1}}} \right) \\
&= \frac{\varphi' \underline{\Lambda}^2 \varphi}{\lambda_1 \underline{\lambda}_{1,1}} \quad (\text{since } \Gamma'_\lambda \Gamma_\lambda = I_K) \\
&\leq \frac{\underline{\lambda}_{1,1}^2}{\lambda_1 \underline{\lambda}_{1,1}} \|\varphi\|_2^2 \quad (\text{given that, under Scenario 2, } \underline{\lambda}_{1,1} = \lambda_{\max}(\underline{\Lambda})) \\
&= \frac{\underline{\lambda}_{1,1}}{\lambda_1} \|\varphi\|_2^2 \\
&\leq c^* \|\varphi_1\|_2^2 \quad (\text{since part (a) of Lemma D-3 implies that there exists a constant } c^* > 1 \\
&\quad \text{such that } 0 < \frac{1}{c^*} \leq \frac{\underline{\lambda}_{1,1}}{\lambda_1} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large}) \\
&\leq K \bar{c}^2 c^* \quad (\text{given that under Assumption S2-1, there exists a constant } \\
&\quad \bar{c} > 1 \text{ such that } \max_{1 \leq g \leq K} |\varphi_{1,g}| \leq \bar{c} < \infty) \\
&\leq C_2 < \infty \quad (\text{for some positive constant } C_2 \geq K \bar{c}^2 c^*). \quad \square
\end{aligned}$$

Lemma E-16: Let

$$D_{\lambda, \kappa} = \begin{pmatrix} \lambda_1^{\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{\kappa - \frac{1}{2}} \end{pmatrix} \quad \text{for } \kappa \in \{1, \dots, K\}$$

Suppose that Assumptions (i)-(iii), given at the beginning of Appendix E, are satisfied.

- (a) Under Scenario 1, where $1 \leq \kappa < K$, suppose that Assumptions S1-1 and S1-2 both hold. Then, there exists some positive constant $\bar{C}_1$ such that

$$\lambda_{\max}(D_{\lambda, \kappa}^{-1} G' G D_{\lambda, \kappa}^{-1}) = \|G D_{\lambda, \kappa}^{-1}\|_2^2 \leq \|G D_{\lambda, \kappa}^{-1}\|_F^2 \leq \bar{C}_1 < \infty$$

and

$$\left\| \frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right\|_2 \leq \bar{C}_1 < \infty$$

for all N and T sufficiently large.

Under Scenario 2, where $\kappa = K$, suppose that Assumptions S2-1 and S2-2 both hold. Then, there exists some positive constant $\overline{C}_2$ such that

$$\lambda_{\max} (D_{\lambda,K}^{-1} G' G D_{\lambda,K}^{-1}) = \|G D_{\lambda,K}^{-1}\|_2^2 \leq \|G D_{\lambda,K}^{-1}\|_F^2 \leq \overline{C}_2 < \infty$$

and

$$\left\| \frac{D_{\lambda,K}^{-1} G' \overline{\Sigma}_X G D_{\lambda,K}^{-1}}{\lambda_1} \right\|_2 \leq C_2^* < \infty$$

for all N and T sufficiently large.

- (b) Under Scenario 1, where $1 \leq \kappa < K$, suppose that Assumptions S1-1 and S1-2 both hold. Then, there exists some positive constant $C_1^\dagger$ such that

$$\left\| \frac{D_{\lambda,\kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 \leq C_1^\dagger < \infty$$

for all N and T sufficiently large.

Under Scenario 2, where $\kappa = K$, suppose that Assumptions S2-1 and S2-2 both hold. Then, there exists some positive constant $C_2^\dagger$ such that

$$\left\| \frac{D_{\lambda,K}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 \leq C_2^\dagger < \infty$$

for all N and T sufficiently large.

Proof of Lemma E-16

Under Scenario 1, to show the first result of part (a), note that

$$\begin{aligned}
& \lambda_{\max} (D_{\lambda, \kappa}^{-1} G' G D_{\lambda, \kappa}^{-1}) \\
&= \|GD_{\lambda, \kappa}^{-1}\|_2^2 \\
&\leq \|GD_{\lambda, \kappa}^{-1}\|_F^2 = \text{tr} \{D_{\lambda, \kappa}^{-1} G' G D_{\lambda, \kappa}^{-1}\} \\
&= \text{tr} \left\{ \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_{Xy} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \end{pmatrix} \right. \\
&\quad \times \left(\bar{\sigma}_{Xy} \quad \bar{\Sigma}_X \bar{\sigma}_{Xy} \quad \cdots \quad \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \right) \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \Bigg\} \\
&= \text{tr} \left\{ \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_{Xy} \bar{\sigma}_{Xy} & \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa} \bar{\sigma}_{Xy} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa} \bar{\sigma}_{Xy} & \cdots & \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{2(\kappa-1)} \bar{\sigma}_{Xy} \end{pmatrix} \right. \\
&\quad \times \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \Bigg\} \\
&= \text{tr} \left\{ \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa-\frac{1}{2})} \end{pmatrix} \right. \\
&\quad \times \begin{pmatrix} \lambda_1^{-\frac{1}{2}} \bar{\sigma}'_{Xy} \bar{\sigma}_{Xy} & \lambda_1^{-\frac{3}{2}} \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \lambda_1^{-(\kappa-\frac{1}{2})} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \\ \lambda_1^{-\frac{1}{2}} \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} & \lambda_1^{-\frac{3}{2}} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} & \cdots & \lambda_1^{-(\kappa-\frac{1}{2})} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa} \bar{\sigma}_{Xy} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{-\frac{1}{2}} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} & \lambda_1^{-\frac{3}{2}} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa} \bar{\sigma}_{Xy} & \cdots & \lambda_1^{-(\kappa-\frac{1}{2})} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{2(\kappa-1)} \bar{\sigma}_{Xy} \end{pmatrix} \Bigg\}
\end{aligned}$$

$$\begin{aligned}
&= \text{tr} \left\{ \begin{pmatrix} \lambda_1^{-1} \bar{\sigma}'_{Xy} \bar{\sigma}_{Xy} & \lambda_1^{-2} \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} & \cdots & \lambda_1^{-\kappa} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} \\ \lambda_1^{-2} \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} & \lambda_1^{-3} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^2 \bar{\sigma}_{Xy} & \cdots & \lambda_1^{-(\kappa+1)} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa} \bar{\sigma}_{Xy} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{-\kappa} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} & \lambda_1^{-(\kappa+1)} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa} \bar{\sigma}_{Xy} & \cdots & \lambda_1^{-(2\kappa-1)} \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{2(\kappa-1)} \bar{\sigma}_{Xy} \end{pmatrix} \right\} \\
&= \sum_{j=1}^{\kappa} \frac{\bar{\sigma}'_{Xy} \bar{\Sigma}_X^{2(j-1)} \bar{\sigma}_{Xy}}{\lambda_1^{2j-1}} \\
&\leq \sum_{j=1}^{\kappa} \left(\frac{\lambda_1}{\lambda_1} \right)^{2(j-1)} \frac{\bar{\sigma}'_{Xy} \bar{\sigma}_{Xy}}{\lambda_1} \\
&= \sum_{j=1}^{\kappa} \frac{\bar{\sigma}'_{Xy} \bar{\sigma}_{Xy}}{\lambda_1} \\
&\leq \kappa C_1 \left(\text{given that, by part (a) of Lemma E-15, } \frac{\bar{\sigma}'_{Xy} \bar{\sigma}_{Xy}}{\lambda_1} \leq C_1 \text{ for some constant } C_1 > 0 \right) \\
&\leq \bar{C}_1 < \infty \text{ for some constant } \bar{C} \geq \kappa C_1 \text{ and for all } N \text{ and } T \text{ sufficiently large.}
\end{aligned}$$

Now, to show the second result of part (a) under Scenario 1, note that

$$\begin{aligned}
&\left\| \frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right\|_2 \\
&= \sqrt{\lambda_{\max} \left[\left(\frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^2 \right]} \left(\text{given that } \frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \text{ is symmetric} \right) \\
&= \sqrt{\left[\lambda_{\max} \left(\frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right) \right]^2} \\
&= \lambda_{\max} \left(\frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right) \left(\text{given that } \frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \text{ is positive definite as shown} \right. \\
&\quad \left. \text{in the proof of part (b) of Lemma E-2} \right) \\
&\leq \frac{\lambda_{\max}(\bar{\Sigma}_X)}{\lambda_1} \lambda_{\max}(D_{\lambda, \kappa}^{-1} G' G D_{\lambda, \kappa}^{-1}) \\
&= \lambda_{\max}(D_{\lambda, \kappa}^{-1} G' G D_{\lambda, \kappa}^{-1}) \quad (\text{since } \lambda_1 = \lambda_{\max}(\bar{\Sigma}_X)) \\
&\leq \bar{C}_1 \quad (\text{by the first result given above under Scenario 1})
\end{aligned}$$

The two results under Scenario 2 can be proved in the same manner as the proof given for Scenario 1, except that we set $\kappa = K$ and apply the result given in part (b) of Lemma E-15 instead of part (a) of Lemma E-15.

To show part (b), we first consider Scenario 1 where $1 \leq \kappa < K$. In this case, we can write

$$\frac{D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} = \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(\kappa - \frac{1}{2})} \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_{Xy} \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \end{pmatrix} \frac{\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} = \begin{pmatrix} \bar{\sigma}'_{Xy} \bar{\sigma}_{Xy} / \lambda_1 \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X \bar{\sigma}_{Xy} / \lambda_1^2 \\ \vdots \\ \bar{\sigma}'_{Xy} \bar{\Sigma}_X^{\kappa-1} \bar{\sigma}_{Xy} / \lambda_1^\kappa \end{pmatrix} \quad (179)$$

so that

$$\begin{aligned}
\left\| \frac{D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 &= \sum_{g=1}^{\kappa} \left(\frac{\bar{\sigma}'_{Xy} \bar{\Sigma}_X^{g-1} \bar{\sigma}_{Xy}}{\lambda_1^g} \right)^2 \\
&= \sum_{g=1}^{\kappa} \left(\frac{\varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \bar{\Sigma}_X^{g-1} \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\lambda_1^g \underline{\lambda}_{1,1}} \right)^2 \\
&\quad \left(\text{since, in this case, } \bar{\sigma}_{Xy} = \frac{\Gamma_1 \underline{\Lambda}_1}{\underline{\lambda}_{1,1}} \varphi_1 = \frac{\Gamma_{\lambda,1} \underline{\Lambda}_1}{\sqrt{\underline{\lambda}_{1,1}}} \varphi_1, \text{ where } \Gamma_{\lambda,1} = \frac{\Gamma_1}{\sqrt{\underline{\lambda}_{1,1}}} \right) \\
&\leq \sum_{g=1}^{\kappa} \left(\frac{\lambda_1}{\lambda_1} \right)^{2(g-1)} \left(\frac{\varphi'_1 \underline{\Lambda}_1 \Gamma'_{\lambda,1} \Gamma_{\lambda,1} \underline{\Lambda}_1 \varphi_1}{\lambda_1 \underline{\lambda}_{1,1}} \right)^2 \quad (\text{since } \lambda_1 = \lambda_{\max}(\bar{\Sigma}_X)) \\
&= \kappa \left(\frac{\varphi'_1 \underline{\Lambda}_1^2 \varphi_1}{\lambda_1 \underline{\lambda}_{1,1}} \right)^2 \quad (\text{since } \Gamma'_{\lambda,1} \Gamma_{\lambda,1} = I_{\kappa}) \\
&\leq \kappa \frac{\underline{\lambda}_{1,1}^4}{\lambda_1^2 \underline{\lambda}_{1,1}^2} \|\varphi_1\|_2^2 \quad (\text{since } \lambda_{\max}(\underline{\Lambda}_1^2) = [\lambda_{\max}(\underline{\Lambda}_1)]^2 = \underline{\lambda}_{1,1}^2) \\
&= \kappa \|\varphi_1\|_2^2 \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right)^2 \\
&\leq \kappa^2 \left(\max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \right)^2 \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right)^2 \\
&\leq \kappa^2 \bar{c}^2 \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right)^2 \quad (\text{given Assumption S1-1}) \\
&\leq \kappa^2 \bar{c}^2 (c^*)^2 \quad \text{for all } N \text{ and } T \text{ sufficiently large} \\
&\quad (\text{given that part (a) of Lemma D-3 implies that there exists an absolute constant } \\
&\quad c^* > 1 \text{ such that } 0 < \frac{1}{c^*} < \frac{\underline{\lambda}_{1,1}}{\lambda_1} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large}) \\
&\leq C_1^\dagger < \infty \quad \left(\text{for some positive constant } C_1^\dagger \geq \kappa^2 \bar{c}^2 (c^*)^2 \right)
\end{aligned}$$

The proof under Scenario 2 where $\kappa = K$ can be obtained in the same way. In particular, in this case

$$\frac{D_{\lambda,K}^{-1}G'\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} = \begin{pmatrix} \lambda_1^{-\frac{1}{2}} & 0 & \cdots & 0 \\ 0 & \lambda_1^{-\frac{3}{2}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_1^{-(K-\frac{1}{2})} \end{pmatrix} \begin{pmatrix} \bar{\sigma}'_{Xy} \\ \bar{\sigma}'_{Xy}\bar{\Sigma}_X \\ \vdots \\ \bar{\sigma}'_{Xy}\bar{\Sigma}_X^{K-1} \end{pmatrix} \frac{\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} = \begin{pmatrix} \bar{\sigma}'_{Xy}\bar{\sigma}_{Xy}/\lambda_1 \\ \bar{\sigma}'_{Xy}\bar{\Sigma}_X\bar{\sigma}_{Xy}/\lambda_1^2 \\ \vdots \\ \bar{\sigma}'_{Xy}\bar{\Sigma}_X^{K-1}\bar{\sigma}_{Xy}/\lambda_1^K \end{pmatrix} \quad (180)$$

so that

$$\begin{aligned} \left\| \frac{D_{\lambda,K}^{-1}G'\bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2^2 &= \sum_{g=1}^K \left(\frac{\bar{\sigma}'_{Xy}\bar{\Sigma}_X^{g-1}\bar{\sigma}_{Xy}}{\lambda_1^g} \right)^2 \\ &= \sum_{g=1}^K \left(\frac{\varphi'\underline{\Lambda}_1\Gamma'_\lambda\bar{\Sigma}_X^{g-1}\Gamma_\lambda\underline{\Lambda}\varphi}{\lambda_1^g\underline{\lambda}_{1,1}} \right)^2 \quad \left(\text{since, in this case, } \underline{\Lambda} = \text{diag}(\underline{\lambda}_{1,1}, \dots, \underline{\lambda}_{1,K}) \right), \\ &\quad \varphi = (\varphi_{1,1}, \dots, \varphi_{1,K}), \text{ and } \bar{\sigma}_{Xy} = \frac{\Gamma\underline{\Lambda}}{\underline{\lambda}_{1,1}}\varphi = \frac{\Gamma_\lambda\underline{\Lambda}}{\sqrt{\underline{\lambda}_{1,1}}}\varphi, \text{ where } \Gamma_\lambda = \frac{\Gamma}{\sqrt{\underline{\lambda}_{1,1}}} \Bigg) \\ &\leq \sum_{g=1}^K \left(\frac{\lambda_1}{\lambda_1} \right)^{2(g-1)} \left(\frac{\varphi'\underline{\Lambda}\Gamma'_\lambda\Gamma_\lambda\underline{\Lambda}\varphi}{\lambda_1\underline{\lambda}_{1,1}} \right)^2 \\ &= K \left(\frac{\varphi'\underline{\Lambda}^2\varphi}{\lambda_1\underline{\lambda}_{1,1}} \right)^2 \quad (\text{since } \Gamma'_\lambda\Gamma_\lambda = I_K) \\ &= K \left(\frac{\underline{\lambda}_{1,1}^2}{\lambda_1^2} \right) \left(\frac{\underline{\lambda}_{1,1}^2}{\underline{\lambda}_{1,1}^2} \right) \|\varphi\|_2^2 \\ &\quad (\text{since, under Scenario 2, } \lambda_{\max}(\underline{\Lambda}^2) = [\lambda_{\max}(\underline{\Lambda})]^2 = \underline{\lambda}_{1,1}^2) \\ &\leq K^2 \left(\max_{1 \leq g \leq K} |\varphi_{1,g}| \right)^2 \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right)^2 \\ &\leq K^2 \bar{c}^2 \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right)^2 \quad (\text{given Assumption S2-1}) \\ &\leq K^2 \bar{c}^2 (c^*)^2 \quad \text{for all } N \text{ and } T \text{ sufficiently large} \\ &\quad (\text{given that part (a) of Lemma D-3 implies that there exists an absolute constant} \\ &\quad c^* > 1 \text{ such that } 0 < \frac{1}{c^*} < \frac{\underline{\lambda}_{1,1}}{\lambda_1} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large}) \\ &\leq C_2^\dagger < \infty \quad \left(\text{for some positive constant } C_2^\dagger \geq K^2 \bar{c}^2 (c^*)^2 \right). \quad \square \end{aligned}$$

Remark E-17: Consider a diffusion index model written in the form

$$y_{t+h} = \alpha + (\bar{F}_t - \bar{\mu}_F)' \bar{\varphi} + \varepsilon_{t+h} \quad (181)$$

$$X_t = \mu_X + \bar{\Gamma} (\bar{F}_t - \bar{\mu}_F) + u_t, \quad (182)$$

where

$$\bar{\varphi} = \begin{pmatrix} \bar{\varphi}_1 \\ \vdots \\ \bar{\varphi}_K \end{pmatrix}, \bar{F}_t = \begin{pmatrix} \bar{F}_{t,1} \\ \vdots \\ \bar{F}_{t,K} \end{pmatrix}, \text{ and } \bar{\mu}_F = \begin{pmatrix} \bar{\mu}_{F,1} \\ \vdots \\ \bar{\mu}_{F,K} \end{pmatrix}.$$

Suppose that $K - \kappa$ components of the $K \times 1$ coefficient vector $\bar{\varphi} = (\bar{\varphi}_1, \dots, \bar{\varphi}_K)'$ are equal to zero, whereas the remaining $\kappa \times 1$ components are nonzero, so that only $\kappa < K$ of the available factors are relevant for predicting the target variable of interest y_{t+h} . Within this context, we want to show that it is always possible to transform the model given by expressions (181) and (182) above into the form

$$\begin{aligned} y_{t+h} &= \alpha + (F_{1,t} - \mu_{F,1})' \varphi_1 + \varepsilon_{t+h} \\ X_t &= \mu_X + \Gamma_1 (F_{1,t} - \mu_{F,1}) + \Gamma_2 (F_{2,t} - \mu_{F,2}) + u_t. \end{aligned}$$

where the $\kappa \times 1$ coefficient vector $\varphi_1 = (\varphi_{1,1}, \dots, \varphi_{1,\kappa})'$ contain those components of $\bar{\varphi} = (\bar{\varphi}_1, \dots, \bar{\varphi}_K)'$ which are nonzero and where the vector $F_{1,t}$ contains those κ factors which are relevant for the purpose of predicting the target variable of interest y_{t+h} .

To formalize this situation more carefully, we need to develop some additional notation. To proceed, first sort the indices $\{1, \dots, K\}$ into two groups:

$$\{r_1, \dots, r_\kappa\} \text{ and } \{j_1, \dots, j_{K-\kappa}\},$$

where $\{r_1, \dots, r_\kappa\} \cup \{j_1, \dots, j_{K-\kappa}\} = \{1, \dots, K\}$. Next, let

$$\bar{\varphi}_{r_g} \neq 0, \text{ for all } g \in \{1, \dots, \kappa\} \text{ and } \bar{\varphi}_{j_k} = 0, \text{ for all } k \in \{1, \dots, K - \kappa\},$$

so that $\bar{\varphi}_{r_k}$ and $\bar{\varphi}_{j_k}$ are, respectively, the r_k -th and the j_k -th component of the coefficient vector $\bar{\varphi}$ given in expression (181)⁹. Now, define

$$\begin{aligned} \varphi_1 &= \begin{pmatrix} \bar{\varphi}_{r_1} \\ \vdots \\ \bar{\varphi}_{r_\kappa} \end{pmatrix} = \begin{pmatrix} \varphi_{1,1} \\ \vdots \\ \varphi_{1,\kappa} \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} \varphi_{j_1} \\ \vdots \\ \varphi_{j_{K-\kappa}} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\ F_{1,t} &= \begin{pmatrix} \bar{F}_{t,r_1} \\ \vdots \\ \bar{F}_{t,r_\kappa} \end{pmatrix}, \text{ and } F_{2,t} = \begin{pmatrix} \bar{F}_{t,j_1} \\ \vdots \\ \bar{F}_{t,j_{K-\kappa}} \end{pmatrix} \end{aligned}$$

⁹Note that our notations here allow the non-zero components $\bar{\varphi}_{r_1}, \dots, \bar{\varphi}_{r_\kappa}$ to be an arbitrary subset of the components of the coefficient vector $\bar{\varphi} = (\bar{\varphi}_1, \dots, \bar{\varphi}_K)'$.

so that $F_{1,t}$ is the subvector of factors which have nonzero coefficients and $F_{2,t}$ is the subvector of factors that are irrelevant for forecasting y_{t+h} because they all have coefficients equal to zero. In addition, let

$$\varphi_{K \times 1} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ 0 \end{pmatrix}$$

and note that φ is just a repermuted version of the original coefficient vector $\bar{\varphi}$, so that the nonzero coefficients now appear as the first κ components of φ , and the remaining components are all equal to zero. It follows that there exists a $K \times K$ permutation matrix $\mathcal{P}$ such that

$$\begin{aligned} \mathcal{P}\bar{F}_t &= F_t = \begin{pmatrix} F_{1,t} \\ F_{2,t} \end{pmatrix}, \mathcal{P}\bar{\varphi} = \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \\ \mathcal{P}\bar{\mu}_F &= \mu_F = \begin{pmatrix} \mu_{F,1} \\ \mu_{F,2} \end{pmatrix}, \text{ and } \bar{\Gamma}\mathcal{P}' = \Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix} \end{aligned}$$

and, using the fact that permutation matrices are orthogonal, we can transform the equations (181) and (182) as follows:

$$\begin{aligned} y_{t+h} &= \alpha + (\bar{F}_t - \bar{\mu}_F)' \bar{\varphi} + \varepsilon_{t+h} \\ &= \alpha + (\bar{F}_t - \bar{\mu}_F)' \mathcal{P}' \mathcal{P} \bar{\varphi} + \varepsilon_{t+h} \\ &= \alpha + (F_t - \mu_F)' \varphi + \varepsilon_{t+h} \\ &= \alpha + (F_{1,t} - \mu_{F,1})' \varphi_1 + (F_{2,t} - \mu_{F,2})' \varphi_2 + \varepsilon_{t+h} \\ &= \alpha + (F_{1,t} - \mu_{F,1})' \varphi_1 + \varepsilon_{t+h} \quad (\text{since } \varphi_2 = 0) \end{aligned} \tag{183}$$

$$\begin{aligned} X_t &= \mu_X + \bar{\Gamma} (\bar{F}_t - \bar{\mu}_F) + u_t \\ &= \mu_X + \bar{\Gamma} \mathcal{P}' \mathcal{P} (\bar{F}_t - \bar{\mu}_F) + u_t \\ &= \mu_X + \Gamma (F_t - \mu_F) + u_t \\ &= \mu_X + \Gamma_1 (F_{1,t} - \mu_{F,1}) + \Gamma_2 (F_{2,t} - \mu_{F,2}) + u_t. \end{aligned} \tag{184}$$

which gives the desired representation.

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