

Supplemental Appendix to “Regularized Partial Least Squares Estimation of an Approximate Envelope Model for Supervised Dimension Reduction in Time Series Forecasting”¹

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Abstract

This supplemental appendix contains the proof of Theorem 3 as well as the proof of part (a) of Theorems 5 and 6. Due to similarity in the proofs of parts (a) and (b) of Theorems 5 and 6, we have placed the proof of part (b) of Theorems 5 and 6 in an Online Appendix. In addition, the proofs of Theorems 1-2, 4, and 7 and the proofs of all the supporting lemmas are also given in the Online Appendix. The Online Appendix can be found at John Chao’s website (<http://econweb.umd.edu/~chao/Research/research.html>) and also at Norman Swanson’s website (<http://norman-swanson.github.io/papers.htm>).

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JEL classification: C32, C33, C38, C52, C53, C55, C58.

Proof of Theorem 3: To proceed, note first that, by making use of the factor model representation given by $X_t = \mu_X + \Gamma (F_t - \mu_F) + u_t$, we can write $(X_T - \mu_X)' \beta = (F_T - \mu_F)' \Gamma' \beta + u_T' \beta$. Now, consider first the term $u_T' \beta$. For this term, note that, since $\beta = (\Gamma_1 / \sqrt{\Delta_{1,1}}) (\varphi_1 / \sqrt{\Delta_{1,1}})$ in this case,

$$\begin{aligned} & E \left[(u_T' \beta)^2 \right] \\ &= E \left[\left(\frac{u_T' \Gamma_1 \varphi_1}{\Delta_{1,1}} \right)^2 \right] = \frac{\varphi_1'}{\sqrt{\Delta_{1,1}}} \frac{\Gamma_1'}{\sqrt{\Delta_{1,1}}} E [u_T u_T'] \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \frac{\varphi_1}{\sqrt{\Delta_{1,1}}} \\ &= \frac{\varphi_1'}{\sqrt{\Delta_{1,1}}} \frac{\Gamma_1'}{\sqrt{\Delta_{1,1}}} \Sigma_{u,T} \frac{\Gamma_1}{\sqrt{\Delta_{1,1}}} \frac{\varphi_1}{\sqrt{\Delta_{1,1}}} \\ &\leq \lambda_{\max}(\Sigma_{u,T}) \frac{\varphi_1'}{\sqrt{\Delta_{1,1}}} \frac{\Gamma_1' \Gamma_1}{\Delta_{1,1}} \frac{\varphi_1}{\sqrt{\Delta_{1,1}}} \end{aligned}$$

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$$\begin{aligned}
&= \lambda_{\max}(\Sigma_{u,T}) \frac{\|\varphi_1\|_2^2}{\underline{\lambda}_{1,1}} \left(\text{given that } \varphi_1' \varphi_1 = \|\varphi_1\|_2^2 \text{ and given that } \frac{\Gamma_1' \Gamma_1}{\underline{\lambda}_{1,1}} = I_\kappa \right. \\
&\quad \left. \text{by condition (i) since this is just the upper left } \kappa \times \kappa \text{ submatrix of } \frac{\Gamma' \Gamma}{\underline{\lambda}_{1,1}} = I_K \right) \\
&\leq \frac{C^* \kappa \bar{c}^2}{\underline{\lambda}_{1,1}} \left(\text{since, by condition (i) of this theorem and by Assumption S1-1, there exist} \right. \\
&\quad \left. \text{constants } C^* \text{ and } \bar{c} \text{ such that } C^* \geq \sup_t \lambda_{\max}(\Sigma_{u,t}) \text{ and } \kappa \bar{c}^2 \geq \kappa \left(\max_{1 \leq g \leq \kappa} |\varphi_{1,g}| \right)^2 \geq \|\varphi_1\|_2^2 \right) \\
&= O\left(\frac{1}{\underline{\lambda}_{1,1}}\right)
\end{aligned}$$

so that, upon application of the Markov's inequality, we deduce that

$$|u_T' \beta| = \left| \frac{u_T' \Gamma_1 \varphi_1}{\underline{\lambda}_{1,1}} \right| = O_p\left(\frac{1}{\sqrt{\underline{\lambda}_{1,1}}}\right). \quad (1)$$

Next, consider the term $(F_T - \mu_F)' \Gamma' \beta$. For this term, note that

$$\begin{aligned}
&(F_T - \mu_F)' \Gamma' \beta \\
&= (F_{1,T} - \mu_{F,1})' \Gamma_1' \frac{\Gamma_1}{\sqrt{\underline{\lambda}_{1,1}}} \left(\frac{\varphi_1}{\sqrt{\underline{\lambda}_{1,1}}} \right) + (F_{2,T} - \mu_{F,2})' \Gamma_2' \frac{\Gamma_1}{\sqrt{\underline{\lambda}_{1,1}}} \left(\frac{\varphi_1}{\sqrt{\underline{\lambda}_{1,1}}} \right) \\
&= (F_{1,T} - \mu_{F,1})' \frac{\Gamma_1' \Gamma_1}{\underline{\lambda}_{1,1}} \varphi_1 + (F_{2,T} - \mu_{F,2})' \frac{\Gamma_2' \Gamma_1}{\underline{\lambda}_{1,1}} \varphi_1 \\
&= (F_{1,T} - \mu_{F,1})' \varphi_1 \left(\text{since, by condition (i), } \frac{\Gamma_1' \Gamma_1}{\underline{\lambda}_{1,1}} = I_\kappa \text{ and } \frac{\Gamma_2' \Gamma_1}{\underline{\lambda}_{1,1}} = 0 \right) \quad (2)
\end{aligned}$$

It follows by expressions (1) and (2) that

$$\begin{aligned}
(X_T - \mu_X)' \beta &= (F_T - \mu_F)' \Gamma' \beta + u_T' \beta \\
&= (F_{1,T} - \mu_{F,1})' \varphi_1 + O_p\left(\frac{1}{\sqrt{\underline{\lambda}_{1,1}}}\right) \\
&= (F_{1,T} - \mu_{F,1})' \varphi_1 + O_p\left(\frac{1}{\sqrt{\lambda_1}}\right), \quad (3)
\end{aligned}$$

where the last equality above follows by part (a) of Lemma D-3, given in Appendix D of the Online Appendix, which shows that there exists an absolute constant $c^* > 1$ such that

$$0 < \frac{1}{c^*} \leq \frac{\lambda_1}{\underline{\lambda}_{1,1}} \leq c^* < \infty \text{ for all } N \text{ and } T \text{ sufficiently large. } \square$$

Proof of Part (a) of Theorem 5:

To proceed, first write $\widehat{\beta}_{\text{drpls}} - \beta = \widehat{\beta}_{\text{drpls}} - \bar{\beta} + \bar{\beta} - \beta$, where

$$\begin{aligned}\widehat{\beta}_{\text{drpls}} &= \widehat{G}_S \left[\widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S + \frac{1}{\widehat{\lambda}_1} \widehat{D}_{\lambda, \kappa}^2 \right]^{-1} \widehat{G}'_S \widehat{\sigma}_{Xy}^S, \quad \beta = \frac{L_1 \phi}{\sqrt{\Delta_{1,1}}} \text{ (by Assumption 4-4(a))}, \\ \bar{\beta} &= G (G' \bar{\Sigma}_X G)^{-1} G' \bar{\sigma}_{Xy}, \text{ and } \widehat{D}_{\lambda, \kappa} = \text{diag} \left(\widehat{\lambda}_1^{\frac{1}{2}} \quad \widehat{\lambda}_1^{\frac{3}{2}} \quad \dots \quad \widehat{\lambda}_1^{\kappa - \frac{1}{2}} \right) \text{ with } \widehat{\lambda}_1 = \lambda_{\max} \left(\widehat{\Sigma}_X^S \right).\end{aligned}$$

Consider first the term $\widehat{\beta}_{\text{drpls}} - \bar{\beta}$. Let $D_{\lambda, \kappa} = \text{diag} \left(\lambda_1^{\frac{1}{2}} \quad \lambda_1^{\frac{3}{2}} \quad \dots \quad \lambda_1^{\kappa - \frac{1}{2}} \right)$, and note that

$$\begin{aligned}& \widehat{\beta}_{\text{drpls}} - \bar{\beta} \\ &= \widehat{G}_S \left[\widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S + \frac{1}{\widehat{\lambda}_1} \widehat{D}_{\lambda, \kappa}^2 \right]^{-1} \widehat{G}'_S \widehat{\sigma}_{Xy}^S - G (G' \bar{\Sigma}_X G)^{-1} G' \bar{\sigma}_{Xy} \\ &= \widehat{G}_S D_{\lambda, \kappa}^{-1} \left[D_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda, \kappa}^{-1} + \frac{1}{\widehat{\lambda}_1} D_{\lambda, \kappa}^{-1} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-1} \right]^{-1} D_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\sigma}_{Xy}^S \\ &\quad - G D_{\lambda, \kappa}^{-1} (D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1})^{-1} D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{Xy} \\ &= \left[(\widehat{G}_S - G) D_{\lambda, \kappa}^{-1} + G D_{\lambda, \kappa}^{-1} \right] \\ &\quad \times \left\{ \left[D_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda, \kappa}^{-1} + \frac{1}{\widehat{\lambda}_1} D_{\lambda, \kappa}^{-1} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-1} \right]^{-1} - (D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1})^{-1} + (D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1})^{-1} \right\} \\ &\quad \times \left[D_{\lambda, \kappa}^{-1} (\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \bar{\sigma}_{Xy}) + D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{Xy} \right] - G D_{\lambda, \kappa}^{-1} (D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1})^{-1} D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{Xy} \\ &= \frac{1}{\sqrt{\lambda_1}} (\widehat{G}_S - G) D_{\lambda, \kappa}^{-1} \left\{ \left[\frac{D_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda, \kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1} \frac{D_{\lambda, \kappa}^{-1} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-1}}{\lambda_1} \right]^{-1} - \left(\frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \\ &\quad \times D_{\lambda, \kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \\ &\quad + \frac{1}{\sqrt{\lambda_1}} \left[(\widehat{G}_S - G) D_{\lambda, \kappa}^{-1} \left\{ \left[\frac{D_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda, \kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1} \frac{D_{\lambda, \kappa}^{-1} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-1}}{\lambda_1} \right]^{-1} - \left(\frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \right. \\ &\quad \times \left. \frac{D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \\ &\quad + \frac{1}{\sqrt{\lambda_1}} (\widehat{G}_S - G) D_{\lambda, \kappa}^{-1} \left(\frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} D_{\lambda, \kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right]\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\sqrt{\lambda_1}} \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \frac{D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \\
& + \frac{1}{\sqrt{\lambda_1}} G D_{\lambda, \kappa}^{-1} \left(\left\{ \left[\frac{D_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda, \kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda, \kappa}^{-1} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-1}}{\lambda_1} \right]^{-1} - \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \right. \\
& \quad \left. \times D_{\lambda, \kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \right) \\
& + \frac{1}{\sqrt{\lambda_1}} G D_{\lambda, \kappa}^{-1} \left\{ \left[\left(\frac{D_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda, \kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda, \kappa}^{-1} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} - \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right] \right. \\
& \quad \left. \times \frac{D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\} \\
& + \frac{1}{\sqrt{\lambda_1}} G D_{\lambda, \kappa}^{-1} \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} D_{\lambda, \kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \\
& + \frac{1}{\sqrt{\lambda_1}} G D_{\lambda, \kappa}^{-1} \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \frac{D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \\
& - \frac{1}{\sqrt{\lambda_1}} G D_{\lambda, \kappa}^{-1} \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \frac{D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}}
\end{aligned}$$

Since the last two terms cancel each other out, we further obtain

$$\begin{aligned}
& \widehat{\beta}_{\text{drpls}} - \overline{\beta} \\
& = \frac{1}{\sqrt{\lambda_1}} \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \left\{ \left(\frac{D_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda, \kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda, \kappa}^{-1} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} - \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \\
& \quad \times D_{\lambda, \kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \\
& + \frac{1}{\sqrt{\lambda_1}} \left[\left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \left\{ \left(\frac{D_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda, \kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda, \kappa}^{-1} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} - \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\} \right. \\
& \quad \left. \times \frac{D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \\
& + \frac{1}{\sqrt{\lambda_1}} \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} D_{\lambda, \kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\sqrt{\lambda_1}} \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \frac{D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \\
& + \frac{1}{\sqrt{\lambda_1}} G D_{\lambda, \kappa}^{-1} \left(\left[\left(\frac{D_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda, \kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda, \kappa}^{-1} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} - \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right] \right. \\
& \quad \left. \times D_{\lambda, \kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \right) \\
& + \frac{1}{\sqrt{\lambda_1}} G D_{\lambda, \kappa}^{-1} \left\{ \left[\left(\frac{D_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda, \kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda, \kappa}^{-1} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} - \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right] \right. \\
& \quad \left. \times \frac{D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\} \\
& + \frac{1}{\sqrt{\lambda_1}} G D_{\lambda, \kappa}^{-1} \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} D_{\lambda, \kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \\
& = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4 + \mathcal{T}_5 + \mathcal{T}_6 + \mathcal{T}_7, \text{ (say)}.
\end{aligned}$$

Analyzing each term on the right-hand side of the above expression, we see that, by making use of parts (c), (d), and (h) of Lemma B-4 given in Appendix B of the Online Appendix, we obtain

$$\begin{aligned}
\|\mathcal{T}_1\|_2 & \leq \frac{1}{\sqrt{\lambda_1}} \left\{ \left\| \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \right\|_2 \right. \\
& \quad \times \left\| \left[\frac{D_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda, \kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda, \kappa}^{-1} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-1}}{\lambda_1} \right]^{-1} - \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \\
& \quad \times \left\| D_{\lambda, \kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \right\|_2 \Bigg\} \\
& = O\left(\frac{1}{\sqrt{\lambda_1}}\right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& \quad \times O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{v+1}} \right\} \right) \\
& \quad \times O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)
\end{aligned}$$

$$\begin{aligned}
&= O_p \left(\max \left\{ \frac{c_0^3(N)}{\lambda_1^{\frac{7}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{3}{2}(1-q)}, \frac{[a(N)]^{\frac{3}{2}}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{3}{2}(1-\frac{q}{2})}, \frac{c_0^2(N) \sqrt{a(N)}}{\lambda_1^3} \left[\frac{\ln N}{T} \right]^{1-q} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \right. \\
&\quad \frac{c_0(N) a(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})}, \frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{1-q} \frac{1}{\lambda_1^{v+1}}, \\
&\quad \left. \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \frac{1}{\lambda_1^{v+1}}, \frac{a(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})} \frac{1}{\lambda_1^{v+1}} \right\} \Bigg) \\
&= o_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right),
\end{aligned}$$

where the last equality above follows from parts (a)-(f), and (j) of Lemma D-10 of the Online Appendix. Consider next $\mathcal{T}_2$. Applying parts (b), (c), and (h) of Lemma B-4, we get

$$\begin{aligned}
\|\mathcal{T}_2\|_2 &\leq \frac{1}{\sqrt{\lambda_1}} \left\{ \left\| \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \right\|_2 \right. \\
&\quad \times \left\| \left[\frac{D_{\lambda, \kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X \widehat{G}_S D_{\lambda, \kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_v} \frac{D_{\lambda, \kappa}^{-1} \widehat{D}_{\lambda, \kappa}^2 D_{\lambda, \kappa}^{-1}}{\lambda_1} \right]^{-1} - \left(\frac{D_{\lambda, \kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \\
&\quad \times \left\| \frac{D_{\lambda, \kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \Bigg\} \\
&= O \left(\frac{1}{\sqrt{\lambda_1}} \right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
&\quad \times O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{v+1}} \right\} \right) O(1) \\
&= O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{1-q}, \frac{a(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})}, \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \right. \\
&\quad \left. \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \frac{1}{\lambda_1^{v+\frac{3}{2}}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \frac{1}{\lambda_1^{v+1}} \right\} \Bigg) \\
&= o_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right),
\end{aligned}$$

where the last equality above follows from parts (d), (e), (f), (g), and (h) of Lemma D-10, given in Appendix D of the Online Appendix, and from the fact that $1/\lambda_1 = o(1)$. Now,

consider $\mathcal{T}_3$. Applying parts (c) and (d) of Lemma B-4 as well as expression (66), given in the proof of Lemma B-4 part (f) of the Online Appendix, we obtain

$$\begin{aligned}
& \|\mathcal{T}_3\|_2 \\
& \leq \frac{1}{\sqrt{\lambda_1}} \left\| \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \right\|_2 \left\| \left(\frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \left\| D_{\lambda, \kappa}^{-1} \left[\frac{\widehat{G}_S' \widehat{\sigma}_{Xy}^S - G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \right\|_2 \\
& = O\left(\frac{1}{\sqrt{\lambda_1}}\right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& \quad \times O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& = O_p \left(\max \left\{ \frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left(\frac{\ln N}{T} \right)^{(1-q)}, \frac{a(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})}, \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right),
\end{aligned}$$

where the last equality above follows from parts (d), (e), and (f) of Lemma D-10, given in Appendix D of the Online Appendix. For the term $\mathcal{T}_4$, we can apply parts (b) and (c) of Lemma B-4 as well as expression (66) of the Online Appendix² to get

$$\begin{aligned}
\|\mathcal{T}_4\|_2 & \leq \frac{1}{\sqrt{\lambda_1}} \left\| \left(\widehat{G}_S - G \right) D_{\lambda, \kappa}^{-1} \right\|_2 \left\| \left(\frac{D_{\lambda, \kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda, \kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \left\| \frac{D_{\lambda, \kappa}^{-1} G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \\
& = O\left(\frac{1}{\sqrt{\lambda_1}}\right) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) O(1) O(1) \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right).
\end{aligned}$$

In addition, for the term $\mathcal{T}_5$, we can apply parts (a), (d), and (h) of Lemma B-4 of the Online

²See the proof of Lemma B-4 part (f) given in Appendix B of the Online Appendix.

Appendix to obtain

$$\begin{aligned}
& \|\mathcal{T}_5\|_2 \\
& \leq \frac{1}{\sqrt{\lambda_1}} \left\{ \|GD_{\lambda,\kappa}^{-1}\|_2 \left\| \left[\frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \widehat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \right. \\
& \quad \times \left. \left\| D_{\lambda,\kappa}^{-1} \left[\frac{\widehat{G}'_S \widehat{\sigma}_{Xy}^S - G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \right\|_2 \right\} \\
& = O\left(\frac{1}{\sqrt{\lambda_1}}\right) O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{v+1}} \right\} \right) \\
& \quad \times O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
& = O_p \left(\max \left\{ \frac{c_0(N) \sqrt{a(N)}}{\lambda_1^2} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{c_0^2(N)}{\lambda_1^{\frac{5}{2}}} \left[\frac{\ln N}{T} \right]^{(1-q)}, \frac{a(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{(1-\frac{q}{2})}, \right. \\
& \quad \left. \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}} \frac{1}{\lambda_1^{v+\frac{3}{2}}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \frac{1}{\lambda_1^{v+\frac{3}{2}}} \right\} \right) \\
& = o_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right),
\end{aligned}$$

where the last equality above follows from parts (d), (e), (f), (g), and (i) of Lemma D-10 and from the fact that $1/\lambda_1 = o(1)$. Turning our attention now to the term $\mathcal{T}_6$, we can apply parts (a), (b), and (h) of Lemma B-4 in Appendix B of the Online Appendix to obtain

$$\begin{aligned}
& \|\mathcal{T}_6\|_2 \\
& \leq \frac{1}{\sqrt{\lambda_1}} \left\{ \|GD_{\lambda,\kappa}^{-1}\|_2 \left\| \left[\frac{D_{\lambda,\kappa}^{-1} \widehat{G}'_S \widehat{\Sigma}_X^S \widehat{G}_S D_{\lambda,\kappa}^{-1}}{\lambda_1} + \frac{1}{\widehat{\lambda}_1^v} \frac{D_{\lambda,\kappa}^{-1} \widehat{D}_{\lambda,\kappa}^2 D_{\lambda,\kappa}^{-1}}{\lambda_1} \right]^{-1} - \left(\frac{D_{\lambda,\kappa}^{-1} G' \overline{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \right. \\
& \quad \times \left. \left\| \frac{D_{\lambda,\kappa}^{-1} G' \overline{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right\|_2 \right\} \\
& = O\left(\frac{1}{\sqrt{\lambda_1}}\right) O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{v+1}} \right\} \right) O(1)
\end{aligned}$$

$$\begin{aligned}
&= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{v+\frac{3}{2}}} \right\} \right) \\
&= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right),
\end{aligned}$$

where the last equality above follows from the fact that $\lambda_1^{\frac{3}{2}}/\lambda_1^{v+\frac{3}{2}} = o(1)$ for all $v > 0$. Finally, consider the term $\mathcal{T}_7$. For this term, we can apply parts (a) and (d) of Lemma B-4 as well as expression (66) of the Online Appendix to get

$$\begin{aligned}
\|\mathcal{T}_7\|_2 &\leq \frac{1}{\sqrt{\lambda_1}} \|GD_{\lambda,\kappa}^{-1}\|_2 \left\| \left(\frac{D_{\lambda,\kappa}^{-1} G' \bar{\Sigma}_X G D_{\lambda,\kappa}^{-1}}{\lambda_1} \right)^{-1} \right\|_2 \left\| D_{\lambda,\kappa}^{-1} \left[\frac{\hat{G}'_S \hat{\sigma}_{Xy}^S - G' \bar{\sigma}_{Xy}}{\sqrt{\lambda_1}} \right] \right\|_2 \\
&= O \left(\frac{1}{\sqrt{\lambda_1}} \right) O(1) O(1) O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
&= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right)
\end{aligned}$$

Combining the results given by the above calculations, we see that

$$\begin{aligned}
\|\hat{\beta}_{\text{drpls}} - \bar{\beta}\|_2 &\leq \|\mathcal{T}_1\|_2 + \|\mathcal{T}_2\|_2 + \|\mathcal{T}_3\|_2 + \|\mathcal{T}_4\|_2 + \|\mathcal{T}_5\|_2 + \|\mathcal{T}_6\|_2 + \|\mathcal{T}_7\|_2 \\
&= \|\mathcal{T}_6\|_2 + (\|\mathcal{T}_4\|_2 + \|\mathcal{T}_7\|_2) + (\|\mathcal{T}_1\|_2 + \|\mathcal{T}_2\|_2 + \|\mathcal{T}_3\|_2 + \|\mathcal{T}_5\|_2) \\
&= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right) \\
&\quad + O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
&\quad + o_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})} \right\} \right) \\
&= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right) \\
&= o_p(1),
\end{aligned}$$

where the last equality above follows from the fact that $\lambda_1 \rightarrow \infty$ as $N, T \rightarrow \infty$ since $\lambda_1 \asymp \underline{\lambda}_{1,1}$ by Lemma D-3(a) and $\underline{\lambda}_{1,1} \rightarrow \infty$ as $N, T \rightarrow \infty$ by part (a) of Assumption 4-2 and also from the rate conditions given in Assumption 6-3, i.e., $[c_0(N)/\lambda_1] [\ln N/T]^{(1-q)/2} = o(1)$ and

$\left[\sqrt{a(N)/\lambda_1} \right] [\ln N/T]^{\frac{1}{2}(1-\frac{q}{2})} = o(1)$, as $N, T, \lambda_1 \rightarrow \infty$.

Next, note that, for the second term $\bar{\beta} - \beta$, part (b) of Theorem 2 shows that $\|\bar{\beta} - \beta\|_2 = O\left(\lambda_1^{-\frac{3}{2}}\right)$. It follows immediately by the triangle inequality that

$$\begin{aligned} \|\hat{\beta}_{\text{drpls}} - \beta\|_2 &\leq \|\hat{\beta}_{\text{drpls}} - \bar{\beta}\|_2 + \|\bar{\beta} - \beta\|_2 \\ &= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right) + O \left(\frac{1}{\lambda_1^{\frac{3}{2}}} \right) \\ &= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right) = o_p(1). \quad \square \end{aligned}$$

Proof of Part (a) of Theorem 6:

To proceed, note first that, in the case where the point forecast is based on $\hat{\beta}_{\text{drpls}}$, we can write the deviation of the point forecast from the conditional mean function of the forecast equation as follows:

$$\begin{aligned} &\hat{y}_{T+h}^{\text{drpls}} - (\alpha + [X_T - \mu_X]' \beta) \\ &= \bar{y}_{T_h} + [X_T - \bar{X}_{T_h}]' \hat{\beta}_{\text{drpls}} - (\alpha + [X_T - \mu_X]' \beta) \\ &= \alpha + (\bar{X}_{T_h} - \mu_X)' \beta + \bar{\varepsilon}_{T_h} + [X_T - \bar{X}_{T_h}]' \hat{\beta}_{\text{drpls}} - (\alpha + [X_T - \mu_X]' \beta) \\ &= [X_T - \mu_X]' (\hat{\beta}_{\text{drpls}} - \beta) - (\bar{X}_{T_h} - \mu_X)' (\hat{\beta}_{\text{drpls}} - \beta) + \bar{\varepsilon}_{T_h}, \end{aligned} \quad (4)$$

where $\bar{X}_{T_h} = T_h^{-1} \sum_{t=1}^{T_h} X_t$ and $\bar{\varepsilon}_{T_h} = T_h^{-1} \sum_{t=1}^{T_h} \varepsilon_{t+h}$ with $T_h = T - h$.

Consider the first term on the right-hand side of expression (4). To proceed, note first that

$$\begin{aligned} &\left| [X_T - \mu_X]' (\hat{\beta}_{\text{drpls}} - \beta) \right| \\ &\leq \|X_T - \mu_X\|_2 \|\hat{\beta}_{\text{drpls}} - \beta\|_2 \quad (\text{by the Cauchy-Schwarz inequality}) \\ &= \sqrt{[X_T - \mu_X]' [X_T - \mu_X]} \sqrt{(\hat{\beta}_{\text{drpls}} - \beta)' (\hat{\beta}_{\text{drpls}} - \beta)} \\ &= \sqrt{\text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \}} \sqrt{(\hat{\beta}_{\text{drpls}} - \beta)' (\hat{\beta}_{\text{drpls}} - \beta)} \end{aligned}$$

Next, note that, by Assumptions 4-1 and 6-4, we have, for all N and T ,

$$\begin{aligned} &E \left(N^{-1} \text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \} \right) \\ &= \frac{1}{N} \text{tr} \{ E \left([X_T - \mu_X] [X_T - \mu_X]' \right) \} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{N} \text{tr} \{ E ([W_T + u_T] [W_T + u_T]') \} \\
&= \frac{1}{N} \text{tr} \{ E (W_T W_T') + E (W_T u_T') + E (u_T W_T') + E (u_T u_T') \} \\
&= \frac{1}{N} \text{tr} \{ \Sigma_{W,T} + \Sigma_{u,T} \} \quad (\text{since } E (W_T u_T') = 0 \text{ and } E (u_T W_T') = 0) \\
&\leq \frac{\text{tr} \{ \Sigma_{W,T} \}}{N} + \frac{N}{N} \lambda_{\max} (\Sigma_{u,T})
\end{aligned}$$

Hence, given that, by Assumption 6-4, $\text{tr} \{ \Sigma_{W,T} \} / N \leq C^\dagger$ for all N, T and, by Assumption 4-1, $\sup_t \lambda_{\max} (\Sigma_{u,t}) \leq \bar{C}$ for some positive constants $C^\dagger$ and $\bar{C}$, it follows that $E (N^{-1} \text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \}) \leq C$ for some constant $C \geq C^\dagger + \bar{C}$. It then further follows that for any $\epsilon > 0$, we can set $C_\epsilon = C/\epsilon$; and, by Markov's inequality, we have, for all N and T sufficiently large,

$$\begin{aligned}
P \left(\left| \frac{\text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \}}{N} \right| \geq C_\epsilon \right) &= P \left(\left| \frac{\text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \}}{N} \right| \geq \frac{C}{\epsilon} \right) \\
&\leq \frac{\epsilon}{C} E (N^{-1} \text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \}) \\
&\leq \frac{\epsilon}{C} C = \epsilon
\end{aligned}$$

so that $\text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \} = O_p(N)$, from which we deduce that $\sqrt{\text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \}} = O_p(\sqrt{N})$. Applying the result of part (a) of Theorem 5, we then obtain

$$\begin{aligned}
&\left| [X_T - \mu_X]' (\hat{\beta}_{\text{drpls}} - \beta) \right| \\
&\leq \sqrt{\text{tr} \{ [X_T - \mu_X] [X_T - \mu_X]' \}} \left\| \hat{\beta}_{\text{drpls}} - \beta \right\|_2 \\
&= O_p(\sqrt{N}) \left[O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right) \right] \\
&= O_p \left(\max \left\{ \sqrt{\frac{N}{\lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{N}{\lambda_1}} \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{\sqrt{N}}{\lambda_1^{\frac{3}{2}}} \right\} \right)
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&[X_T - \mu_X]' (\hat{\beta}_{\text{drpls}} - \beta) \\
&= O_p \left(\max \left\{ \sqrt{\frac{N}{\lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{N}{\lambda_1}} \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{\sqrt{N}}{\lambda_1^{\frac{3}{2}}} \right\} \right). \quad (5)
\end{aligned}$$

Consider now the second term on the right-hand side of expression (4), i.e., the term $(\bar{X}_{T_h} - \mu_X)' (\hat{\beta}_{\text{drpls}} - \beta)$. To proceed with the analysis of this term, first write

$$\begin{aligned}
& (\bar{X}_{T_h} - \mu_X)' (\hat{\beta}_{\text{drpls}} - \beta) \\
&= \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' \begin{pmatrix} L_1 & L_2 \end{pmatrix} \begin{pmatrix} L'_1 \\ L'_2 \end{pmatrix} (\hat{\beta}_{\text{drpls}} - \beta) \quad (\text{since } L \text{ is an orthogonal matrix}) \\
&= \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_1 L'_1 (\hat{\beta}_{\text{drpls}} - \beta) + \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_2 L'_2 (\hat{\beta}_{\text{drpls}} - \beta). \tag{6}
\end{aligned}$$

To analyze the first term on the right-hand side of expression (6) above, note that

$$\begin{aligned}
& \left| \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_1 L'_1 (\hat{\beta}_{\text{drpls}} - \beta) \right| \\
&\leq \left\| L'_1 \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right) \right\|_2 \left\| L'_1 (\hat{\beta}_{\text{drpls}} - \beta) \right\|_2 \\
&= \sqrt{\left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right)' L_1 L'_1 \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right)} \sqrt{(\hat{\beta}_{\text{drpls}} - \beta)' L_1 L'_1 (\hat{\beta}_{\text{drpls}} - \beta)} \\
&= \sqrt{\text{tr} \left\{ L'_1 \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right)' L_1 \right\}} \sqrt{(\hat{\beta}_{\text{drpls}} - \beta)' L_1 L'_1 (\hat{\beta}_{\text{drpls}} - \beta)}
\end{aligned}$$

Let $\lambda_1 = \lambda_{\max}(\bar{\Sigma}_X)$, and note that

$$\begin{aligned}
& \frac{1}{\lambda_1} \text{tr} \left\{ L'_1 \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X] \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [X_s - \mu_X] \right)' L_1 \right\} \\
&= \frac{1}{\lambda_1} \text{tr} \left\{ L'_1 \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [W_t + u_t] \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [W'_s + u'_s] L_1 \right) \right\} \\
&= \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_1 \right) \right\} + \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_1 \right) \right\} + \\
& \quad \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_1 \right) \right\} + \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_1 \right) \right\} \tag{7}
\end{aligned}$$

Now, let $e_{i,\kappa}$ denote a $\kappa \times 1$ elementary vector whose i^{th} component equals 1 and all other

components are 0, and we can apply the inequality $|xy| \leq (x^2/2) + (y^2/2)$ to obtain

$$\begin{aligned}
0 &\leq \frac{1}{\lambda_1} \left| \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_1 \right) \right\} \right| \leq \frac{1}{\lambda_1} \sum_{i=1}^{\kappa} \left| e'_{i,\kappa} \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_1 \right) e_{i,\kappa} \right| \\
&\leq \frac{1}{2\lambda_1} \sum_{i=1}^{\kappa} \left\{ e'_{i,\kappa} \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_1 \right) e_{i,\kappa} + e'_{i,\kappa} \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_1 \right) e_{i,\kappa} \right\} \\
&= \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_1 \right) \right\} \\
&\quad + \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_1 \right) \right\}, \tag{8}
\end{aligned}$$

Similarly, we can show that

$$\begin{aligned}
0 &\leq \frac{1}{\lambda_1} \left| \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_1 \right) \right\} \right| \\
&\leq \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_1 \right) \right\} \\
&\quad + \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_1 \right) \right\} \tag{9}
\end{aligned}$$

Making use of expressions (8) and (9), we can then bound expression (7) as follows

$$\begin{aligned}
0 &\leq \frac{1}{\lambda_1} \text{tr} \left\{ L'_1 \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X] \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [X_s - \mu_X] \right)' L_1 \right\} \\
&\leq \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_1 \right) \right\} + \left| \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_1 \right) \right\} \right| \\
&\quad + \left| \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_1 \right) \right\} \right| + \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_1 \right) \right\} \\
&\leq \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_1 \right) \right\} + \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_1 \right) \right\} \\
&\quad + \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_1 \right) \right\} + \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_1 \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_1 \right) \right\} + \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_1 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_1 \right) \right\} \\
& = \frac{2}{\lambda_1} \text{tr} \left\{ \frac{1}{T_h^2} \left(\sum_{t=1}^{T_h} L'_1 W_t \right) \left(\sum_{s=1}^{T_h} W'_s L_1 \right) \right\} + \frac{2}{\lambda_1} \text{tr} \left\{ \frac{1}{T_h^2} \left(\sum_{t=1}^{T_h} L'_1 u_t \right) \left(\sum_{s=1}^{T_h} u'_s L_1 \right) \right\} \quad (10)
\end{aligned}$$

Consider the first term on the right-hand side of expression (10). Let $l_{1,g}$ denote the g^{th} column of L_1 ; and, taking expectation, we have

$$\begin{aligned}
0 & \leq \frac{2}{\lambda_1} E \left[\text{tr} \left\{ \frac{1}{T_h^2} \left(\sum_{t=1}^{T_h} L'_1 W_t \right) \left(\sum_{s=1}^{T_h} W'_s L_1 \right) \right\} \right] \\
& = \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_1 E [W_t W'_s] L_1 \} \\
& = \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \sum_{g=1}^{\kappa} l'_{1,g} E [W_t W'_s] l_{1,g} \\
& \leq \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} |l'_{1,g} E [W_t W'_s] l_{1,g}| \\
& \leq \frac{2}{\lambda_1} \sum_{g=1}^{\kappa} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \sqrt{E \{ (l'_{1,g} W_t)^2 \}} \sqrt{E \{ (l'_{1,g} W_s)^2 \}} \quad (\text{by CS inequality}) \\
& = \frac{2}{\lambda_1} \sum_{g=1}^{\kappa} \left(\frac{1}{T_h} \sum_{t=1}^{T_h} \sqrt{E \{ (l'_{1,g} W_t W'_t l_{1,g}) \}} \right)^2 \\
& \leq \frac{2}{\lambda_1} \sum_{g=1}^{\kappa} \left(\sqrt{\frac{1}{T_h} \sum_{t=1}^{T_h} E \{ (l'_{1,g} W_t W'_t l_{1,g}) \}} \right)^2 \quad (\text{by Jensen's inequality}) \\
& = \frac{2}{\lambda_1} \sum_{g=1}^{\kappa} l'_{1,g} \left(\frac{1}{T_h} \sum_{t=1}^{T_h} E [W_t W'_t] \right) l_{1,g} = \frac{2}{\lambda_1} \sum_{g=1}^{\kappa} l'_{1,g} \bar{\Sigma}_W l_{1,g} \quad \left(\text{letting } \bar{\Sigma}_W = \frac{1}{T_h} \sum_{t=1}^{T_h} E [W_t W'_t] \right) \\
& = \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \frac{2}{\lambda_{1,1}} \sum_{g=1}^{\kappa} \lambda_{1,g} \quad (\text{since } l'_{1,g} \bar{\Sigma}_W l_{1,g} = l'_{1,g} L_1 \underline{\Delta}_1 L'_1 l_{1,g} = e'_{g,\kappa} \underline{\Delta}_1 e_{g,\kappa} = \lambda_{1,g}) \\
& \leq 2 \left(\frac{\lambda_{1,1}}{\lambda_1} \right) \kappa \quad \text{for all } N, T \text{ sufficiently large (since part (a) of Assumption 4-2 implies} \\
& \quad \text{that } \frac{\lambda_{1,g}}{\lambda_{1,1}} \leq \frac{\lambda_{1,1}}{\lambda_{1,1}} = 1 \text{ for all } g \in \{1, \dots, \kappa\} \text{ and for all } N, T \text{ sufficiently large)} \\
& \leq 2c^* \kappa \quad \left(\text{as Lemma D-3(a) implies that } 0 < \frac{1}{c^*} < \frac{\lambda_{1,1}}{\lambda_1} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large} \right)
\end{aligned}$$

$\leq C < \infty$ for some constant $C \geq 2c^*\kappa$.

It follows that for any $\epsilon > 0$, we can set $C_\epsilon = C/\epsilon$; and, by Markov's inequality, we have

$$\begin{aligned}
& P \left(\left| \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_1' W_t W_s' L_1 \} \right| \geq C_\epsilon \right) \\
&= P \left(\left| \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_1' W_t W_s' L_1 \} \right| \geq \frac{C}{\epsilon} \right) \\
&\leq \frac{\epsilon}{C} E \left[\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_1' W_t W_s' L_1 \} \right] \\
&\leq \frac{\epsilon}{C} C = \epsilon,
\end{aligned}$$

so that

$$\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_1' W_t W_s' L_1 \} = O_p(1) \quad (11)$$

Next, consider the second term on the right-hand side of expression (10). For this term, we can take expectation and get

$$\begin{aligned}
0 &\leq \frac{2}{\lambda_1} E \left[\text{tr} \left\{ \frac{1}{T_h^2} \left(\sum_{t=1}^{T_h} L_1' u_t \right) \left(\sum_{s=1}^{T_h} u_s' L_1 \right) \right\} \right] \\
&= \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_1' E [u_t u_s'] L_1 \} \\
&= \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \sum_{g=1}^{\kappa} l'_{1,g} E [u_t u_s'] l_{1,g} \\
&= \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h} l'_{1,g} E [u_t u_t'] l_{1,g} + \frac{4}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} E \{ (l'_{1,g} u_t) (l'_{1,g} u_{t+m}) \} \\
&\leq \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h} l'_{1,g} E [u_t u_t'] l_{1,g} + \frac{4}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} |E \{ (l'_{1,g} u_t) (l'_{1,g} u_{t+m}) \}| \\
&\leq \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h} E |l'_{1,g} u_t|^2 \\
&\quad + \frac{4}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) [\alpha(m)]^{1-\frac{1}{2}-\frac{1}{4}} \sqrt{E |l'_{1,g} u_t|^2} \left(E |l'_{1,g} u_{t+m}|^4 \right)^{\frac{1}{4}}
\end{aligned}$$

(by Corollary 14.3 of Davidson (1994))

$$\begin{aligned}
&\leq \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h} E |l'_{1,g} u_t|^2 + \frac{4}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{g=1}^{\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) [\alpha(m)]^{\frac{1}{4}} \sqrt{E |l'_{1,g} u_t|^2} (C^\bullet)^{\frac{1}{4}} \\
&\quad \text{(for some constant } C^\bullet > 0 \text{ by applying Lemma D-6 part (b) of the Online Appendix,} \\
&\quad \text{setting } \delta = l_{1,g} \text{ and noting that } \|l_{1,g}\|_2 = \sqrt{l'_{1,g} l_{1,g}} = 1) \\
&\leq \frac{2}{\lambda_1} \frac{\kappa T_h}{T_h^2} (C^\bullet)^{\frac{1}{2}} + \frac{4}{\lambda_1} \frac{\kappa (T_h - 1)}{T_h^2} 2 \left(2^{\frac{1}{2}} + 1 \right) (C^\bullet)^{\frac{1}{2}} \sum_{m=1}^T [\alpha(m)]^{\frac{1}{4}} \\
&\quad \left(\text{applying Liapunov's inequality to show that } \sqrt{E |l'_{1,g} u_t|^2} \leq \left(E |l'_{1,g} u_t|^4 \right)^{\frac{1}{4}} \leq (C^\bullet)^{\frac{1}{4}} \right) \\
&\leq \frac{2 (C^\bullet)^{\frac{1}{2}}}{\lambda_1} \frac{\kappa}{T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \right] \\
&\quad \text{(by Assumption 4-3 part (b))} \\
&\leq \frac{2 (C^\bullet)^{\frac{1}{2}}}{\lambda_1} \frac{\kappa}{T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \left(\text{since } \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \leq C^\# < \infty \text{ for some} \right. \\
&\quad \left. \text{constant } C^\# > 0 \text{ as shown in Lemma D-11 of Appendix D given in the Online Appendix} \right) \\
&\leq \frac{C^\dagger}{\lambda_1 T_h} \left(\text{for some constant } C^\dagger \geq 2 (C^\bullet)^{\frac{1}{2}} \kappa \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \right)
\end{aligned}$$

It follows that for any $\epsilon > 0$, we can set $C_\epsilon^\dagger = C^\dagger/\epsilon$; and, by Markov's inequality, we have

$$\begin{aligned}
&P \left(\left| \lambda_1 T_h \left[\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_1 u_t u'_s L_1 \} \right] \right| \geq C_\epsilon^\dagger \right) \\
&= P \left(\left| \lambda_1 T_h \left[\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_1 u_t u'_s L_1 \} \right] \right| \geq \frac{C^\dagger}{\epsilon} \right) \\
&\leq \frac{\epsilon}{C^\dagger} \lambda_1 T_h E \left(\left[\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_1 u_t u'_s L_1 \} \right] \right) \\
&\leq \frac{\epsilon}{C^\dagger} \lambda_1 T_h \frac{C^\dagger}{\lambda_1 T_h} = \epsilon
\end{aligned}$$

so that

$$\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_1 u_t u'_s L_1 \} = O_p \left(\frac{1}{\lambda_1 T} \right) \quad (12)$$

Combining the results given in expressions (11) and (12), we see that

$$\begin{aligned}
0 &\leq \frac{1}{\lambda_1} \text{tr} \left\{ L_1' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X] \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [X_s - \mu_X] \right)' L_1 \right\} \\
&\leq \frac{2}{\lambda_1} \left[\text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_1' W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W_s' L_1 \right) \right\} + \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_1' u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_1 \right) \right\} \right] \\
&= \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_1' W_t W_s' L_1 \} + \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_1' u_t u_s' L_1 \} \\
&= O_p(1) + O_p \left(\frac{1}{\lambda_1 T} \right) = O_p(1),
\end{aligned}$$

from which it follows that $\sqrt{\text{tr} \left\{ L_1' \left(T_h^{-1} \sum_{t=1}^{T_h} [X_t - \mu_X] \right) \left(T_h^{-1} \sum_{t=1}^{T_h} [X_s - \mu_X] \right)' L_1 \right\}} = O_p(\sqrt{\lambda_1})$. Applying the result of part (a) of Theorem 5 above, we then obtain

$$\begin{aligned}
&\left| \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_1 L_1' (\hat{\beta}_{\text{drpls}} - \beta) \right| \\
&\leq \sqrt{\text{tr} \left\{ L_1' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right)' \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right) L_1 \right\}} \sqrt{(\hat{\beta}_{\text{drpls}} - \beta)' L_1 L_1' (\hat{\beta}_{\text{drpls}} - \beta)} \\
&\leq \sqrt{\text{tr} \left\{ L_1' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right)' \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right) L_1 \right\}} \|\hat{\beta}_{\text{drpls}} - \beta\|_2 \\
&= O_p(\sqrt{\lambda_1}) \left[O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right) \right] \\
&= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1} \right\} \right). \tag{13}
\end{aligned}$$

Next, consider the term $\left[T_h^{-1} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_2 L_2' (\hat{\beta}_{\text{drpls}} - \beta)$. By a similar argument, we have, by the CS inequality,

$$\left| \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_2 L_2' (\hat{\beta}_{\text{drpls}} - \beta) \right| \leq \left\| L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right) \right\|_2 \left\| L_2' (\hat{\beta}_{\text{drpls}} - \beta) \right\|_2$$

$$\begin{aligned}
&= \sqrt{\left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X\right)' L_2 L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X\right)} \sqrt{\left(\widehat{\beta}_{\text{drpls}} - \beta\right)' L_2 L_2' \left(\widehat{\beta}_{\text{drpls}} - \beta\right)} \\
&= \sqrt{\text{tr} \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X\right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X\right)' L_2 \right\}} \sqrt{\left(\widehat{\beta}_{\text{drpls}} - \beta\right)' L_2 L_2' \left(\widehat{\beta}_{\text{drpls}} - \beta\right)}
\end{aligned}$$

Next, note that

$$\begin{aligned}
&\frac{1}{\lambda_1} \text{tr} \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X]\right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [X_s - \mu_X]\right)' L_2 \right\} \\
&= \frac{1}{\lambda_1} \text{tr} \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [W_t + u_t]\right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [W_s' + u_s'] L_2\right) \right\} \\
&= \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' W_t\right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W_s' L_2\right) \right\} + \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' W_t\right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2\right) \right\} \\
&\quad + \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' u_t\right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W_s' L_2\right) \right\} \\
&\quad + \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' u_t\right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2\right) \right\} \tag{14}
\end{aligned}$$

Now, let $e_{i,N-\kappa}$ denote a $(N - \kappa) \times 1$ elementary vector whose i^{th} component equals 1 and all other components are 0, and we can write

$$\begin{aligned}
0 &\leq \frac{1}{\lambda_1} \left| \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' W_t\right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2\right) \right\} \right| \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^{N-\kappa} \left| e_{i,N-\kappa}' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' W_t\right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2\right) e_{i,N-\kappa} \right| \\
&\leq \frac{1}{\lambda_1} \sum_{i=1}^{N-\kappa} \left\{ \frac{1}{2} e_{i,N-\kappa}' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' W_t\right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W_s' L_2\right) e_{i,N-\kappa} \right. \\
&\quad \left. + \frac{1}{2} e_{i,N-\kappa}' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' u_t\right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u_s' L_2\right) e_{i,N-\kappa} \right\} \quad \left(\text{since } |xy| \leq \frac{x^2}{2} + \frac{y^2}{2} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_2 \right) \right\} \\
&\quad + \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right) \right\}
\end{aligned} \tag{15}$$

Similarly, we can show that

$$\begin{aligned}
0 &\leq \frac{1}{\lambda_1} \left| \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_2 \right) \right\} \right| \\
&\leq \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_2 \right) \right\} \\
&\quad + \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right) \right\}
\end{aligned} \tag{16}$$

Making use of expressions (15) and (16), we can then bound expression (14) as follows

$$\begin{aligned}
0 &\leq \frac{1}{\lambda_1} \text{tr} \left\{ L'_2 \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X] \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [X_s - \mu_X] \right)' L_2 \right\} \\
&\leq \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_2 \right) \right\} + \left| \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right) \right\} \right| \\
&\quad + \left| \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_2 \right) \right\} \right| + \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right) \right\} \\
&\leq \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_2 \right) \right\} + \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_2 \right) \right\} \\
&\quad + \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right) \right\} + \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_2 \right) \right\} \\
&\quad + \frac{1}{2\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right) \right\} + \frac{1}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right) \right\} \\
&= \frac{2}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W'_s L_2 \right) \right\} + \frac{2}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right) \right\} \tag{17}
\end{aligned}$$

Consider the first term on the right-hand side of expression (17). Let $l_{2,j}$ be the j^{th}

column of L_2 , and, taking expectation, we have

$$\begin{aligned}
0 &\leq \frac{2}{\lambda_1} \text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L_2' W_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} W_s' L_2 \right) \right\} \\
&= \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' E [W_t W_s'] L_2 \} = \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \sum_{j=1}^{N-\kappa} l_{2,j}' E [W_t W_s'] l_{2,j} \\
&\leq \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{j=1}^{N-\kappa} \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} |l_{2,j}' E [W_t W_s'] l_{2,j}| \\
&\leq \frac{2}{\lambda_1} \sum_{j=1}^{N-\kappa} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \sqrt{E \{ (l_{2,j}' W_t)^2 \}} \sqrt{E \{ (l_{2,j}' W_s)^2 \}} \quad (\text{by CS inequality}) \\
&= \frac{2}{\lambda_1} \sum_{j=1}^{N-\kappa} \left(\frac{1}{T_h} \sum_{t=1}^{T_h} \sqrt{E \{ (l_{2,j}' W_t W_t' l_{2,j}) \}} \right)^2 \\
&\leq \frac{2}{\lambda_1} \sum_{j=1}^{N-\kappa} \left(\sqrt{\frac{1}{T_h} \sum_{t=1}^{T_h} E \{ (l_{2,j}' W_t W_t' l_{2,j}) \}} \right)^2 \quad (\text{by Jensen's inequality}) \\
&= \frac{2}{\lambda_1} \sum_{j=1}^{N-\kappa} l_{2,j}' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} E [W_t W_t'] \right) l_{2,j} = \frac{2}{\lambda_1} \sum_{j=1}^{N-\kappa} l_{2,j}' \bar{\Sigma}_W l_{2,j} \quad \left(\text{given that } \bar{\Sigma}_W = \frac{1}{T_h} \sum_{t=1}^{T_h} E [W_t W_t'] \right) \\
&= \frac{2}{\lambda_1} \sum_{j=1}^{N-\kappa} \underline{\lambda}_{2,j} \quad (\text{since } l_{2,j}' \bar{\Sigma}_W l_{2,j} = l_{2,j}' L_2 \underline{\Delta}_2 L_2' l_{2,j} = e_{j,N-\kappa}' \underline{\Delta}_2 e_{j,N-\kappa} = \underline{\lambda}_{2,j}) \\
&= \frac{2}{\lambda_1} \left[\sum_{h=1}^{\varrho} \underline{\lambda}_{2,h} + \sum_{j=\varrho+1}^{N-\kappa} \frac{c_j}{m(N)} \right] \quad \left(\text{by Assumption 4-2(b), } \underline{\lambda}_{2,j} = \frac{c_j}{m(N)} \text{ for } j = \varrho + 1, \dots, N - \kappa \right) \\
&= 2 \left(\frac{\underline{\lambda}_{1,1}}{\lambda_1} \right) \left[\sum_{h=1}^{\varrho} \frac{\underline{\lambda}_{2,h}}{\underline{\lambda}_{1,1}} + \frac{N}{\underline{\lambda}_{1,1} m(N)} \left(\frac{1}{N} \sum_{j=\varrho+1}^{N-\kappa} c_j \right) \right] \\
&\leq 2c^* [C_2 + C_1] \quad (\text{applying Assumption 4-2(b) and also applying Lemma D-3 part(a) which} \\
&\quad \text{implies that } 0 < \frac{1}{c^*} < \frac{\underline{\lambda}_{1,1}}{\lambda_1} \leq c^* < \infty \text{ for all } N, T \text{ sufficiently large for some constant } c^* > 1) \\
&\leq C < \infty \quad (\text{for some constant } C \geq 2c^* [C_2 + C_1])
\end{aligned}$$

It follows that for any $\epsilon > 0$, we can set $C_\epsilon = C/\epsilon$; and, by Markov's inequality, we have

$$P \left(\left| \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L_2' W_t W_s' L_2 \} \right| \geq C_\epsilon \right)$$

$$\begin{aligned}
&= P \left(\left| \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_2 W_t W'_s L_2 \} \right| \geq \frac{C}{\epsilon} \right) \\
&\leq \frac{\epsilon}{C} E \left[\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_2 W_t W'_s L_2 \} \right] \\
&\leq \frac{\epsilon}{C} C = \epsilon
\end{aligned}$$

so that

$$\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_2 W_t W'_s L_2 \} = O_p(1) \quad (18)$$

Next, consider the second term on the right-hand side of expression (17). For this term, we can take expectation and get

$$\begin{aligned}
0 &\leq \frac{2}{\lambda_1} E \left[\text{tr} \left\{ \left(\frac{1}{T_h} \sum_{t=1}^{T_h} L'_2 u_t \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} u'_s L_2 \right) \right\} \right] \\
&= \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_2 E [u_t u'_s] L_2 \} \\
&= \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \sum_{k=1}^{N-\kappa} l'_{2,k} E [u_t u'_s] l_{2,k} \\
&= \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h} l'_{2,k} E [u_t u'_t] l_{2,k} + \frac{4}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} E \{ (l'_{2,k} u_t) (l'_{2,k} u_{t+m}) \} \\
&\leq \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h} l'_{2,k} E [u_t u'_t] l_{2,k} + \frac{4}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} |E \{ (l'_{2,k} u_t) (l'_{2,k} u_{t+m}) \}| \\
&\leq \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h} E |l'_{2,k} u_t|^2 \\
&\quad + \frac{4}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) [\alpha(m)]^{1-\frac{1}{2}-\frac{1}{4}} \sqrt{E |l'_{2,k} u_t|^2} \left(E |l'_{2,k} u_{t+m}|^4 \right)^{\frac{1}{4}} \\
&\quad \text{(by Corollary 14.3 of Davidson (1994))} \\
&\leq \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h} E |l'_{2,k} u_t|^2 + \frac{4}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{k=1}^{N-\kappa} \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) [\alpha(m)]^{\frac{1}{4}} \sqrt{E |l'_{2,k} u_t|^2} (C^\bullet)^{\frac{1}{4}} \\
&\quad \left(\text{applying Lemma D-6 part (b) by setting } \delta = l_{2,k}, \text{ noting that } \|l_{2,k}\|_2 = \sqrt{l'_{2,k} l_{2,k}} = 1 \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2}{\lambda_1} \frac{(N - \kappa) T_h}{T_h^2} (C^\bullet)^{\frac{1}{2}} + \frac{4}{\lambda_1} \frac{(N - \kappa) (T_h - 1)}{T_h^2} 2 \left(2^{\frac{1}{2}} + 1 \right) (C^\bullet)^{\frac{1}{2}} \sum_{m=1}^T [\alpha(m)]^{\frac{1}{4}} \\
&\quad \left(\text{applying Liapunov's inequality to show that } \sqrt{E |l'_{2,k} u_t|^2} \leq \left(E |l'_{2,k} u_t|^4 \right)^{\frac{1}{4}} \leq (C^\bullet)^{\frac{1}{4}} \right) \\
&\leq \frac{2 (C^\bullet)^{\frac{1}{2}}}{\lambda_1} \frac{N - \kappa}{T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \right] \text{ by Assumption 4-3 part (b)} \\
&\leq \frac{2 (C^\bullet)^{\frac{1}{2}}}{\lambda_1} \frac{N - \kappa}{T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \left(\text{since } \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \leq C^\# < \infty \text{ for some} \right. \\
&\quad \left. \text{constant } C^\# > 0 \text{ as shown in Lemma D-11 in Appendix D of the Online Appendix} \right) \\
&< \frac{C^\dagger N}{\lambda_1 T_h} \left(\text{for some constant } C^\dagger \geq 2 (C^\bullet)^{\frac{1}{2}} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \text{ and given that } \kappa \geq 1 \right)
\end{aligned}$$

It follows that for any $\epsilon > 0$, we can set $C_\epsilon^\dagger = C^\dagger/\epsilon$; and, by Markov's inequality, we have

$$\begin{aligned}
&P \left(\left| \frac{\lambda_1 T_h}{N} \left[\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_2 u_t u'_s L_2 \} \right] \right| \geq C_\epsilon^\dagger \right) \\
&= P \left(\left| \frac{\lambda_1 T_h}{N} \left[\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_2 u_t u'_s L_2 \} \right] \right| \geq \frac{C^\dagger}{\epsilon} \right) \\
&\leq \frac{\epsilon}{C^\dagger} \frac{\lambda_1 T_h}{N} E \left(\left[\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_2 u_t u'_s L_2 \} \right] \right) \\
&\leq \frac{\epsilon}{C^\dagger} \frac{\lambda_1 T_h}{N} \frac{C^\dagger N}{\lambda_1 T_h} = \epsilon
\end{aligned}$$

so that

$$\frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_2 u_t u'_s L_2 \} = O_p \left(\frac{N}{\lambda_1 T} \right) \quad (19)$$

Combining the results given in expressions (18) and (19), we see that

$$\begin{aligned}
&\frac{1}{\lambda_1} \text{tr} \left\{ L'_2 \left(\frac{1}{T_h} \sum_{t=1}^{T_h} [X_t - \mu_X] \right) \left(\frac{1}{T_h} \sum_{s=1}^{T_h} [X_s - \mu_X] \right)' L_2 \right\} \\
&\leq \frac{2}{\lambda_1} \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_2 W_t W'_s L_2 \} + \frac{1}{\lambda_1} \left(\frac{2}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} \text{tr} \{ L'_2 u_t u'_s L_2 \} \\
&= O_p(1) + O_p \left(\frac{N}{\lambda_1 T} \right)
\end{aligned}$$

from which it follows that $\sqrt{\text{tr} \left\{ L_2' \left(T_h^{-1} \sum_{t=1}^{T_h} [X_t - \mu_X] \right) \left(T_h^{-1} \sum_{t=1}^{T_h} [X_s - \mu_X] \right)' L_2 \right\}} = O_p \left(\max \left\{ \sqrt{\lambda_1}, \sqrt{\frac{N}{T}} \right\} \right)$. Applying the result of part (a) of Theorem 5 above, we then obtain

$$\begin{aligned}
& \left| \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_2 L_2' \left(\hat{\beta}_{\text{drpls}} - \beta \right) \right| \\
& \leq \sqrt{\text{tr} \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right)' \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right) L_2 \right\}} \sqrt{\left(\hat{\beta}_{\text{drpls}} - \beta \right)' L_2 L_2' \left(\hat{\beta}_{\text{drpls}} - \beta \right)} \\
& \leq \sqrt{\text{tr} \left\{ L_2' \left(\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right)' \left(\frac{1}{T_h} \sum_{s=1}^{T_h} X_s - \mu_X \right) L_2 \right\}} \left\| \hat{\beta}_{\text{drpls}} - \beta \right\|_2 \\
& = O_p \left(\max \left\{ \sqrt{\lambda_1}, \sqrt{\frac{N}{T}} \right\} \right) \\
& \quad \times O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \right\} \right) \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1} \right. \right. \\
& \quad \left. \left. \sqrt{\frac{N}{T}} \frac{c_0(N)}{\lambda_1^{\frac{3}{2}}} \left[\frac{\ln N}{T} \right]^{\frac{1-q}{2}}, \sqrt{\frac{N}{T}} \frac{\sqrt{a(N)}}{\lambda_1} \left[\frac{\ln N}{T} \right]^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1^{\frac{3}{2}}} \sqrt{\frac{N}{T}} \right\} \right) \\
& = O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \sqrt{\frac{N}{T\lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \right. \right. \\
& \quad \left. \left. \sqrt{\frac{N}{T\lambda_1}} \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1}, \frac{1}{\lambda_1^{\frac{3}{2}}} \sqrt{\frac{N}{T}} \right\} \right). \tag{20}
\end{aligned}$$

Making use of expressions (6), (13), and (20), we see that

$$\begin{aligned}
& (\bar{X}_{T_h} - \mu_X)' \left(\hat{\beta}_{\text{drpls}} - \beta \right) \\
& = \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_1 L_1' \left(\hat{\beta}_{\text{drpls}} - \beta \right) + \left[\frac{1}{T_h} \sum_{t=1}^{T_h} X_t - \mu_X \right]' L_2 L_2' \left(\hat{\beta}_{\text{drpls}} - \beta \right)
\end{aligned}$$

$$\begin{aligned}
&= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1} \right\} \right. \\
&\quad \left. + O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \sqrt{\frac{N}{T\lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \right. \right. \\
&\quad \left. \left. \sqrt{\frac{N}{T\lambda_1}} \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1}, \frac{1}{\lambda_1^{\frac{3}{2}}} \sqrt{\frac{N}{T}} \right\} \right) \right) \\
&= O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \sqrt{\frac{N}{T\lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \right. \right. \\
&\quad \left. \left. \sqrt{\frac{N}{T\lambda_1}} \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1}, \frac{1}{\lambda_1^{\frac{3}{2}}} \sqrt{\frac{N}{T}} \right\} \right) \tag{21}
\end{aligned}$$

Finally, consider the last term on the right-hand side of expression (4), i.e., the term $\bar{\varepsilon}_{T_h} = T_h^{-1} \sum_{t=1}^{T_h} \varepsilon_{t+h}$. For this term, it is easy to see that, under Assumption 4-3 part (b); we have

$$\begin{aligned}
0 &\leq E[\bar{\varepsilon}_{T_h}^2] = \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h} \sum_{s=1}^{T_h} E[\varepsilon_{t+h} \varepsilon_{s+h}] \\
&= \frac{1}{T_h^2} \sum_{t=1}^{T_h} E[\varepsilon_{t+h}^2] + 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} E\{\varepsilon_{t+h} \varepsilon_{t+h+m}\} \\
&\leq \frac{1}{T_h^2} \sum_{t=1}^{T_h} E[\varepsilon_{t+h}^2] + 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} |E[\varepsilon_{t+h} \varepsilon_{t+h+m}]| \\
&\leq \frac{1}{T_h^2} \sum_{t=1}^{T_h} E[\varepsilon_{t+h}^2] + 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) [\alpha(m)]^{1-\frac{1}{2}-\frac{1}{4}} \sqrt{E|\varepsilon_{t+h}|^2} (E|\varepsilon_{t+h+m}|^4)^{\frac{1}{4}} \\
&\quad \text{(by Corollary 14.3 of Davidson (1994))} \\
&\leq \frac{1}{T_h^2} \sum_{t=1}^{T_h} E|\varepsilon_{t+h}|^2 + 2 \left(\frac{1}{T_h} \right)^2 \sum_{t=1}^{T_h-1} \sum_{m=1}^{T_h-t} 2 \left(2^{\frac{1}{2}} + 1 \right) [\alpha(m)]^{\frac{1}{4}} \sqrt{E|\varepsilon_{t+h}|^2} (C^\bullet)^{\frac{1}{4}} \\
&\quad \text{(applying the first result given in part (b) of Lemma D-6 of the Online Appendix)} \\
&\leq \frac{T_h}{T_h^2} (C^\bullet)^{\frac{1}{2}} + 2 \frac{(T_h-1)}{T_h^2} 2 \left(2^{\frac{1}{2}} + 1 \right) (C^\bullet)^{\frac{1}{2}} \sum_{m=1}^T [\alpha(m)]^{\frac{1}{4}} \\
&\quad \left(\text{applying Liapunov's inequality to show that } \sqrt{E|\varepsilon_{t+h}|^2} \leq (E|\varepsilon_{t+h}|^4)^{\frac{1}{4}} \leq (C^\bullet)^{\frac{1}{4}} \right)
\end{aligned}$$

$$\begin{aligned}
&\leq (C^\bullet)^{\frac{1}{2}} \frac{1}{T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \right] \quad (\text{by Assumption 4-3 part (b)}) \\
&\leq (C^\bullet)^{\frac{1}{2}} \frac{1}{T_h} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \quad \left(\text{since } \sum_{m=1}^T \exp \left(-\frac{cm^{\gamma_2}}{4} \right) \leq C^\# < \infty \text{ for some} \right. \\
&\quad \left. \text{constant } C^\# > 0 \text{ as shown in Lemma D-11 in Appendix D of the Online Appendix} \right) \\
&\leq \frac{C^\dagger}{T_h} \quad \left(\text{for some constant } C^\dagger \geq (C^\bullet)^{\frac{1}{2}} \left[1 + 4 \left(2^{\frac{1}{2}} + 1 \right) a^{\frac{1}{4}} C^\# \right] \right)
\end{aligned}$$

It follows that for any $\epsilon > 0$, we can set $C_\epsilon^\# = \sqrt{C^\dagger/\epsilon}$, we have by Markov's inequality

$$P \left(\left| \sqrt{T_h} \bar{\varepsilon}_{T_h} \right| \geq C_\epsilon^\# \right) = P \left(|T_h \bar{\varepsilon}_{T_h}^2| \geq (C_\epsilon^\#)^2 \right) \leq \frac{\epsilon}{C_\epsilon^\dagger} T_h E \left[\bar{\varepsilon}_{T_h}^2 \right] \leq \frac{\epsilon}{C_\epsilon^\dagger} T_h \frac{C^\dagger}{T_h} = \epsilon$$

so that, as $T \rightarrow \infty$, given that h is a (fixed) finite integer, we have

$$\bar{\varepsilon}_{T_h} = \frac{1}{T_h} \sum_{t=1}^{T_h} \varepsilon_{t+h} = O_p \left(\frac{1}{\sqrt{T}} \right). \quad (22)$$

Putting the results given in expressions (5), (21), and (22) together, we see that, since $\lambda_1 \lesssim N$,

$$\begin{aligned}
&\widehat{y}_{T+h}^{\text{drpls}} - [\alpha + (X_T - \mu_X)' \beta] \\
&= [X_T - \mu_X]' \left(\widehat{\beta}_{\text{drpls}} - \beta \right) - (\bar{X}_{T_h} - \mu_X)' \left(\widehat{\beta}_{\text{drpls}} - \beta \right) + \bar{\varepsilon}_{T_h} \\
&= O_p \left(\max \left\{ \sqrt{\frac{N}{\lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{N}{\lambda_1}} \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{\sqrt{N}}{\lambda_1^{\frac{3}{2}}} \right\} \right) \\
&\quad + O_p \left(\max \left\{ \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \sqrt{\frac{N}{T\lambda_1}} \frac{c_0(N)}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \right. \right. \\
&\quad \left. \left. \sqrt{\frac{N}{T\lambda_1}} \sqrt{\frac{a(N)}{\lambda_1}} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{1}{\lambda_1}, \frac{1}{\lambda_1^{\frac{3}{2}}} \sqrt{\frac{N}{T}} \right\} \right) + O_p \left(\frac{1}{\sqrt{T}} \right) \\
&= O_p \left(\max \left\{ \frac{c_0(N) \sqrt{N}}{\lambda_1^{\frac{3}{2}}} \left(\frac{\ln N}{T} \right)^{\frac{1-q}{2}}, \frac{\sqrt{a(N)N}}{\lambda_1} \left(\frac{\ln N}{T} \right)^{\frac{1}{2}(1-\frac{q}{2})}, \frac{\sqrt{N}}{\lambda_1^{\frac{3}{2}}} \right\} \right) + O_p \left(\frac{1}{\sqrt{T}} \right),
\end{aligned}$$

which shows the required result for part (a) of Theorem 6. $\square$